# WHAT DOES CULTURE HAVE TO DO WITH KNOWLEDGE? A CONTRIBUTION TO SOLVING THE RIDDLE OF THE ONTOLOGICAL CONSTITUTION OF KNOWLEDGE FROM A SOCIOCULTURAL PERSPECTIVE

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## Abstract

Sociocultural research has been successful in helping us to better understand how social, cultural, and political contexts shape the teaching and learning of mathematics. Against rationalist and empiricist accounts of knowledge, sociocultural research has forcefully argued that knowledge is culturally situated. The results contributed by ethnomathematics in recent years support this idea. However, there still remains the problem of theoretically explaining how knowledge is anchored in culture. Such an explanation requires a clear theoretical account of the ontological constitution of knowledge. In the first part of this paper, I explore this problem by offering an overview of knowledge as conceived in one of the contemporary sociocultural theories in mathematics education: the theory of objectification. In the second part I turn to the importance of clarifying the conceptions of knowledge we use in mathematics education research. My argument is that, since learning is always about learning something, it is virtually impossible to understand learning from an educational viewpoint if the question of knowledge has not been elucidated first.

Keywords: Ontology of mathematics, Hegel, dialectics, theory of objectification, knowledge, contradiction

## 1. INTRODUCTION

The epistemology of mathematics has been the object of a broad range of investigations in mathematics education (Artigue, 1995; Brousseau, 1989; Chevallard, 2006; Glaeser, 1981; Kuzniak & Vivier, 2019). These investigations have contributed to clarify several aspects of the complexity of mathematical knowledge (Barbin, 2009; Fried, 2009; Sierpinska, 1985) and its implications in mathematics teaching and learning (Fauvel & Maanen, 2000; Furinghetti, 1977; Guillemette & Radford, 2022). However, educational reflections have shown limited interest in tackling the *ontological* question of the *nature* of knowledge in general, and mathematical knowledge in particular (among a few notable exceptions are Otte (2003), de Freitas et al. (2017), and Margolinas and Bessot (2023)).

Although, in recent times, the role of cultural contexts has been recognized in discussions about mathematical knowledge, cultural contexts often appear as merely contextual backdrops where knowledge unfolds—that

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is, as a simple *arrière-fond* (see, e.g., Glas, 1993). Consequently, mathematical knowledge tends to be understood either as organically unrelated to culture (as in Piaget's epistemology) or only loosely connected to it. In a fundamental sense, implicitly or explicitly, mathematics appears as transcending its cultural and historical context.

However, this transcendental conception of mathematics vis-à-vis culture becomes increasingly problematic when set against sociocultural perspectives, which advance a fundamentally different conception of knowledge—one in which knowledge is understood as *inherently cultural*.[1] Within this framework, culture is conceived as a constitutive component of knowledge itself. The substantial body of research emerging from ethnomathematics in recent decades offers both empirical and theoretical support for this view (D'Ambrosio, 2006; Rosa et al., 2017). Nevertheless, despite these contributions, a significant theoretical challenge persists: formulating a coherent and robust conception of knowledge that adequately reflects its culturally embedded ontological character.[2]

This paper is based on the idea that what we consider knowledge to be (the *ontological* question) and the idea of how we come to learn about knowledge (the *epistemological* question) are fundamentally intertwined. Accordingly, the first part of the paper engages with the ontological question of the constitutive role of culture in mathematics knowledge as developed within Vygotskian cultural-historical theory, with particular reference to the theory of objectification (Radford, 2021).[3] In the second part, the paper addresses the importance of clarifying the concept of knowledge that is adopted in mathematics education research. I argue that the conception of knowledge we adopt in our educational theories and research approaches greatly influences our theoretical conception of learning and its practical investigation. The clarifications of the conceptions of knowledge that are embedded in mathematics education theories may help us to better understand similarities, contradictions, and differences between theories, a point that may lead to an enrichment of our field both in its theoretical and practical dimensions.

## 2. A NEO-HEGELIAN MATERIALIST DIALECTICAL PERSPECTIVE

The conception of knowledge advanced in this paper aims to move beyond the conceptions of knowledge conveyed by the empiricist, Platonist, and rationalist paradigms. Empiricism, both classical and contemporary, typically treats mathematical knowledge as a subjective construction (see, e.g., von Glasersfeld, 1995). In contrast, contemporary Platonism posits a transcendental ontology in which "numbers, functions and other

1 Lizcano (2009) presents an interesting investigation of differences between ancient Greek and Chinese thought that shows how cultural conceptions of reality lead to two different forms of mathematical thinking.

2 Within the French tradition of didactique des mathématiques, the work of Chevallard and Brousseau both acknowledge the role of culture in mathematics, particularly in terms of the functional role of social institutions (see, e.g., Chevallard, 1990; Brousseau, 2006). However, neither explicitly addresses the question of how culture—as the societal process that deals with the question of cultural meanings and beliefs about the nature of reality and its individuals—contributes to the ontological constitution of knowledge.

3 To avoid misunderstandings, I would like to clarify that the goal of this paper is not to provide a survey of sociocultural theories in mathematics education or their views on the role of culture in learning mathematics as other works have done (for example, Atweh et al., (2001) and Bishop (1988)). The goal of this paper is to explore, from an educational perspective, a precise ontological question: the constitutive role of culture in mathematics knowledge.

abstract entities exist in their own right" (Brown, 2012, p. 3). Drawing on the legacies of Descartes and Kant, contemporary rationalism considers mathematical knowledge as teleologically oriented, rooted in a universally shared faculty of reason that enables the apprehension of fundamental mathematical truths (e.g., BonJour, 1998). This paper moves beyond these paradigms by positing that knowledge is inherently consubstantial with its social, cultural, and historical contexts. To do so, I shall turn to the dialectical approach Hegel articulated in his work, although with important modifications.

Hegel has the merit of having suggested a *processual* view of knowledge. For him, in contradistinction with the array of a-historical thinkers that preceded him, such as Berkeley, Hume, Descartes, and Kant, knowledge is neither the product of a rationalist *res cogitans* nor of an empiricist *res sensibilis*. Knowledge, for him, is a historical entity that unfolds in deep connection with the activities human individuals carry out in their specific contexts (Hegel, 1977).

However, as Marx (1988) noticed, in Hegel's account, the processual view of knowledge and the idea of knowledge as produced in human activity remain *abstract*; that is, removed from the specific cultural context and the concrete sentient labouring individuals.

In her book on dialectics, Pagès reminds us that

> One of Hegel's "greatnesses" is that he thought in terms of dialectics, in terms of the negation of negation, the production of individuals by themselves … Hegelian dialectics is also linked to the recognition of the role of [activity or] human labor … Nevertheless, dialectical work, the work of the negative, remains, for Hegel, that of the Concept or the Spirit. This is abstract for Marx, when it is necessary, on the contrary, to consider "real, acting individuals, as they are conditioned by a determined development of their productive forces and the corresponding mode of relations." (Pagès, 2015, p. 49)[4]

The abstract aura that surrounds Hegel's dialectics is Hegel's idealist view. And it is precisely this view that Marx (1998) endeavoured to fix through the Spinozist monistic view he sketches in his *Theses on Feuerbach* and develops further in the *German Ideology*.

In Hegelian materialist dialectics, knowing and being are mutually constitutive. This dialectical co-determination implies that, as individuals produce knowledge, new forms of subjectivity and consciousness emerge—forms that, in turn, make possible the envisioning of further modes of knowledge production. Marx, however, introduces a critical transformation of this framework: such processes unfold not in the abstract realm of ideas, but within a concrete, cultural-historical dialectics shaped by the interplay of economic, political, aesthetic, and other societal forces. In this shift, we move from an idealist dialectics to a materialist one.

## 3. KNOWLEDGE

In this section, I present an overview of a dialectical materialist conception of knowledge that allows us to

4 In this passage, Pagès cites Marx's (1968) *L'idéologie allemande* [German Ideology], p. 50.

understand the ontological constitutive role of culture.

### 3.1 The roots of knowledge

Following Hegel, Marx, and Vygotsky, the origins of knowledge are understood as grounded in human efforts to respond to their material, intellectual, spiritual, aesthetic, and social needs. Knowledge, hence, does not emerge from simply doing things; it emerges in concrete, sensuous, and practical *activity*.

As Leont'ev (1978) emphasized, it is through collective, goal-directed human activity that knowledge is produced—activity that is historically situated and socially mediated. Lektorskii (1978) encapsulated these ideas by arguing that the origin, structure, and functioning of knowledge are grounded in "the mechanism giving rise to it, namely, a specific system of activity" (p. 52). This perspective on knowledge underscores the inseparability of cognition from the historically and culturally mediated practices through which human beings engage with the world.

Thus, the concept of a circle is not an a priori Kantian category—that is, it does not exist prior to, or independently of, the individuals' lived experience of the world. For instance, the proposition "the sun is round" is not the result of a deduction derived from innate categories of thought. On the contrary, the origins of the concept of a circle are to be found in concrete, practical activity. As Mikhailov (1980) insightfully observed, "People could see the sun as round only because they rounded clay with their hands" (p. 199). Likewise, our ability to perceive lines, rectangles, and other geometric forms in the environment stems from the manual shaping of materials by our ancestors—who "gave shape to stone, sharpened its edges, [and] gave it facets" (p. 199). In opposition to Aristotelian philosophy, Mikhailov argues that "The meaning of the words 'border,' 'facet,' 'line' does not come from *abstracting* the general external features of things" (p. 199), but rather emerges from historically situated, purposeful human activity.

To be understood, the genesis and subsequent transformations of knowledge need to be located within the framework of cultural production and consumption. To satisfy their various needs (not only of survival but also needs created in and by society), people produce and consume. To give a contemporary example, the postmodern forms of capitalist production create new needs of consumption (e.g., financialization), which leads to new forms of production and the ensuing new commodification products revolving around the increasing dominance of speculative and risk markets.

Conceived in this way, knowledge in general, and mathematical knowledge in particular, is not a psychological entity sunk into the individual's mind; neither is it a construction through which "a teleological [universal] reason running throughout all historicity announces itself," as Husserl claims knowledge to be in his *The Origins of Geometry* (Husserl, in Derrida, 1989, p. 180). Knowledge is a cultural-historical entity that must be understood as movement, as a "*process*" (Hegel, 2012, p. 262) imbricated in the life of individuals and produced in that life.

### 3.2 A dialectical materialist conception of knowledge

Hence, from the perspective advanced here, knowledge is neither a speculative construct of the mind nor a mere correspondence with, or representation of, an external reality. Knowledge does not simply reflect reality; it *refracts* it. In other words, rather than re-producing or re-presenting an allegedly independent reality, the knowledge that individuals generate in joint activity carries the imprint of their understanding of a reality that they are simultaneously producing. This entails a dialectical unity of knowing and becoming—a unity through which reality itself is actively constituted. In constituting reality, knowledge also generates particular forms of intelligibility.

In more specific terms, within this framework (Radford, 2021), knowledge is understood *immanently*: as a dialectical system: as a dynamic, dialectical constellation of historically and culturally constituted forms of thinking, speaking, acting, and symbolizing that refract cultural views about the nature of reality and its individuals.

### 3.3 Knowing how to fish

I now turn to a contemporary example from a fishing community in Macau, a city located in Rio Grande do Norte in the northeastern region of Brazil. Macau is home to two fishing associations that play a vital social, economic, and cultural role in the community. Through an intergenerational process of knowledge transmission, passed down orally and through direct interaction with the natural environment and local ecosystem, the fishers have developed a complex constellation of artisanal fishing techniques. These techniques are informed by astronomical observations, beliefs about the nature of the world, the role of the wind, water depth, and the seasonal movement of fish.

> To carry out artisanal fishing, fishermen use small and medium-sized boats powered by engines, oars, and paddles, and artisanal equipment such as nets, handlines, and cast nets, which they can store in their huts, take home, or, when they do not need to be repaired or cleaned, leave on the boats. They also use other types of equipment to finish fishing, such as ice to preserve the fish and coolers. (Beserra, 2025, p. 29)

Knowing how to fish is this constellation of forms of thinking, doing, speaking, and interacting with nature that Macau fish harvesters resort to in their everyday life. These forms do not live outside, in a Platonic world; they live at the very heart of Macau's life. They are embedded in an economic sphere of production and consumption. Indeed, fishers sell their fish or offer their catch to community merchants, who act as sales centres. Usually, the merchants' houses are located in a port where boats can anchor, which facilitates the commercial transaction. After selling their fish, the fishers can return to continue fishing. Like all knowledge, Macau fishing knowledge is anchored in the politics of living.

## 4. THE ONTOLOGICAL CONSTITUTION OF KNOWLEDGE

In the previous section I presented a dialectical materialist conception of knowledge. Knowledge, it was argued, emerges from *cultural-historical activity*, which, in the neo-Hegelian dialectical approach, is much more than simply doing things or carrying out a series of actions. German language makes a distinction that can help us grasp activity in the sense conveyed here. German language distinguishes between *Aktivität* and *Tätigkeit* (Roth & Radford, 2011). While the former refers to being busy with something (as reading a paper or watching television), the latter refers to those processes where individuals interact collectively in a strong social sense and they do so in the collective pursuit of a same object—the object of the activity in Leont'ev's (1978) sense. Cultural-historical activity (*Tätigkeit*) always unfolds through cultural-historical mediations that affect what the individuals produce in activity (e.g., knowledge). In other words, the objects produced in the activity are not independent of the specificities of the activity. So, if we want to understand knowledge, we need to turn to the cultural-historical activities from where it is produced.

While the preceding ideas provide a general sense of the ontology of knowledge as conceived within dialectical materialism, we must go further to apprehend more fully its ontological constitution—its very *substance*, so to speak—and thereby contribute to the task before us: unravelling the riddle of the cultural constitution of knowledge.

The ontological substance of knowledge—the very "stuff" of which it is constituted—encompasses both its modes of existence and the nature of its various relationships to individuals' reality. To apprehend this substance more fully, I first examine the modes of existence of knowledge, then turn to a reinterpretation of the relationship between the abstract and the concrete, and finally address the roles of difference and contradiction as key elements of the ontology of knowledge.

### 4.1 Knowledge and its modes of existence

***Potentiality***

One of the trickiest questions that arises in discussions about knowledge is the one concerning its modes of existence. To address this question, it might be worthwhile to start with something with which we are familiar: language.

Language already exists in the culture in which we are born. With its grammar, vocabulary, figures of speech, etc., the language through which we think, express ourselves, and communicate with others, is already there, in our culture. Its mode of existence is *disposition*. Language is a general cultural disposition that *empowers* us to say things. And so is knowledge. Knowledge empowers us to think and do things in certain ways. To use a Hegelian term, we can say that knowledge, like language, is a *potentiality*.

***Actuality or the manifestation of knowledge***

When we talk, language as potentiality acquires life. It is *manifested* in what we say. To distinguish between potentiality and its manifestation, Hegel used the term *actuality*. Following Aristotle, in Hegel's philosophy, potentiality (δύναμις, dunamis) refers to a *capacity* or *empowerment* to do something (Aristotle, 1984; see Metaphysics, 1048a). Actuality (ένέργεια, energía), by contrast, is the concrete happening of that which, before being put in motion, before being manifested, was potentiality.

Potentiality and actuality constitute two orders of existence of language and knowledge; the elucidation of their relationship is central to the understanding of the concept of knowledge that I am sketching here.

Let us start by noticing that the relationship between potentiality and actuality is not a causal or functional one (that is, it is neither "if A, then B," nor "x → f(x)). The manifestation of knowledge cannot be dictated by its potentiality. We need to keep in mind that potentiality means *empowerment*. Thus, in the case of language, language as potentiality empowers us to think and talk, but it cannot tell us *what* to think or say. The same is true of knowledge.

When in the 19$^{th}$ century Évariste Galois engaged in the investigation of the n$^{th}$-degree equations, he did not start from scratch, from nothing. There was a longstanding mathematical tradition that appeared to him as potentiality. Drawing on this cultural-historical potentiality, he was able to imagine new approaches, centred on the idea of substitution and permutation, and pose the investigation of equations in new terms, thereby going beyond what was known so far (for a recent edition of his work, see Galois (1951)). In doing so, potentiality was transformed. When the Macau fishers started using GPS technologies to orient themselves in navigation, they did not start from scratch. The new technologies were understood within the cultural-historical category of potentiality that already existed in their culture and, resorting to it, new concrete actions appeared, going beyond what was known so far.

Instead of being a direct, mechanical, or causal relationship, the relationship between potentiality and its manifestation is a dialectical one. One of its chief characteristics is *overdetermination*: this relationship is overdetermined by the individuals' biography, the social and the cultural context, and the activity in which individuals feel, act, think, and talk. The overdetermined relationship between potentiality and its manifestation is materialized in a dialectic interplay between the various forms that compose knowledge (forms of thinking, acting, speaking, symbolizing, etc.). Its understanding leads us to pay attention to the role of language, and in particular to the meaning of the verb *to be*, which now acquires a new sense that I discuss below.

### 4.2 The dialectic of forms

In Western classical ontologies, it is often assumed that objects of knowledge can be grasped discursively, in particular through the verb *to be*, as when we say "a circle *is* …" This theoretical tradition goes back to ancient Greece and the belief that what can be known is what does not change. Parmenides, for example, was one of the earliest to argue that true knowledge must concern what is eternal and unchanging, as change leads to contradiction and illusion (see *Fragment 8*, in Kirk, Raven, and Schofield, (2013)).

The Greek belief in the centrality of language to ontology finds its best expression in Plato, for whom the cosmos itself is structured according to logos (see, e.g., *Timaeus*). This ontological commitment to language is also evident in Euclid's *Elements*, where mathematical inquiry begins with explicit definitions of the objects under study. "A circle *is* …" "A number *is* …."

The Hegelian perspective I draw on takes a different path. Language certainly plays a central role in the constellation of forms that constitute knowledge. However, language cannot grasp and express knowledge completely. There is always a *surplus* that is unspeakable or unutterable. An object of knowledge cannot be grasped through the abstract determinations of essence, which is what the discursive schema subject-predicate "*A* is *B*" does. As Hegel remarks, "Predicates cannot exhaust what they are attached to" (Hegel, 1991, p. 67). I see a flower. I say: the flower is white. I add: the flower is small, etc. Predicates exhibit "on their own account, a restricted content, and they show themselves to be inappropriate to the fullness of the representation . . . which they do not at all exhaust" (Hegel, 1991, p. 68). The same can be said of perception (Hegel, 1991, p. 78).

To conceive of knowledge as a constellation of forms requires approaching the constellation dialectically—that is, as a dynamic configuration in which forms are not isolated but continually interact, mutually influence, and transform one another. This ongoing interplay not only reshapes the individual forms (e.g., language, thinking, perceptomotor processes) but also reconfigures the constellation as a whole. The issue at stake, however, is not the *relevance* of language, perception, tactility, gesture, movement, or voice—these remain essential, as they are the means through which humans engage with and make sense of knowledge. Rather, the key is to remain attentive to the pervasive *dialectical* structure that organizes and animates the forms of knowledge.

In conducting our classroom research, these ideas lead us to pay attention to the dialectic between language, gesture, perception, matter, and symbols in the ways knowledge is manifested in the classroom. Figure 1 shows an example where a small group of Grade 9 14-15-year-old students seeks the formula of the general term of a sequence of figures. The figures are made up of two rows of circles (the first one having three circles on the bottom row and two on the top row, etc.). After an intense discussion, through a coordinated sequence of indexical gestures and words accompanied by perceptual actions, one of the students brings to the group's attention a numeric-spatial structure from where the formula can be perceived. Rhythm is a crucial semiotic component in the creative movement of mathematical imagination.

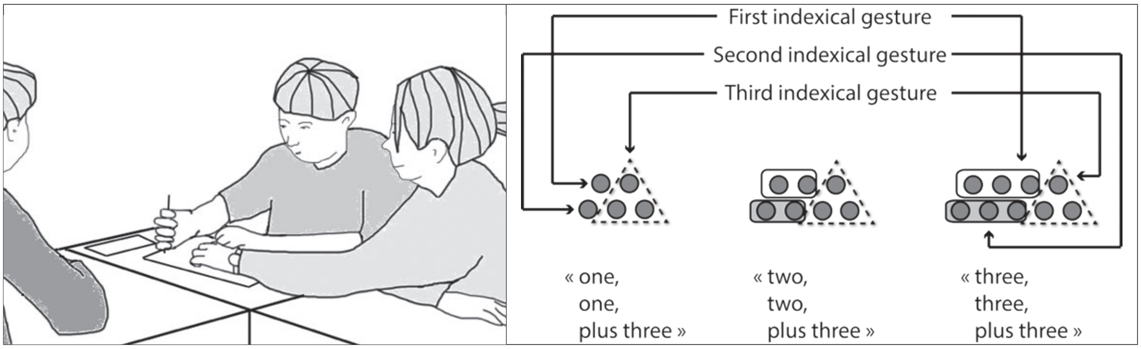


Figure 1. The dialectic of forms of knowledge in the investigation of a figural sequence (Radford, 2015)

### 4.3 The abstract and the concrete

The view that knowledge is produced through practical activity invites a reconsideration of the relationship between the concrete and the abstract. Ontologically, knowledge is constituted as both abstract *and* concrete. Let me attempt to elaborate on this idea.[5]

Referring to the geometrical circle, Ilyenkov describes the situation as follows: the idea of a circle appears through "the conscious state of our body *identical in form with the thing [the circle] outside the body*" (1977, p. 69; emphasis in the original). We see here knowledge as an abstract and concrete entity at work. Its abstract nature is *unveiled* and *fixed* in its semiotic, material, and bodily *concreteness*, in the singularity of the circle produced, *without therefore vanishing* or disappearing from it.

"This," Ilyenkov tells us, "can be represented quite clearly."

> When I describe a circle with my hand on a piece of paper (in real space), my body . . . comes into a state fully identical with the form of the circle outside my body, into a state of real *action* in the form of a circle. My body (my hand) really describes a circle, and the awareness of this state (i.e., of the form of my own action in the form of the thing) is [the circle]. (Ilyenkov,1977, p. 69; emphasis in the original)

5 For a more detailed elaboration, see Radford (2024).

As we see, there are three elements in Ilyenkov's account:

a) embodied activity,

b) the circle as an ideal form, and

c) consciousness as the link between a) and b)—consciousness as a state of 'full identity' (1977, p. 69) between the action of the body and the ideal form of the circle.

The situation described by Ilyenkov might seem incomprehensible. How can there be full identity between two such different things as a bodily action and an ideal form? Indeed, from the perspective of Aristotelian logic, these two elements are incommensurable. But this is precisely not the case in dialectical materialism, which relies here on an idea of Spinoza and another of Hegel.

For Spinoza, corporeal material action and the idea of action are two sides of the same coin. In his *Ethics*, Spinoza considers the body as an extensive thing that expresses itself continuously through modes of extension. Spinoza says: "a mode of extension and the idea of that mode are one and the same thing, though expressed in two ways" (1989, p. 83). For example, the circle drawn by the hand—which is a mode of extension—and "the idea of a circle [...] are one and the same thing displayed through different attributes" (Spinoza, 1989, p. 83).

Evidently, not just any mode of extension, not just any figure that the hand draws, is suitable. A triangle would not work. There must be *adequacy* between the two. However, the adequacy mentioned here is not about one between an external reality and its representation (see Section 3.2 above). *It is this adequacy between the cultural-historical idea of circle and the concrete sensuous subjective action* that Ilyenkov thematizes in Hegelian terms, speaking of "full identity" between idea and bodily action. This identity must be understood in terms of *differences.* As Magee (2010, p. 30) explains, "'A is A', a thing is what it is." However, Magee continues, "Hegel points out […] that identity is a meaningless abstraction without the concept of difference." In the previous passage, Ilyenkov (1977, p. 69) quickly adds that it is not only identity but "consciousness" (p. 70) of this state of identity/difference. It is in this sense that Ilyenkov should be understood when he speaks of a *state of consciousness* of full identity between idea and bodily action—where bodily action merges with history and both make something new, an unrepeatable repeatable and a repeatable unrepeatable.

This "state of consciousness" (which we can also see in our Grade 9 example above) is related to what later in this article I refer to as learning. In the meantime, in the next section, I pursue the idea of difference that is crucial to understanding the ontological constitution of knowledge.

### 4.4 The ontology of difference

Commenting on Bergson's philosophy, Deleuze reminds us that, for Bergson, philosophy should have "a positive and direct relationship with things" and attempt to grasp things from what they are (Deleuze, 2002, p. 44). With Bergson,

> philosophy intends to establish, or rather restore, *a different* relationship with things, and thus a *different kind of* knowledge—a knowledge and relationship that science hid from us, deprived us of, because it only allowed us to conclude and infer, without ever presenting or giving us the thing in itself. (Deleuze,

> 2002, p. 29; emphasis in the original)

This task can only be accomplished if philosophy returns to the things themselves in their own individuality, avoiding to see them through general concepts that are alien to them:

> Instead of diluting his thought in the general, the philosopher must concentrate it on the individual... The object of metaphysics is to recapture in individual existences, and to follow back to the source from which it emanates, the particular ray which, conferring on each of them its own particular nuance, thereby links it to the universal light. (Bergson, cited in Deleuze, 2002, p. 33)

According to Bergson, what makes this task possible is that the human mind is architectonically equipped with the capacity to "settle itself into mobile reality, adopt its ever-changing direction, and finally grasp it intuitively" (Bergson, 1975, p. 40). Intuition, in this view, is thought to grant access to things as they are in themselves, without the interference of human, cultural, or historical conceptual mediation.

The neo-Hegelian dialectical materialist view of objects I present here abandons this long-standing empiricist philosophical tradition that assumes that the essentiality of objects lies in their own singularity, their "own particular nuance" (Bergson, cited in Deleuze, 2002, p. 33). I follow an opposite path: I assume that the singularity of objects and subjects, their essence, does not reside *within* themselves. The nature of their essence is not of the order of *interiority*, but rather of *exteriority*. Indeed, in the theoretical path I am following, objects and subjects are conceived of as *relational* through and through. Their relational nature is to be understood in terms of an *ontology of difference* that brings to the fore the Other as constitutive of things and beings.

Hegel theorized this constitutive nature of otherness through the ideas of negation and contradiction. What this means is that, from the outset, the terms of all empirical diversities are conceived as *interrelated*. In other words, each term of an empirical diversity, each individuality, is linked to *its others*. Thus, *A* is linked to *B*, *C*, *D*, etc. Between *A* and its others there is a relation of non-coincidence. In Hegel's jargon, there is a basic *opposition*. However, the opposition marks only an *external* characteristic of the empirical diversity. Underneath this basic opposition lies a more profound one: instead of being self-sufficient, each term of the empirical diversity turns out to be *dependent* on its others. To refer to this ontological dependence Hegel uses the terms *negation* and *contradiction* in a specific sense that contravenes those of classical logic. To say that *B*, *C*, *D*, *negate A* means that the essence of *A* contains its opposites; that is to say, what A is not, $\bar{A}$. Classical logic excludes the possibility for something to be A and $\bar{A}$ at the same time. As Parmenides put it in Fragment VIII of his *Poem*, "Being is or it is not" (Kirk, Raven, & Schofield, 2013, p. 250). To this logic of exclusion, Hegel offers a dialectical logic, a logic of inclusion. The result is a completely different conception of contradiction, which, in fact, appears as the motor of the movement and transformation of things. Contradiction, indeed, is the transformative movement that allows the empirical diversity to become an ontologically related diversity where each term depends on the others and where to know something about *A* entails understanding *A* in its relationship with the other terms of the diversity. Thus, the fishers of Macau come to know what a fish is not only by closely examining it, meticulously observing how it grows, what it does, etc. (its own *nuances*), but also by distinguishing it from what it is not (a shellfish, for instance). The same goes for the sky constellations they use to navigate in the sea. A constellation is apprehended in itself as it is put in relationship with a different one.

Let us return to our example of the circle.

Circle is the name of that geometric object that is constituted through relations of commonality and differences with other objects. The circle necessarily includes things that are different from itself, like the circle's centre, and that by not being the circle, stand in opposition to it. This opposition is transformed in something else, a contradiction, which in its dialectical movement, leads to something that now incorporates both the circle as mere determinacy and a point (its centre). Like human beings, an object of knowledge is a differential entity of

> the *positive* and the *negative*: the positive is the identical relation to self in such a way that it is *not* the negative, while the negative is what is distinct on its own account in such a way that it is *not* the positive. (Hegel, 1991, pp. 184–185; emphasis in the original)

That is why knowledge, as I am outlining it here, is the bearer of its negation: it is the bearer of that which is not itself and which, not being itself, is a constitutive *part* of itself. The inevitably negative nature of knowledge—its distinction from what it is not—other geometric forms, such as the point (the circle's centre)—is built in itself and is part of its being.

This is true of human beings too. In our Macau example, fish harvesters are bearers of what they are not: those who harvest shellfish. In this relational ontology of difference "The living die," Hegel says, "and they do so simply because, insofar as they live, they bear the germ of [its other, namely] death within themselves" (Hegel, 1991, p. 149).

In the pursuit of comprehending the ontological substance of knowledge, I now turn to contradiction as an inherent part of knowledge's constitution.

### 4.5 Knowledge as the bearer of contradictions

Because knowledge is anchored in the politics of living, knowledge is ineluctably a bearer of the societal contradictions from where it arises. I present below two short examples.

***Macau fishing***

In the case of the Macau fishing community, some of the contradictions arise from a distinction between *what* is fished or caught, and *who* does it. This two-fold contradiction distinguishes, on the one hand, between fish and shellfish (*marisco*), as these types of catch are not considered the same. On the other hand, there is a question of gender. Usually, harvesting shellfish is carried out by women.

> The marisqueiras (shellfishwomen) initially had their rights delayed, as they were engaged in shellfish fishing and were not initially recognized as fishermen. Having won their rights, they prefer to be called fisherwomen, seeking to avoid exclusion due to the specificity of their fishing activity. (Beserra, 2025, p. 35)

Like all forms, forms of knowledge encompass social relationships of production that have become objects of thought, consciousness, attention, and discourse. Notwithstanding the fact that they are produced by individuals, these forms become independent of them. In the eyes of those who have access to them (the fishermen and women in our example) these forms acquire the apparent banality of everyday life. And to get there, to acquire the social recognizance they enjoy, these forms have undergone a lengthy process of social institutionalization. What this means is that there is now a socially recognized activity that produces and reproduces these forms—for example, choosing the right cast net to catch a specific fish, choosing the right boat, the right time of day, the right location, throwing the cast net in a certain way, etc.

To give another example about the societal contradictions contained in knowledge I turn to the Renaissance concept of number.

***The legitimate way to count***

The Renaissance concept of number emerged from the influence of previous Greek and Arabian mathematics traditions and was crafted out of various societal practices, such as the humanist-oriented practice of aristocratic patrons and manuscript collectors (Rose, 1975) and the commerce-oriented practices studied in the Abbacus Schools (Høyrup, 2018).

Societal practices and their ensuing mathematical knowledge exerted a dialectical influence on each other in various ways. There was, for instance, a political battle about the legitimate way to count—should it be done with Arabian numerals or through counters? As Struik (1968, p. 293) notes: "The struggle raged between the merchant guilds and the remnants of the ancient feudal aristocracy," showing how mathematical knowledge is the bearer of the societal contradictions from where it emerges.

Instead of being pure abstract knowledge, mathematics knowledge in general bears the imprint of the social and cultural contradictions from where it is produced. As Ilyenkov states, as soon as one attempts to conceive of knowledge as purified of its concrete history and "purified of all the traces of palpable corporeality, it turns out that this attempt is fundamentally doomed to failure, that after such a purification there will be nothing but transparent emptiness, an indefinable vacuum" (2012, p. 177).

### 4.6 The body

Now, it is important to notice that the body that describes the circle in Ilyenkov's example (or the body that casts a net into the Atlantic Ocean from a Macau fish harvester's boat) is not the *simpliciter* body (e.g., that of the acultural and ahistorical individual of other philosophies), but the body of the *social-historical individual.* In the precise event of bringing the cultural-historical concept of circle into life, we find in fact the *social-historical* individual's body moving, sensing, and living—living not as a monad among monads creating their own worlds but as part of organized matter in a movement through which subjective experience moves dialectically between personal senses and cultural meanings.

In the next section, I move to the question of the conceptions of knowledge in theories in mathematics education. The idea is to stress the importance of discussing and critically examining the conceptions of knowledge that we bring in our theorizations and ensuing practices. Indeed, a thorough discussion of conceptions of knowledge used in the various theories developed in mathematics education can be highly beneficial to better understand the theories we resort to in doing research. Such a discussion can also help us to better understand similarities and differences between our theories and can also lead us to ask new questions.

## 5. REFLECTING ON THE CONCEPTION OF KNOWLEDGE IN THEORIES IN MATHEMATICS EDUCATION

As mentioned in the introduction, theoretical reflections on knowledge have mainly focused on *epistemological* issues (Artigue, 1995; Brousseau, 2006; Chevallard, 2006; Gascón, 2003). This does not mean, however, that concepts of knowledge have been absent from discussion. Thus, in Brousseau's work, knowledge is considered to emerge in response to a "situation" conceived of within a Piagetian framework: A "situation" is "a model of interaction of a subject with a certain environment that determines a given knowledge as the resource available to the subject to reach or preserve a favorable state in this environment" (Brousseau, 2000, p. 10). Brousseau makes the hypothesis that for each mathematical knowledge there exists a "fundamental situation" *that characterizes it and differentiates it from the others*" (Brousseau, 2000, p. 13; emphasis in the original). In this context, knowledge "is what achieves the equilibrium between a subject and an environment, what the subject brings into play when it invests a situation" (Margolinas & Bessot, 2023, p. 42). In Chevallard's work, by contrast, knowledge is conceptualized along the lines of *praxeologies*; that is, "organised way[s] of doing and thinking contrived within a given society (Chevallard, 2006, p. 23).

To move forward in understanding differences and similarities about our conceptions of knowledge we might need to focus on ontological issues in more explicit ways. We can ask to what extent the concepts of knowledge are different in the theory of didactical situations (Brousseau, 2002), the anthropological theory of the didactic (Chevallard, 1985), the ontosemiotic approach (Godino, 2024), commognitive theory (Sfard, 2008), the theory of mathematical working spaces (Kuzniak et al. 2022), and the theory of objectification (Radford, 2021), for example. It is at this juncture that it might be important to investigate the *ontological substance* of the concept of knowledge that theories advance. What I have called in this paper the ontological substance of knowledge still seems to me to be an open *problématique*—a "hard core" problem, to use Gascón's (2003, p. 44) Lakatosian term—one that has the potential to advance theoretical research in mathematics education.

However, it would be misleading to regard such an investigation as a pure philosophical exercise. On the contrary: as suggested above, our epistemological stances (i.e., stances about how students come to know) are intimately connected with our ontological stances (i.e., assumptions about the *nature* of knowledge). The proposed investigation might shed new light on the role of culture in the ontology of knowledge. Is culture a constitutive, hard-core element of knowledge (as argued in this paper) or merely an environmental background that knowledge ultimately transcends, as Piaget (Piaget & García, 1989) famously claimed? Such investigation may also illuminate the deep configurations of our theories and reveal new insights into the ways learning is conceptualized within them.

Here is an example. Contemporary mathematics education theories seem to agree with the importance of providing room for the students to engage in mathematics learning. As Lesh et al. (2003) put it: "hardly any modern theories of learning would argue students passively receive knowledge from teachers" (p. 212). Or as Chevallard (2006, p. 30) notes, the old pedagogical regime, where the teacher was "regarded as an authoritative source," is over. It seems that, at least theoretically (Labaree, 2005), we have overcome the instruction-based learning paradigm. However, the students' engagement with mathematics that contemporary educational theories emphasize can vary from theory to theory. A great deal of the differences about the meaning of "engagement" can rest on what is understood by the *nature* of knowledge. Thus, in their critique of constructivism, Lesh et al. (2003) contend that not all knowledge can be considered a construction, à la Piaget. They suggest a model and modelling perspective and argue that models (or conceptual systems in their terminology) involve processes that the concept of construction cannot capture (e.g., skills). Their view of knowledge intends to move us away from the constructivist "questions related to the relationship between truth and reality to more pragmatic questions related to the usefulness of a particular model" (p. 229). Within this pragmatic modelling perspective, learning is not about constructing. It is rather about having "Students express, test, revise, refine, and extend their own ways of thinking" (p. 218).

Interesting as it is, this perspective, however, still seems to remain within the empiricist view of knowledge. It does not seem to overcome the fundamental problems of constructivism. Indeed, like constructivism, in this modelling perspective, the subjective experience of the world is the theoretical category that *explains* learning. And like all empiricist theories, it reduces the social to the subjective experience individuals undergo through mere interaction (a critique of the social that Lerman's (1992) seminal work articulates). It is not clear how the students' personal senses are affected by cultural curricular conceptions of mathematics as, for instance, the styles of argumentation and student participation that are expected in the mathematics classroom. All happens as if students are *naturally* endowed with a spirit of modelling builders (see also Popkewitz's (2004) critique; for a more recent elaboration, see Valero & Knijnik, (2015)). More broadly, we can ask: How, in this pragmatic and, more generally, empiricist line of thought, are the inherent societal contradictions that permeate the school reflected in the conceptions of teaching and learning, and in the conceptions of teachers and students? What is the role of culture here? What is the role of history? Should we still think of society as a set of abstract instrumental institutions and of culture as taming governing mechanisms as is often done following the early sociological schools that arose out of the 18th century European Enlightenment? I would like to argue that an inquiry into the ontological substance of our concepts of knowledge can be helpful to reveal subtle yet important differences in our stances about learning and how, within our theories, learning is nurtured in the classrooms.

## 6. CONCLUDING REMARKS

Although sociocultural research has significantly advanced our understanding of how social, cultural, and political contexts shape the teaching and learning of mathematics (Atweh et al., 2001; Bishop, 1988), the problem of explaining how knowledge is anchored in culture persists. In this paper, I have argued that any attempt to unravel the riddle of the relationship between knowledge and culture must address the ontology of mathematical knowledge. In this spirit, the first part of the paper offered an account of how knowledge is conceived within the theory of objectification. From this perspective, knowledge is neither the result of the sensory impression with which objects mark the mind (Hume, 1965), nor the mere pragmatic usefulness of deeds or representations (Lesh et al., 2003; von Glasersfeld, 1995), nor the equilibrating mechanisms between a subject and an environment (Piaget, 1970). Knowledge "is the form of a thing created by social-human labour, reproducing forms of the objective material world" (Ilyenkov, 2012, p. 191). Or to put it in a Spinozist vein, knowledge is "the form of labour realised in the substance of nature" (Ilyenkov, 2012, p. 191), labour that is embodied and realized in it. More precisely: knowledge is a culturally and historically constituted dialectic system of forms of thinking, doing, reflecting, symbolizing, and languaging, arising from joint collective labour in its intertwinements with cultural beliefs about reality and its individuals.

At this point, we can ask the question of what would mathematics teaching look like if one fully embraces the idea of knowledge as presented in this paper. Although this question is too vast to be answered here, I limit myself to mention two aspects.

First, conceiving knowledge as a cultural-historical entity opens pathways for moving beyond the student-centred, individualist conceptions of learning that position the students as the constructors of their own knowledge. As elaborated in the theory of objectification, learning can be conceptualized as a process of *encountering* cultural-historical knowledge. This encounter unfolds as a collective classroom process in which cultural-historical knowledge—initially existing as potentiality—becomes progressively materialized within the teaching-learning activity (*Tätigkeit*). In this process (which is inherently full of tensions and contradictions) knowledge acquires concrete determinations through the dialectic of its forms—forms of thinking, languaging, acting, gesturing, perceiving, symbolizing, feeling, and so forth—and thereby becomes an object of consciousness and thought (for an example, see Radford, 2022). This process (which we term process of objectification) arises from the interweaving of teacher's and students' personal senses with cultural meanings that carry within them their historicity, politics, aesthetics, and societal contradictions. From this perspective, rather than a technical process of knowledge diffusion, learning appears as a *transformative* process. As Vološinov (1976) put it, "Human thought never reflects merely the object under scrutiny. It also reflects, along with that object, the being of the scrutinizing subject, [their] concrete social existence" (p. 26), which is a way to express the dialectics between knowing and becoming.

Thus, seeing the encounter in dialectical terms—that is, in terms of a dialectics between what is to be encountered (knowledge) and those encountering it (learners)—we are invited to rethink learning itself and to reimagine the relationship between knowledge, teachers, and learners. Teachers lose their aura as patriarchs of knowledge and the messiahs of the students: they become learners as well. Teachers and students appear as *subjectivities in the making*. Teachers and students appear working *together* (although not doing exactly the same things), entangled with knowledge, not only understanding and transforming it, but also disturbing it, subverting it. The school appears as the social institution that, through the various pedagogical models it fosters, organizes with more or less success the encounter with knowledge.

This last point leads me to the second part of my answer, which revolves around the claim made above that mathematics appears in the classroom through the cultural and historical mediations of teaching-learning activity. Its appearance is hence correlated to the manner in which teaching-learning activity unfolds. This is why alienating teaching-learning activities unavoidably leads to alienating learning, as in direct teaching, where teachers and students are disempowered (Radford, 2012). The practical implications here are about imagining and implementing the kind of teaching and learning activity that would be conducive to an emancipative and inclusive school and classroom *praxis*—which in my recent work is termed *joint labour* (2021).

There is still much to be done in mathematics education in order to better grasp the impact of our conceptions of knowledge in teaching, learning, and researching, although some pioneer work has already been started (see, e.g., Margolinas and Bessot, 2023; Scheiner, 2020).

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