TEACHING AND LEARNING MATHEMATICS THROUGH PRODUCTIVE AMBIGUITY, STRUGGLE, AND FAILURE: AN INTEGRATED CONCEPTUAL FRAMEWORK

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Abstract

This conceptual article presents an integrated framework for mathematics education positioning productive ambiguity, struggle, and failure as essential components of meaningful learning. Productive ambiguity invites learners to explore and interpret uncertainty, fostering curiosity and collaborative reasoning. Productive struggle focuses on sustained effort in navigating complex tasks, emphasizing the development of connections between procedural and conceptual understanding. Productive failure positions setbacks as valuable opportunities for reflection, enhancing understanding and preparing learners for future challenges. The conceptual framework comprises three interconnected phases, with ambiguity initiating exploration, struggle promoting active engagement, and failure encouraging critical reflection. These phases operate in a dynamic and recursive manner, offering a flexible model for iterative learning. Practical hypothetical classroom scenarios demonstrate the application of this framework, showcasing its potential to reimagine mathematics education as an exploratory and reflective practice.

Keywords: productive ambiguity, productive struggle, productive failure, school mathematics, conceptual framework

INTRODUCTION

A common critique of mathematics classrooms in their most traditional form includes their reliance on rigid procedures and formulaic routines, overshadowing the subject's potential to spark curiosity, foster creativity, and nurture resilience (Lockhart, 2009; Su, 2020). Methods focused on rote memorization and algorithmic problem-solving restrict students' engagement with deeper mathematical concepts and critical reasoning (Boaler, 2022; Lithner, 2008). Within this critique, a narrow approach reinforces the harmful belief that mistakes signify failure, undermining learners' confidence and their ability to tackle complex, real-world problems (Kapur, 2008; Luzano, 2024). In an era that demands adaptability and analytical thinking, the critique suggests that mathematics education must undergo a fundamental transformation (Gree et al., 2024; Jansen, 2023).

Within such a critique, this paper advocates for a conceptual shift in mathematics education, proposing a framework built around three interconnected ideas: productive ambiguity, productive struggle, and

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productive failure. Terms like ambiguity, struggle, and failure often carry negative connotations in everyday language, suggesting something to be avoided or eliminated. Nevertheless, in the context of mathematics education research, extensive literature on each of these concepts suggests that these should be celebrated as vital components of the teaching and learning process. Although each concept is individually well-documented, their combined impact remains underexplored. Together, they redefine mathematics as an exploratory and dynamic discipline, advancing not only mathematical proficiency but also essential skills such as resilience, creativity, and critical thinking.

This conceptual article explores the theoretical underpinnings of productive ambiguity, struggle, and failure, drawing connections among these concepts to present an integrative framework for mathematics education that responds to recent calls to move beyond isolated theory use, and instead engage in the deliberate coordination and integration of theoretical constructs across traditions (Artigue, 2023; Bikner-Ahsbahs et al., 2023; Kidron, 2023), as well as contribute meaningfully to the development of theoretical frameworks (Mizoguchi, 2025). In this respect, our work aligns with Kirshner's (2004) perspective on the importance of enculturation[1] in mathematics education. Kirshner emphasizes that learning extends beyond the acquisition of knowledge and skills to include the adoption of cultural dispositions through immersion in community practices. Similarly, our approach seeks to advocate for learning environments where students not only acquire conceptual understanding but also engage deeply with the sociocultural norms and practices of ambiguity, struggle, and failure, promoting a holistic development aligned with the goals of enculturation. Structured around three phases – ambiguity as a driver of exploration, struggle as a means of sustained engagement, and failure as a catalyst for reflection and refinement – our framework highlights how these phases operate both sequentially and recursively to enhance mathematical learning. Rather than making a proposal for classroom practice, our use of practical scenarios illustrates the application of these concepts in classrooms, offering educators actionable insights to reimagine mathematics as a space for inquiry and innovation.

This paper presents a conceptual framework intended to guide iterative, design-oriented educational research. Instead of proposing a validated model, the framework introduces theoretical conjectures that can inform both pedagogical design and classroom practice. Foundational scholarship in design-based research emphasizes the importance of generating and refining theoretical insight through cycles of enactment, reflection, and revision (Cobb et al., 2003; Design-Based Research Collective, 2003; Sandoval, 2004). More recent work reinforces this view, highlighting the role of early-stage theoretical articulation in shaping usable, practice-informed knowledge (Hoadley & Campos, 2022; Yang & Lee, 2025). Making the core design conjectures explicit at this stage supports the development of a robust conceptual foundation and sets the stage for future empirical inquiry into the role of frameworks that guide mathematics education practice.

PRODUCTIVE AMBIGUITY

Productive ambiguity arises when unclear or multifaceted meanings challenge learners to explore, interpret,

1 We acknowledge Alan Bishop's extensive contributions to the field of mathematical enculturation (e.g., Bishop, 1991). However, we find Kirshner's perspective on enculturation more pertinent to our work, as it directly addresses the practical issues of teaching and learning in common school contexts. In contrast, Bishop's anthropological frameworks are more closely aligned with the domain of ethnomathematics.

and resolve uncertainty. Rather than hindering understanding, ambiguity redefines mathematics as a dynamic and exploratory discipline (Sterner, 2022) and has the potential to encourage exploration and discovery in learning environments (Oliveri, 2011). A seminal contribution to this line of thinking is Gray and Tall's (1994) analysis of the duality between *process* and *concept* in arithmetic, which highlights how the inherent ambiguity of mathematical symbols (e.g., "3 + 2" denoting both the process of addition and the resulting sum) functions as a powerful feature of mathematical thought rather than as a defect. They describe this amalgam of process, product, and symbol as the "procept", emphasizing how successful mathematical thinkers flexibly shift between these interpretations. Our use of productive ambiguity resonates with this insight, but extends it beyond symbolic notation to encompass definitional, representational, and linguistic forms that arise in classroom practice.

Definitional ambiguity occurs when terms have multiple valid interpretations depending on context. For example, Bergman et al. (2024) examined the term *trapezoid*, which can refer to a shape with either one or two pairs of parallel sides, depending on the definition. Students tasked with reconciling this discrepancy engaged in mathematical reasoning, clarifying assumptions and refining their conceptual understanding. Similarly, Kercher et al. (2022) investigated tasks where teachers evaluated alternative definitions of continuity, such as whether a continuous function requires an unbroken graph or simply the absence of breaks. These exercises encouraged teachers to deliberate and justify their reasoning, raising a deeper appreciation for the flexible and evolving nature of mathematical concepts. Barwell (2005) provides another example, where students interpreted the term "sharing" in a mathematical context. Although it often has social connotations, the students negotiated its meaning in division, enriching their understanding through discussion. Together, these cases illustrate how confronting definitional ambiguity sharpens analytical skills and highlights the dynamic interplay between intuitive and formal reasoning.

Representational ambiguity arises when mathematical representations, such as diagrams or symbols, support multiple interpretations. For instance, Marmur and Zazkis (2022) explored how unconventional fractions like $\frac{1}{6.5}$ stimulated debate among prospective teachers about their validity as fractions. While some viewed these as violating traditional definitions, others recognized their structural equivalence to common fractions. This debate enhanced participants' understanding of fractional properties and flexibility in interpretation. Similarly, Feldman et al. (2020) investigated how students interpreted ambiguous motion graphs depicting distance and time. Without explicit instructions, students devised multiple interpretations – focusing on slope, area under the curve, or specific data points – mirroring real-world problem-solving where clarity emerges through exploration. Representations often act as both specific instances and general exemplars, as Giardino (2017) emphasizes, noting that a triangle in Euclidean geometry simultaneously represents a unique figure and all triangles. Foster (2011) extended this idea, categorizing ambiguity into symbolic and paradigmatic, showing how ambiguous representations prompt alternative problem-solving strategies that foster creativity and adaptability. Symbolic ambiguity refers to when the same symbol represents different ideas or concepts; for example, the × sign indicates the multiplication of real numbers or the cross product of vectors. Paradigmatic ambiguity refers to different assumptions about mathematical representations. For example, the expression "5 + 2" may refer to both the process of addition and the object resulting from this process (see also Gray & Tall, 1994).

Linguistic ambiguity often stems from words or symbols with multiple meanings. For instance, Barwell

(2005) reports on how students grappled with the phrase “more than” in inequality problems, discovering its nuanced mathematical implications through negotiation. Along the same lines, Peterson et al. (2020) introduced the idea of “clarifiable ambiguity”, where imprecise statements like “the graph goes up” required learners to specify whether they referred to slope, direction, or numerical values. Resolving such ambiguities fosters precise communication and strengthens reasoning. In turn, Sterner (2022) explored polysemy, where terms like “function” hold different meanings across disciplines, such as a relationship between variables in mathematics or a biochemical role in biology. Tasks requiring students to define function across contexts encourage adaptability and highlight connections between mathematical and scientific language. For Priestley (2013), analogies and metaphors, being inherently ambiguous, are powerful for linking mathematical concepts to familiar experiences, such as comparing proportional relationships to recipes. Such rhetorical ambiguity fosters creativity and helps students bridge the gap between abstract structures and everyday applications.

Encouraging productive ambiguity in the mathematics classroom is important because it transforms uncertainty into a powerful driver of inquiry, critical thinking, and collaboration. Ambiguity challenges learners to navigate complex ideas without predetermined answers, and cultivates a deeper engagement with mathematical ideas, prompting learners to reconcile differing perspectives and explore novel connections (Foster, 2011; Sterner, 2022). This approach redefines mathematics as an exploratory, dynamic discipline rather than a static collection of rules, helping students build confidence in tackling open-ended problems (Feldman et al., 2020; Marmur & Zazkis, 2022). The study of mathematics shifts from learning canonical procedures and facts toward dialogue and construction of understanding, and the teaching of mathematics toward skillful orchestration of productive conversations and the formation of classroom community in which mathematics is something that people *do* rather than a set of methods or a collection of lenses on the world (Díez-Palomar et al., 2021; Thanheiser, 2023). Moreover, embracing ambiguity creates a classroom environment that values curiosity and creativity, where students collaboratively negotiate meaning and refine their reasoning (Barwell, 2005; Peterson et al., 2020). Such an environment equips learners with the skills to approach real-world challenges with both rigor and flexibility, addressing the complex, ambiguous nature of problems they will encounter outside the classroom. In this way, productive ambiguity not only enhances mathematical understanding but also nurtures a mindset of lifelong learning and innovation, aligning with broader educational goals to prepare students as adaptable, creative, critical thinkers (Suzawa, 2013).

Productive ambiguity emerges most powerfully when teachers deliberately design or frame tasks to invite multiple interpretations while resisting the urge to close discussion prematurely. Teachers introduce or highlight uncertainty by designing tasks with multiple valid interpretations (whether definitional, representational, or linguistic) while resisting the urge to prematurely close down discussion (Barwell, 2005; Peterson et al., 2020). Their role is to frame ambiguity as an invitation to inquiry, supporting students to articulate assumptions, justify reasoning, and engage in collective negotiation of meaning. This orchestration resonates with research on classroom scaffolding, where teachers balance agency with guidance to sustain meaningful inquiry (Hiebert & Grouws, 2007; Warshauer, 2015b). In this way, ambiguity is transformed from a potential source of confusion into a resource for sense-making, critical thinking, and collaborative reasoning. We note that the selection of tasks as well as the balancing of agency with guidance are themselves rife with ambiguities for the teacher. If the teacher approaches their pedagogy as itself a confluence of ambiguity, struggle and failure, that is, embracing the framework we offer, we imagine a parallel and complementary growth and development for them as well.

PRODUCTIVE STRUGGLE

Productive struggle focuses on the intellectual development that emerges from interacting with challenging tasks. Defined as the deliberate and effortful engagement with meaningful tasks requiring persistence and cognitive effort, productive struggle allows mistakes and difficulties to become valuable learning opportunities. Through this process, learners develop resilience, critical thinking skills, and a stronger grasp of mathematical concepts (Warshauer, 2015a; Zeybek, 2016). As a theoretical tool, it reframes mathematics as a dynamic field that values effort, exploration, and persistence (Casler-Failing & Collins, 2022; Warshauer, 2015a). Rather than emphasizing immediate correctness, productive struggle encourages an iterative process where learners confront uncertainty, work through complex problems, and build a deeper understanding—not just of mathematical content, but also of the relative usefulness of different modes of inquiry and ways of thinking (Boaler, 2022).

At its foundation, productive struggle involves engaging with tasks that challenge students' existing knowledge while encouraging exploration of mathematical concepts and structures. This approach moves beyond rote or algorithmic solutions, inviting learners to investigate underlying relationships and conceptual frameworks for a more comprehensive understanding (Russo et al., 2021). It often includes navigating obstacles, identifying and addressing errors, and refining strategies to gain meaningful insights (Casler-Failing, 2024; Granberg, 2016). For example, Granberg (2016) observed students using GeoGebra to address errors in solving linear function problems, demonstrating how their struggles led to refined knowledge and problem-solving strategies. Collaborative discussions further enhance this reflective process, enabling learners to share diverse perspectives and improve their reasoning (Chen et al., 2024; Crawford, 2024).

The relationship between productive struggle and positive educational outcomes is well-supported in mathematics education research. Engaging with high-cognitive-demand tasks can help learners connect procedural fluency with conceptual understanding, ultimately strengthening their mathematical reasoning (Casler-Failing, 2024; Warshauer, 2015a). Casler-Failing and Collins (2022) illustrated this with pre-service teachers who grappled with robotics programming tasks that required iterative problem-solving and conceptual reasoning, helping them appreciate the role of struggle in learning. This approach also cultivates resilience and a growth mindset, encouraging learners to view challenges as opportunities for development (Russo et al., 2021). Murawska (2018) described how real-world tasks, such as verifying population density claims, encouraged students to persist and engage in critical thinking. Tasks like these allow students to practice reasoning, modeling, and argumentation, making their learning experiences more meaningful and impactful (Bolyard et al., 2023; Chen et al., 2024). Additionally, in cases where students attempt to solve problems before receiving formal instruction, productive failure has been shown to enhance understanding and support long-term retention of concepts (Biccard, 2024; Kapur, 2016). Biccard (2024) identified mathematical modeling tasks as particularly effective in promoting productive struggle, as they often present multiple solution pathways and ambiguities that require persistence to resolve.

Effective implementation of productive struggle in classrooms requires thoughtful task design, a supportive environment, and appropriate scaffolding. Tasks should be designed to challenge students without overwhelming them, encouraging exploration within their zones of proximal development and providing multiple entry points for engagement (Casler-Failing, 2024; Melani et al., 2024). For example, Melani et al. (2024) found that students working on mathematical modeling tasks initially struggled to connect real-world contexts with mathematical frameworks but ultimately developed deeper conceptual clarity through the process. Thompson (2023) found that "tinkering" and experimentation may not lead to a desired result yet have the potential to generate valuable mathematical discoveries and to cultivate self-images of mathematics doers and creators. Creating a classroom culture where struggle is normalized and mistakes are seen as integral to learning helps build students' confidence in tackling complex problems (Casler-Failing & Collins, 2022; Warshauer, 2015b). Teachers play a key role in maintaining cognitive rigor while supporting students to persist independently, avoiding the temptation to simplify tasks too quickly (Chen et al., 2024; Townsend et al., 2018). Professional development workshops, such as those described by Bolyard et al. (2023), have helped teachers learn how to guide students through challenging tasks while maintaining high standards of engagement and learning. Collaboration and reflection also play a crucial role, allowing students to refine their understanding by engaging with diverse viewpoints and participating in active problem-solving (Bolyard et al., 2023; Crawford, 2024).

Productive struggle contributes to more thoughtful and sustained learning by emphasizing effort, inquiry, and reflection over quick solutions. It encourages resilience and critical thinking, equipping learners with the confidence and adaptability to approach complex problems. Teachers who foster environments where struggle is valued help students develop the perseverance and flexibility they need to succeed in mathematics and beyond (Casler-Failing & Collins, 2022; Kapur, 2016; Warshauer, 2015a). Such teachers would see themselves as engaged in their own productive struggle, grappling with the ambiguities and challenges of teaching in ways that promise confidence and adaptability over time instead of quick solutions or recipes for their teaching.

PRODUCTIVE FAILURE

Productive failure is distinct from productive struggle, though both involve students grappling with challenging tasks. Productive struggle emphasizes persistence, resilience, and iterative refinement of ideas as learners work through difficulties, with the struggle itself fostering deeper understanding (Warshauer, 2015a; Russo et al., 2021). By contrast, productive failure is a pedagogical design in which initial failure is deliberately expected. Students are first immersed in complex, ill-structured problems without prior instruction, leading them to generate incomplete or flawed solutions. These unsuccessful attempts are not incidental but function as a preparatory phase: they activate prior knowledge, expose conceptual gaps, and prime learners to attend more closely to subsequent instruction (Kapur, 2010, 2016; Kapur & Bielaczyc, 2012; Loibl & Leuders, 2019). In this sense, the struggle embedded in productive failure differs from productive struggle. Rather than being an end in itself, it is strategically orchestrated to make later instruction more meaningful and to support long-term conceptual consolidation (Boaler, 2022; Sinha & Kapur, 2021).

In the initial problem-solving phase, students are tasked with devising diverse solutions to novel challenges using their existing knowledge. These attempts often yield flawed or incomplete solutions, activating crucial cognitive processes. Importantly, even when students do not reach a correct solution, the process itself offers valuable opportunities for learning. Struggling with the problem reveals gaps in understanding and sparks the identification of strategies, patterns, or questions that deepen their engagement. This "learning along the way" (Bateson, 1994; Schultz, 2018) cultivates skills such as critical thinking and adaptability, enabling students to tinker with ideas and refine their approaches. By the time students progress to a phase where their knowledge becomes more structured and cohesive, they are better equipped to assimilate new information. This learning sequence (transitioning from open exploration to more structured teacher-led instruction/guidance) has been shown to promote longer-term learning outcomes compared to the traditional instruction-first approach (Loibl & Leuders, 2019; Sinha & Kapur, 2021).

Research consistently demonstrates the effectiveness of productive failure, particularly in STEM education. Kapur's (2010) study with 7th-grade mathematics students in Singapore illustrates the approach's benefits. Students in the productive failure condition tackled problems related to rate and speed without scaffolding before receiving instruction. Despite their initial struggle, these students significantly outperformed their peers in the direct instruction condition in measures of conceptual understanding and the ability to transfer their knowledge to novel problems. Similar effects were observed in studies of standard deviation and linear functions, where initial exploration prepared students to consolidate knowledge more effectively during formal instruction (Granberg, 2016; Kapur, 2014).

Beyond mathematics, productive failure has shown promise in other STEM fields such as physics and computer science. For example, high school students grappling with Newtonian kinematics through ill-structured problems exhibited better transfer of knowledge compared to peers who received direct instruction (Kapur, 2008). Additionally, in computer science education, productive failure interventions have been used to teach topics like pattern recognition and algorithms. While some studies failed to find significant differences in factual retention, they highlighted qualitative benefits such as broader exploration of solution spaces and increased engagement (Steinhorst et al., 2024). However, the approach is not universally effective. Research in non-STEM domains like social sciences has yielded mixed results. Nachtigall et al. (2020) found that productive failure did not consistently outperform direct instruction in teaching social science research methods. These findings suggest that productive failure's effectiveness may be influenced by the structuredness of the domain and the clarity of canonical solutions. Domains with well-defined concepts and methods, such as mathematics and physics, may provide a better fit for productive failure than less-structured fields.

Several key mechanisms explain why productive failure often leads to better learning outcomes. The first is the activation of prior knowledge. During the initial phase, students draw on their existing cognitive frameworks to attempt solutions. These efforts engage their schemas, making them more receptive to new information presented during instruction (Loibl & Leuders, 2019). Second, struggling with problems is an integral part of the learning process, prompting students to engage deeply with the material and recognize the value of effort as a sign of cognitive growth. This productive struggle activates neural pathways, making the brain more receptive to new information and fostering connections that contribute to long-term understanding (Boaler, 2022). Such awareness primes them to focus more attentively on instruction, enabling them to revise and refine their mental models (Kapur & Bielaczyc, 2012; Nachtigall et al., 2020). Another critical mechanism is the recognition of deep features of the targeted concept. In the instructional phase, students' erroneous solutions are contrasted with canonical solutions, helping them identify and understand the fundamental principles underlying the problem (Loibl & Leuders, 2019). This process not only consolidates knowledge but also enhances its transferability to new contexts, as students develop a deeper understanding of how concepts interconnect (Sinha & Kapur, 2021). Collaboration is another feature that amplifies the benefits of productive failure. Group problem-solving activities allow students to pool their knowledge, explore multiple solution paths, and critique each other's reasoning. This collaborative environment fosters the co-construction of knowledge and provides opportunities for peer learning, further enhancing the learning experience (Steinhorst et al., 2024).

The design of productive failure interventions is crucial to their success. Tasks must be carefully crafted to balance complexity and accessibility, ensuring they are challenging enough to provoke failure but not so difficult that students become frustrated or disengaged. This "sweet spot" of complexity allows students to generate diverse solutions without being overwhelmed (Loibl & Leuders, 2019; Sinha & Kapur, 2021). Equally important is the instructional phase that follows problem-solving. This phase should explicitly address the errors and misconceptions evident in students' initial attempts, using them as springboards for deeper learning. Comparing and contrasting erroneous solutions with correct ones is particularly effective in supporting conceptual change and helping students refine their understanding (Loibl & Leuders, 2019). Moreover, providing scaffolds during the instructional phase can help students integrate new knowledge without cognitive overload (Kapur, 2016). On the other hand, tinkering environments (Thompson, 2023) are more open-ended without specific expectations, allowing for a craft mentality that enables experimentation without the pressures of creating something that 'looks like math'. Shifting the core of what counts as mathematics moves the purposes of mathematics education away from learning traditional procedures toward the facilitation of 'being mathematical' in one's orientation to the world. Similarly, a professional perspective on pedagogy for the teacher that reframes teaching "failures" as essential for personal development enables teachers themselves to worry less about failures and to become increasingly aware of deeper features of student learning that can guide them away from proscribed methods toward forms of masterful orchestration of classroom activity.

CONNECTIONS AMONG THE CONCEPTS

Overlapping features

The concepts of productive ambiguity, productive struggle, and productive failure share foundational principles aimed at fostering deeper learning by engaging students with complexity and uncertainty. Central to their overlap is the role of disequilibrium, which challenges students' existing frameworks of understanding and encourages active inquiry. For example, ambiguity frequently arises in tasks that present multiple interpretations or incomplete definitions, such as reconciling differing definitions of a trapezoid (Bergman et al., 2024). Similarly, productive struggle emphasizes sustained engagement with challenging tasks that resist immediate solutions, as observed in iterative problem-solving strategies with tools like GeoGebra (Granberg, 2016). Productive failure complements these approaches by framing mistakes and incomplete solutions as essential steps toward deeper conceptual understanding, as demonstrated in research on rate and speed tasks (Kapur, 2010). All three concepts emphasize critical thinking and active engagement. Ambiguity invites learners to explore and negotiate meanings, while struggle requires them to refine strategies and navigate obstacles (Casler-Failing, 2024). Failure, on the other hand, serves as a catalyst for reflection, preparing students to assimilate new knowledge during subsequent instruction (Loibl & Leuders, 2019). Together, these approaches redefine mathematics as a dynamic discipline rooted in inquiry and persistence, aligning with broader educational goals of cultivating adaptable and creative thinkers (Sterner, 2022). This perspective echoes Gray and Tall's (1994) observation that ambiguity in notation underpins flexible mathematical thinking, and that its absence can lead to rigid, procedural approaches. By integrating ambiguity with struggle and failure, our framework broadens this idea into a more general account of how learners can harness uncertainty productively. Mathematics is in this respect the curricular space for exploiting the motivational aspects as well as supporting learners' appreciation of uncertainty (Meaney, 2017).

Distinctive roles

Despite their interconnectedness, each concept uniquely contributes to the learning process. Productive ambiguity emphasizes exploration and interpretive reasoning, challenging students to navigate uncertainty and refine their understanding through discussion and deliberation (Oliveri, 2011; Sterner, 2022). Productive struggle focuses on the intellectual development that arises from perseverance, helping learners connect procedural fluency with conceptual understanding (Casler-Failing & Collins, 2022; Warshauer, 2015a). Meanwhile, productive failure explicitly incorporates initial setbacks as mechanisms for learning, using errors to activate prior knowledge, highlight conceptual gaps, and facilitate schema refinement during instruction (Kapur, 2016; Loibl & Leuders, 2019).

Integrated process

These concepts can function sequentially or interactively within mathematics learning. In a sequential framework, ambiguity might initiate the process by framing tasks with multiple valid interpretations. Struggle then emerges as students engage with these tasks, encountering obstacles that require sustained effort and refinement. Failure naturally follows as an outcome of these efforts, offering opportunities for reflection and deeper understanding during targeted instruction (Kapur & Bielaczyc, 2012). Alternatively, the concepts often overlap dynamically within a single task. For instance, an ambiguous problem may simultaneously elicit struggle and failure as students wrestle with multiple interpretations, refine their reasoning, and identify misconceptions. These intertwined processes exemplify how ambiguity, struggle, and failure collectively support robust and meaningful learning experiences.

Ambiguity, struggle, and failure may appear in classrooms through different routes. In some cases, they can be deliberately introduced by the teacher through the design of tasks, questions, or orchestrated discussion. In other cases, they arise naturally from students' engagement as they encounter difficulties, conflicting interpretations, or incomplete solution paths. Teachers play a crucial role in recognizing and leveraging both kinds of moments, reframing them as opportunities for inquiry, persistence, and reflection rather than as obstacles to be avoided.

AN INTEGRATIVE FRAMEWORK FOR MATHEMATICS LEARNING

Traditional perspectives on mathematics learning often portray it as a linear progression focused on skill acquisition and knowledge retention. Yet real-world problem-solving rarely follows such predictable paths. In contrast, an integrative framework that embraces productive ambiguity, productive struggle, and productive failure offers a dynamic, recursive approach. This approach reflects recent models of theory generation in mathematics education that prioritize building coherence among diverse conceptual traditions rather than enforcing a single unified theory (Artigue, 2023; Bikner-Ahsbahs et al., 2023), and it emphasizes exploration, persistence, and reflection as essential components in fostering conceptual understanding, critical thinking, and resilience.

Phases of the framework

The proposed framework (see Figure 1) operates through three interconnected phases: ambiguity, struggle, and failure. Together, these phases create a dynamic environment where learners confront complexity, navigate challenges, and transform errors into opportunities for growth.

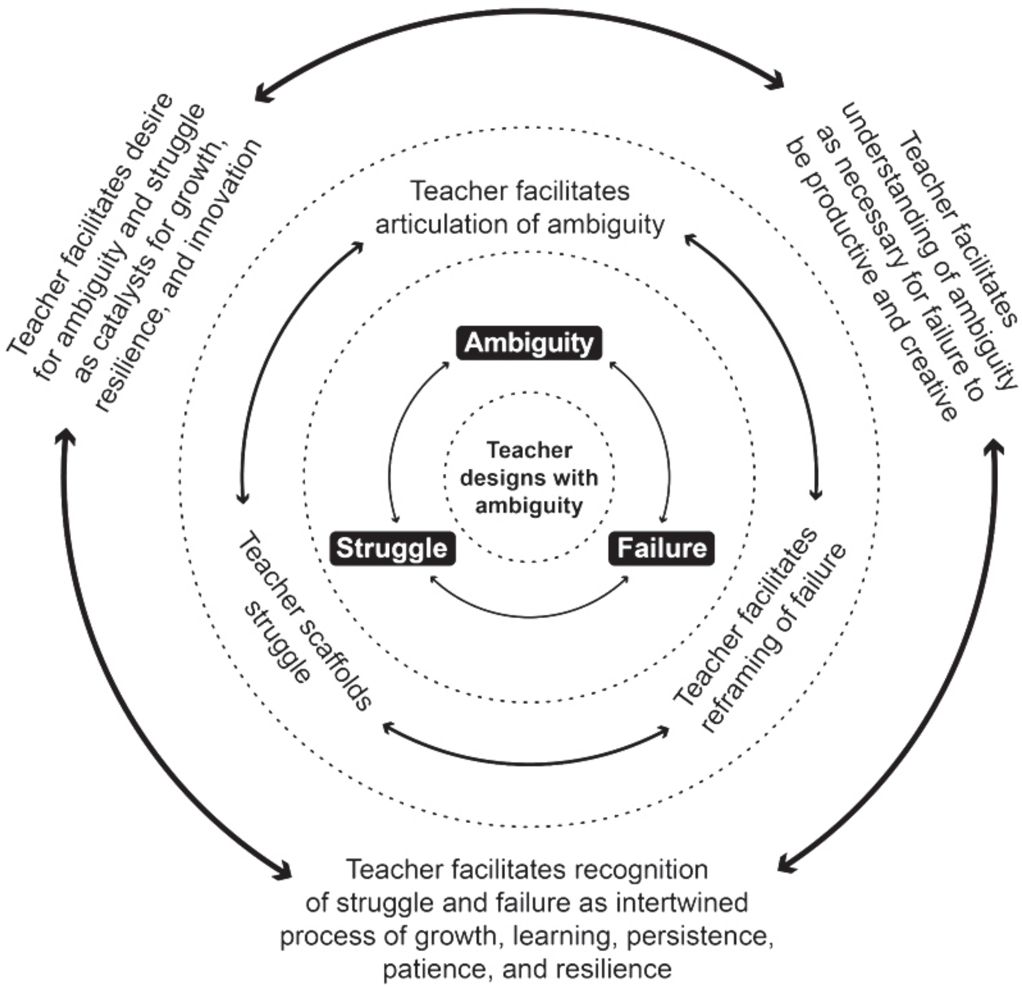

Figure 1. A conceptual framework for productive ambiguity, struggle, and failure

Phase 1 – Productive ambiguity: Opening exploration

Ambiguity invites students into tasks with multiple valid interpretations or undefined parameters, compelling them to confront uncertainty. Far from being a hindrance, this phase redefines mathematics as a dynamic and exploratory discipline, cultivating curiosity and creativity (Sterner, 2022). For instance, tasks involving definitional ambiguity, such as reconciling whether a trapezoid has one or two pairs of parallel sides, challenge learners to deliberate and justify their reasoning, thereby sharpening their conceptual understanding (Bergman et al., 2024). Representational challenges, like interpreting unconventional motion graphs, further encourage learners to explore diverse perspectives and foster flexible thinking (Feldman et al., 2020). These explorations align with mathematical practices such as reasoning abstractly and quantitatively, emphasizing the exploratory nature of the discipline. Teachers play a crucial role here by deliberately introducing or highlighting ambiguity, ensuring that uncertainty provokes inquiry without causing unproductive confusion. In our framework, we place the teacher at the center as working within their own ambiguity, struggles and failures, introducing this ambiguity in order to invite students into these opportunities. Later, the teacher facilitates an emerging desire for ambiguity and struggle as catalysts for growth, resilience and innovation.

Phase 2 – Productive struggle: Sustained engagement

Struggle emerges as students engage deeply with tasks that resist immediate solutions, cultivating persistence and resilience. This phase builds procedural fluency and conceptual understanding through iterative problem-solving and collaborative refinement (Russo et al., 2021). For example, using tools like GeoGebra, students iteratively refine geometric constructions, address errors, and engage in critical discussions that deepen their mathematical reasoning (Granberg, 2016). The intellectual growth facilitated by productive struggle comes from a focus on effort and reflection rather than quick correctness. Learners are encouraged to construct viable arguments and evaluate diverse strategies, strengthening their confidence in approaching mathematics as an evolving and rich discipline (Warshauer, 2015a). Teacher support in this phase involves scaffolding perseverance: monitoring when to intervene, maintaining cognitive rigor, and encouraging students to persist without prematurely simplifying the challenge. Here the teacher facilitates an appreciation for ambiguity and struggle as necessary for failure to be productive and creative.

Phase 3 – Productive failure: Reflection and refinement

Reframing failure as a constructive phase of learning creates critical opportunities for reflection and conceptual breakthroughs. Errors reveal gaps in understanding and prompt targeted instruction that refines conceptual frameworks. Research demonstrates the value of engaging with ill-structured problems, such as designing mathematical models for real-world phenomena, as a means of priming students for deeper learning during subsequent instruction (Kapur, 2010). Contrasting students' erroneous solutions with canonical approaches fosters a deeper understanding of underlying principles and enhances the transferability of knowledge (Loibl & Leuders, 2019). Normalizing failure as an integral part of the learning process supports resilience and a growth mindset, equipping learners to address complex problems confidently. In this phase, teachers help frame errors constructively, guiding students in comparing flawed and canonical solutions so that setbacks become opportunities for generative reflection, and facilitating the appreciation for failure and struggle as intertwined processes of growth, learning, persistence, patience, and resilience.

A cyclic and recursive process

The integrative framework operates as a recursive process with feedback loops among productive ambiguity, productive struggle, and productive failure, enabling continuous refinement and deeper engagement. These phases are not strictly sequential; ambiguity within a task often simultaneously elicits struggle and failure as students explore multiple interpretations, refine strategies, and address misconceptions. Ambiguity sparks exploration, leading to struggle as learners engage deeply with challenging tasks. Struggle, in turn, primes learners for the insights generated through failure, which often loops back into further ambiguity and inquiry. For instance, in tasks requiring mathematical modeling of real-world phenomena, ambiguity in data interpretation leads to struggle with model construction and eventual failure in initial attempts. These failures prompt reflection and refinement, including reframing initial questions, reconsidering the purpose of the mathematical activity, or exploring alternative approaches (Biccard, 2024; Kapur, 2016). Dialogue with peers or instructors often becomes a critical element of this process, enabling learners to articulate reasoning, exchange ideas, and collaboratively refine strategies. Revisiting tasks after initial engagement allows learners to deepen their understanding, build connections between concepts, and refine their problem-solving strategies. This iterative process, which can occur within various instructional models (whether tasks and instruction are distinct or seamlessly integrated), redefines mathematics as a dynamic cycle of inquiry, persistence, and reflection. Such an approach equips learners to tackle complex challenges and fosters a mindset attuned to exploration and growth (Boaler, 2022).

ILLUSTRATIVE SCENARIOS: BRIDGING THEORY AND PRACTICE

To illustrate how productive ambiguity, struggle, and failure can manifest across different areas of school mathematics, we present four hypothetical classroom scenarios spanning the domains of algebra/functions, geometry/measurement, data/modelling, and number/patterns. These domains were chosen to demonstrate breadth rather than exhaustiveness, since together they capture key strands of the mathematics curriculum where uncertainty and challenge can be deliberately harnessed. Mathematical modelling appears in more than one scenario because such tasks naturally invite open interpretation, iterative reasoning, and the possibility of error; nevertheless, the inclusion of geometry- and number-focused tasks shows that our framework applies equally well beyond modelling contexts. The scenarios are not intended as prescriptions for practice but as illustrative contrasts that highlight how ambiguity, struggle, and failure can take different forms across mathematical content areas and grade levels. At the same time, they are grounded in relevant literature and draw on the tradition of critical fiction (e.g., Ryan, 2025; Hrastinski, 2023, 2025), using narrative as a methodological tool to surface the often-overlooked social and cultural dynamics of classroom practice. Together, these scenarios offer a practical lens through which teachers and researchers can reimagine classrooms as dynamic spaces for exploration, persistence, and reflective learning.

To clarify how the cyclical process operates across scenarios, Table 1 summarizes how each example illustrates productive ambiguity, productive struggle, and productive failure in relation to the recursive framework.

Scenario (domain)	Productive ambiguity (entry point)	Productive struggle (challenge)	Productive failure (expected/leveraged errors)
Graphs of linear functions (algebra/functions)	Missing axis labels; uncertain context of variables	Calculating slope and intercept from incomplete data	Misidentifying rise/run; incorrect assumptions about axes
Area and perimeter of a garden (geometry/ measurement)	Terms like "approximately" and "reasonable cost" open to interpretation	Balancing algebraic equations with geometric constraints	Exceeding limits; misinterpreting cost structures
Virus dataset (data/modelling with exponential functions)	Unclear time scale (days vs. weeks)	Constructing and refining exponential models against messy data	Misestimating growth rates; misreading dataset
Tile patterns (numbers/ patterns)	Open-ended invitation: multiple ways to see and extend the pattern	Translating visual pattern into numeric/ general rule	Missing the doubling structure; miscounting totals

Table 1. Alignment of scenarios with productive ambiguity, struggle, and failure

Scenario 1: Exploring graphs of linear functions

In a high school algebra classroom, students are invited to identify the equation of a line based on a partially labeled graph. The graph includes missing axis labels and an ambiguous starting point for the line, prompting students to interpret whether the x-axis represents time, distance, or another variable. This deliberate use of productive ambiguity challenges students to make assumptions and justify their reasoning, promoting interpretive thinking and collaborative discussions. For instance, some groups assume the axis represents time, while others interpret it as distance, leading to a class-wide debate about the role of contextual information in mathematical modeling. This approach aligns with the value of ambiguity in encouraging deeper engagement and understanding, as described by Kapur (2010, 2014) in the productive failure framework.

As the activity continues, students engage in productive struggle by attempting to calculate the slope and y-intercept of the line using incomplete data. They grapple with errors, such as misidentifying the rise and run, and refine their understanding of proportional relationships through persistent effort. Peer collaboration further enhances their problem-solving processes, as groups share strategies and refine their approaches. Such collaborative environments are identified as critical for productive struggle, enabling students to persist and learn through challenge (Hiebert & Grouws, 2007; Warshauer, 2015a).

The teacher compares students' equations and predictions for line transformations with a canonical solution, highlighting common misconceptions and errors. This integration of productive failure allows students to reflect on their flawed assumptions and recalibrate their understanding, consistent with research that highlights the value of error analysis in developing conceptual clarity (Kapur & Bielaczyc, 2012). Through this reflection, students revisit their initial interpretations of the graph's ambiguous axes and redefine their assumptions. This process sparks new cycles of inquiry, where students test refined models and challenge each other's conclusions, reinforcing the recursive nature of learning as identified in productive struggle frameworks (Young et al., 2024).

Scenario 2: Investigating area and perimeter relationships

In a middle school geometry class, students are tasked with designing a rectangular garden with specific area and perimeter constraints, balancing cost and material use. The problem's ambiguous phrasing, such as "approximately 20 square meters" and "reasonable cost", introduces productive ambiguity, requiring students to negotiate definitions and justify their interpretations. Some groups interpret "approximately" as a 10% margin of error, while others use stricter criteria, sparking debates about precision and practicality. This ambiguity primes students to think critically about mathematical definitions and their real-world implications, consistent with Zeybek's (2016) findings on ambiguity in geometry tasks.

As students calculate dimensions that meet the constraints, they encounter productive struggle by grappling with algebraic equations and geometric reasoning. They test various combinations of length and width, iteratively refining their models when calculations exceed the cost constraints. This process strengthens their understanding of the relationship between area and perimeter while encouraging resilience and critical thinking, as emphasized by Kapur (2010) and Warshauer (2015b) in their discussions of high-cognitive-demand tasks.

During the productive failure phase, students present their garden designs and identify errors, such as exceeding perimeter limits or misinterpreting the cost structure. The teacher uses these mistakes to highlight key misconceptions, prompting students to revisit their definitions of "approximately" and "reasonable cost." This renewed focus on ambiguity initiates another cycle of exploration and refinement, as students revise their designs with greater precision and deeper insight into mathematical modeling, a process documented by Young et al. (2024).

Scenario 3: Modeling real-world data with exponential functions

In an advanced high school mathematics class, students analyze a dataset modeling the spread of a virus. The dataset includes missing contextual details, such as whether time is measured in days or weeks, introducing productive ambiguity that compels students to make and justify assumptions. This ambiguity leads to diverse interpretations, as some students assume weekly measurements while others posit daily increments, sparking class discussions on how assumptions influence modeling outcomes. This aligns with research by Kapur (2014) on the value of exploring ill-defined problems in building conceptual understanding.

Students engage in productive struggle as they attempt to create an exponential model predicting future infection rates. They encounter discrepancies between their predictions and observed data, grappling with the complexities of exponential growth. Through iterative adjustments and collaborative discussions, students refine their models, gaining a deeper understanding of the relationship between mathematical representations and real-world phenomena, as highlighted by Russo et al. (2021) in their analysis of teacher strategies during challenging mathematical tasks.

The teacher presents a well-calibrated model and contrasts it with students' predictions, highlighting common errors such as misestimating growth rates or misinterpreting the dataset's structure. This phase of productive failure enables students to reflect on their reasoning and refine their analytical skills. Encouraged by the teacher, students revisit the ambiguous aspects of the dataset, exploring how adjusting assumptions impacts their models. This recursive process allows them to cycle through ambiguity, struggle, and failure again, leading to progressively more accurate and nuanced models.

Scenario 4: Exploring patterns with multiplication

In a 4th-grade classroom, students investigate a visual pattern involving a grid of colored tiles, where the number of tiles doubles with each successive row. The task is designed to be open-ended by allowing for multiple approaches and interpretations: "What patterns do you notice? Can you predict how many tiles will be in the 10th row?" Rather than prescribing a specific method, the question invites students to explore various strategies, such as visualizing the pattern, creating a table, or using mathematical reasoning. This use of productive ambiguity encourages diverse perspectives and sparks discussion about growth patterns and representations, aligning with Granberg's (2016) findings on the value of exploratory tasks in engaging students.

As students attempt to describe the pattern, they engage in productive struggle, grappling with translating visual patterns into mathematical expressions. Many initially miscalculate totals or fail to recognize the doubling structure, requiring persistence and iterative problem-solving to refine their understanding. Peer discussions help clarify misconceptions, as students collaboratively explore different approaches to represent the pattern numerically and visually. These activities reflect the benefits of scaffolding and collaboration in supporting productive struggle, as noted by Warshauer (2015b) and Lemley et al. (2019).

In the final phase, the teacher introduces a symbolic representation of the pattern using powers of two, comparing it to students' earlier strategies. This phase of productive failure highlights the value of errors in deepening understanding, as discussed in Kapur's (2010, 2014) research on productive failure. The teacher encourages students to revisit the ambiguous aspects of the task, such as identifying different ways to predict totals for future rows. This renewed exploration launches another cycle of ambiguity, struggle, and failure, reinforcing the recursive and iterative nature of the learning process, as described by Young et al. (2024).

DISCUSSION

The significance of this conceptual framework lies in its potential to challenge entrenched practices in mathematics education and offer a new way to approach learning. Mathematics classrooms often focus on procedural correctness over deeper inquiry, limiting opportunities for creativity and critical thinking. This framework suggests a different approach, treating ambiguity as a starting point for exploration, struggle as an essential part of engagement, and failure as a steppingstone to understanding. In doing so, we also build on Gray and Tall's (1994) seminal account of ambiguity, positioning our framework as an extension that connects their insights on symbolic duality with broader classroom practices involving struggle and failure. These elements together redefine the learning process as dynamic, ongoing, and reflective. Its importance is particularly clear in today's world, where building resilience and adaptability is critical. The framework aligns with broader goals of helping students develop skills for handling the complexities of real-world problem-solving. It also encourages classrooms to reflect the cyclical nature of real mathematical inquiry, providing practical guidance for anyone working to create more meaningful learning experiences. Our work is important, especially with the concept of VUCA receiving increasing attention in educational discourse (see, e.g., Canzittu, 2022; Sarid & Levanon, 2023; Stein, 2021). VUCA stands for volatility (the rate and unpredictability of change in a situation), uncertainty (a lack of clarity about the present or future outcomes), complexity (the involvement of multiple interconnected variables and factors that make understanding and decision-making difficult), and ambiguity (situations where the meaning of an event or condition is unclear and open to interpretation). Addressing VUCA in educational contexts helps build resilience, adaptability, and critical thinking. These are essential skills that enable teachers and students to navigate challenges and thrive in an ever-changing world.

While our framework emphasizes the centrality of student agency in inquiry, engagement, and reflection, these processes cannot be understood as entirely student-driven. Teacher support remains essential for sustaining productive learning. In the ambiguity phase, teachers design or highlight uncertainty in ways that stimulate curiosity while avoiding unproductive confusion. During struggle, teachers provide scaffolding that sustains perseverance and maintains cognitive rigor without prematurely removing challenge. In the failure phase, teachers frame errors as legitimate learning opportunities and guide students in making reflective connections between flawed and canonical solutions. In this sense, the role of the teacher is not to diminish student-led activity but to orchestrate the conditions under which exploration, engagement, and reflection become generative. This stance resonates with research emphasizing classroom orchestration and scaffolding as mechanisms for balancing agency and guidance (e.g., Hiebert & Grouws, 2007; Kapur & Bielaczyc, 2012; Warshauer, 2015b). Equally important are the broader classroom conditions that allow ambiguity, struggle, and failure to be experienced as productive rather than discouraging. These conditions include the cultivation of a safe classroom climate where risk-taking and mistake-making are normalized (Boaler, 2022; Warshauer, 2015b), the use of scaffolding and inclusive practices that ensure all learners can persist meaningfully with challenging tasks (Casler-Failing & Collins, 2022; Townsend et al., 2018), and the teacher's orchestration of discussions that balance exploration with conceptual rigor (Hiebert & Grouws, 2007). Without such conditions, ambiguity risks collapsing into confusion, struggle may devolve into frustration, and failure can be perceived as deficit rather than opportunity.

Our framework stands out because of its integrative approach. Similar to the way Kidron (2023) shows that comparing how different theories anticipate student thinking can lead to stronger designs, our work combines ideas that are usually studied separately, addressing concerns such as those raised by Mizoguchi (2025) about the need for contributions that advance rather than simply apply educational theory. Productive ambiguity, struggle, and failure have been studied individually in education research, but this is the first clear attempt to connect them into a single framework. This connection offers a more complete picture of how these ideas work together to improve learning. It provides a way to think about mathematics education as an iterative process where students tackle challenges, engage deeply with the material, and reflect on their progress. As Kirshner (2000) highlights, framing pedagogical strategies with clear, theory-based guidance can empower teachers to design tasks that cultivate exploration, persistence, and thoughtful reflection, and also to recognize when to expect their presence. For example, exercises designed for habituation should minimize ambiguity, puzzles designed for the promotion of persistence, curiosity and courage emerge from non-routine activity often unrelated to contemporaneous curricular objectives, and probes selected for better understanding of students' conceptual understanding typically challenge students at the boundaries of their knowledge. Kirshner's (2002) cross-disciplinary approach parallels this framework's focus on ambiguity, struggle, and failure as interrelated components of learning. This approach is not just theoretical; it has practical applications. It gives a structure for designing tasks that encourage exploration, persistence, and thoughtful reflection. Framing mathematics learning as a blend of ambiguity, struggle, and failure offers a clearer understanding of these ideas while showing how they can be used in real classrooms.

The adaptability of the proposed framework depends on its application across diverse cultural contexts, as societal and educational norms strongly influence responses to ambiguity, struggle, and failure (Ansalone, 2009; Galván et al., 2011). In East Asian education systems, where high-stakes performance and rote learning are often emphasized, ambiguity and failure may be viewed negatively. For example, in Japan, while lesson study promotes localized classroom innovation, it frequently lacks formal theorization and may conflate pedagogical practice with academic research (Mizoguchi, 2025). As a result, learners in such contexts often struggle with open-ended tasks that lack clear parameters or definitive answers (Richmond, 2007; Wang & Tai, 2024). In contrast, educational systems emphasizing exploratory and inquiry-based learning, such as those in Nordic countries, tend to normalize uncertainty and regard mistakes as integral to the learning process (Pedersen & Haavold, 2023). In addition to cultural influences, social class significantly shapes students' encounters with ambiguity and failure. Educational experiences often align with the expectations for specific social classes, shaping how individuals respond to uncertainty and problem-solving tasks (Gates, 2019; Lubienski, 2000). For instance, curricula for working-class students tend to focus on following directions and performing repetitive tasks, which require little problem-solving or engagement with ambiguity. In contrast, middle-class curricula often emphasize creativity and inquiry, fostering skills necessary for professional and creative careers. Meanwhile, elite classes are often trained to leverage the work of others, focusing on management and strategic decision-making (Bukodi et al., 2024). These systemic differences indicate that engaging with ambiguity and productive failure aligns most closely with middle-class educational practices, as formal schooling systems often uphold and perpetuate middle-class norms, implicitly or explicitly marginalizing the experiences and cultural capital of lower social classes (Byrne, 2009). Addressing these cultural and social class differences requires intentional framing and scaffolding. Ultimately, the framework's success lies in its capacity to adapt to the unique educational environments, cultural expectations, and social class contexts where it is implemented, ensuring inclusivity and effectiveness across diverse settings.

There are, however, some limitations to this framework. As a conceptual model, it has not yet been tested widely in practice, though similar ideas have been applied in early childhood environments. For instance, Meaney (2017) observed that toddlers signal uncertainty to supervising adults through behaviors such as whining. In response, the adults offer support by maintaining proximity as the child works through their problem. Questions about relevance and scalability in different classroom settings remain for this framework, grounded in theory and examples from the literature. Additionally, there are concepts that could be added to make the framework stronger. For example, using metacognitive strategies might help students manage ambiguity, and building emotional resilience could provide extra support during moments of struggle (for more, see Boaler, 2022). Considering cultural differences – such as how various classroom norms and values shape responses to ambiguity and failure – could also add valuable depth. Practical challenges also need to be addressed. Designing tasks that are both challenging and accessible requires careful planning to ensure students stay engaged without feeling overwhelmed. Teachers may need additional training or resources to use this framework effectively and to create an environment where ambiguity, struggle, and failure are seen as natural parts of the learning process.

The next step is to move this framework from theory to practice. Classroom activities need to be designed with productive ambiguity, struggle, and failure at their core. These tasks should be open-ended and allow for multiple ways of thinking, helping students explore, question, and refine their understanding. Research is needed to examine how this framework impacts students' mathematical reasoning, resilience, and retention of knowledge. Comparative studies across subjects and age groups could provide valuable insights into their broader use. Future work should also consider related factors, such as how collaboration, self-reflection, and emotional support could enhance the framework. Adding these dimensions could make it more adaptable and effective for a wide range of learners.

This framework presents a fresh perspective on mathematics education, placing curiosity, persistence, and reflection at the heart of learning. The integration of productive ambiguity, struggle, and failure offers a structured way to approach complex ideas and make learning more meaningful. Although still in the early stages, it has great potential to change how mathematics is taught and how students experience the subject. Ongoing research and refinement will be essential to fully realize what this framework can achieve.

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