

CONCEPTUAL FIELDS AND CONCEPTUAL POLYSEMY OF MATHEMATICAL OBJECTS WITH APPLICATIONS TO THE MULTIPLICATIVE CONCEPTUAL FIELD

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Abstract

By building on Vergnaud's theory of conceptual fields and his description of concepts as conglomerates of situations, invariants, and representations, we stipulate that mathematical concepts can have precisely three roots: relations in situations, relations in iconic imagery, or invariants in symbol systems. Through conceptual polysemy, several classes of situations and iconic imagery will regularly map to the same mathematical symbol system, which by mathematical virtue must be free of contradictions. Mathematical concepts in school are almost always introduced through situations and iconic imagery and labeled with symbol systems. We stipulate that successful conceptual progression is characterized by an epistemological shift, where the dominant meaning transfers from residing in situations or iconic imagery to residing in relations in symbol systems. We explain how this theory can be put to work when planning primary school teaching, focusing on long-term conceptual progress, by exemplifying with the multiplicative conceptual field.

Keywords: Conceptual fields, polysemy, multiplication, division, fractions, symbol systems

1. INTRODUCTION

Moving from the concrete to the abstract has long been central in elementary mathematics. Because formal notations and reasoning take years to master, early mathematics must be bootstrapped from what is experientially real for children (Freudenthal, 1983). Yet this concrete-to-abstract trajectory is unevenly theorized. Classic accounts frame it as cognitive growth. Bruner (1964) describes enactive engagement with objects, iconic representations that depict phenomena, and symbolic reasoning in systems whose signs bear no causal relation to what they denote. These modes are not sequential and often operate in parallel. Teaching-oriented frameworks, such as the *Concrete-Representational-Abstract* (CRA) model (Witzel et al., 2003) and the *Concrete-Pictorial-Abstract* (CPA) model (Leong et al., 2015), begin with concrete manipulation and progress to abstraction. Fyfe and Nathan (2019) expand the theoretical framework of *concreteness fading* by defining and clarifying the terms abstract, concrete, and fading. A persistent limitation across these approaches is guidance on which concrete or pictorial instantiations best support long-term development, i.e., choices that do not seed later misconceptions several years down the road. Most empirical

studies target short units (e.g., solving linear equations: Ottmar & Landy, 2017; factoring quadratics: Leong et al., 2015), leaving open how early representational choices play out across grades. To our knowledge, large-scale longitudinal evidence is scarce.

Our interest in long-term development arises from designing large-scale interventions in early arithmetic and algebra. The teaching model Thinking, Reasoning, and Reckoning (TRR) was prototyped with six-year-olds, blending inquiry and explicit instruction (Sterner et al., 2020), and later tested in grades 1–3 with more than 1,000 teachers (Helenius, 2022) in a socially rich, Vygotskian classroom design (Ahl et al., 2022). While the CRA-design informed the initial program, the CRA paradigm did not provide the analytical structure needed to plan conceptual progress over the next five to ten years of teaching. What we were striving for was to never introduce concepts in a way that needed to be abandoned at any later stage of development. We therefore sought a framework aligned with CRA’s spirit but offering stronger potential for planning long-term conceptual development.

For designing teaching sequences aimed at a long-horizon conceptual development, we propose a framework based on conceptual polysemy. As we will show, the lens of conceptual polysemy will allow us to trace *epistemological shifts* as meanings migrate from situations and imagery into symbolic system relations.

Several influential theories address long-term development and conceptual change, such as theory replacement (Posner et al., 1982), framework restructuring (Vosniadou & Brewer, 1992), and Knowledge in Pieces (diSessa, 1993, 2018). While powerful, these perspectives originate in science education, where time-dependent causal processes are a central focus. Mathematics foregrounds structures stabilized in symbol systems. For this reason, we turn to Vergnaud. His theories of conceptual fields (2009) and representations (1998) provide a basis for analyzing the concrete-abstract distinction through theorizing the cognition of both the concrete and the abstract. Like the theories mentioned above, Vergnaud includes primitive intuitions (theorems-in-action, concepts-in-action). Our contribution in this article is to further decompose some of Vergnaud’s notions, in a way that we propose extends Vergnaud’s intention of providing a basis for medium- and long-term didactical decisions. In particular, we will propose that the concept of *conceptual polysemy*, which we will introduce, helps when trying to understand how invariants (I) that are first encountered in a plurality of classes of situations (S) and representational iconic schematic imagery (R_{iconic}) in later stages of development converge in a single symbol system (R_{sym}), thereby requiring the consolidation of invariants (I). The notions will be explained below.

The article proceeds as follows. We first introduce polysemy with examples from elementary and advanced mathematics. We then draw on the theory of conceptual fields to argue that mathematical meaning can be bootstrapped from classes of situations, iconic imagery, or relations within symbol systems. Next, we analyze concepts tied to the symbol system a/b . Finally, we present models used to introduce multiplicative concepts in early grades within ongoing interventions, before turning to discussion.

2. POLYSEMY

What is distinctive in our use of polysemy, following Fillmore & Atkins (2000), is the focus on the networked structure of meanings. In mathematics, polysemy is not merely the coexistence of multiple interpretations (as

well-documented), but the coexistence of structurally related, sometimes incompatible, interpretations unified in a symbol system. This structural emphasis adds explanatory power for why certain ways of introducing particular mathematical concepts can both support and hinder progression.

In linguistics, polysemy has been studied theoretically and practically, notably in digital translation. Polysemy differs from other notions of multiple meaning because the senses must be related (Ravin & Leacock, 2000). Fillmore and Atkins (2000) specify that polysemy requires (i) a shared central origin, (ii) a network of related senses, and (iii) that grasping the “inner” sense supports understanding the “outer.” In mathematics, this means that diverse models (e.g., part–whole, measure, quotient) must trace back to common invariants and connect through explicit structural links. A well-chosen core interpretation can then illuminate peripheral meanings, guiding both progression and reconciliation of conflicts. The notion of conceptual polysemy that we will present later aligns with Fillmore and Atkins’ (2000) definition, but differs in that mathematics often involves not only words but superconcepts unifying several related subconcepts.

We consider polysemy fundamental to mathematics because it recurs across cognitive levels. At the developmental level, Case’s and colleagues theory of central conceptual structures (Case, 1998; Case & Griffin, 1990) illustrates how children’s number sense develops when two structures, non-numerical magnitude (i.e., more/less) and counting, are integrated. Between the ages of four and six, this fusion gives rise to the *mental number line*, a polysemic network that combines two previously distinct meanings (Okamoto & Case, 1996).

Lakoff and Núñez (2000) similarly show that whole numbers draw on four distinct embodied metaphors: element collections, object construction/deconstruction, path movement, and segment lengths. These experiential roots overlap structurally, allowing for their subsumption into a single numerical mental concept, while each embodied root contributes a slightly different yet complementary meaning.

At advanced levels, mathematical objects also admit multiple valid definitions. Thurston (1995) hinted at over 37 distinct ways of conceiving the derivative, remarking that the list “has no reason to stop” (p. 30). Likewise, the exponential function can be defined as a particular number, e , raised to the power x ; it can be directly defined as a formal power series or as the unique solution to the differential equation $y' = y$, with $y(0) = 1$. Each definition requires proofs to establish equivalence to other possible characterizations, yet for the mathematician’s mental image, all properties integrate seamlessly.

In summary, polysemy in mathematics spans from early cognitive structures to advanced formal theory. We propose, in a theoretical manner, that all mathematical concepts are polysemic. This can not be proven directly, but it is, in a Popperian sense, a strong scientific claim, since it offers anyone a way of falsifying our claim by presenting a mathematical concept that is not polysemic in the way we propose.

Before turning to exemplifying polysemy in the specific case of concepts tied to the symbol system a/b , we will first introduce key terminology from Vergnaud’s theory of symbol systems. Then we define conceptual polysemy formally.

3. THEORETICAL PROPOSAL: FIELDS, CONCEPTS, AND POLYSEMY

In this section, we integrate three strands of theory, conceptual fields, conceptual polysemy, and our principle

of the three roots of invariants, into a coherent proposal. We follow Vergnaud in defining concepts as triplets $C=(S, I, R)$, explained below, and we situate our notion of *conceptual polysemy* within this structure. By combining these perspectives, we aim to clarify how meanings are introduced, diversified, and consolidated in school mathematics.

3.1 Conceptual fields and concepts

Following Vergnaud (2009), we define a concept as $C=(S, I, R)$ where S is the class of situations in which the concept is operative, I is the set of operational invariants (concepts-in-action, theorems-in-action), and R is the available representations. A *conceptual field* organizes families of such concepts and situation classes (Vergnaud, 1982, 1983, 2009). According to Vergnaud (1998, 2009), it makes no sense to consider concepts in isolation. Instead, concepts are considered in the context of other concepts, so-called conceptual fields. From Vergnaud's work, we will in particular use his view of a concept as consisting of operational invariants (I), situations (S), and representations (R).

Vergnaud picked up the concept of scheme from Piaget. Following Kant, Piaget used the scheme construct as a mediator between the Locke and Hume tradition, where knowledge originates in impressions from our senses, and the rationalist Descartes and Leibniz tradition, where knowledge originates in the inner mental activity of reason. Schemes are mental objects. Vergnaud formulated a theory of schemes involving the notion of the two operational invariants (I) concepts-in-action and theorems-in-action, signifying that one's actions are organized by concept-like and theorem-like structures, even when these cannot be put into words or any other symbolized form.

Vergnaud also emphasized the concept of situations (S) when defining a scheme as "the invariant organization of activity for a certain class of situations" (Vergnaud, 2009, p. 88). What constitutes a class of situations depends on your schemes for handling it. Conversely, what scheme is activated depends on the interpretation of the situation. Schemes and situations are hence dual constructs. Both situations and schemes are mental objects, which makes them individual and non-accessible to an observer, except through indirectly observing individuals' actions and making inferences on underlying schemes. Vergnaud, however, leaps here and acknowledges that to analyze learning and plan teaching on a medium to long-term basis, classes of situations must be treated as if they exist materially. Existing materially, in this sense, means that the situation can be described in general, even in the absence of a particular person holding the situation. Much of Vergnaud's work with conceptual fields, such as additive or multiplicative structures (Vergnaud, 1982, 1983), focuses on identifying and describing sets of additive and multiplicative classes of situations where additive and multiplicative invariants are applicable.

Finally, for Vergnaud, representations (R) are any semiotic means to name any conceptually relevant information. In Vergnaud's theory of representations, he posits that when a concept is represented by any type of semiotic resource in predicative form, as Vergnaud terms it, the schemes surrounding the concept become stronger and more easily generalizable to adjacent situations. Therefore, Vergnaud views representations of a concept as a part of the concept itself.

Using the abovementioned notions, Vergnaud defines a concept $C=C(S, I, R)$ as where S is the class of situations where the concept is relevant, I is the set of operational invariants that an individual can use to deal with these situations, and R is the set of representations, symbolic, verbal, graphical, gestural, etcetera, that

can be used to represent invariants, situations, and procedures. Vergnaud's definition of concepts is the steppingstone for our theorization.

3.2 Three roots of invariants.

In previous work (Ahl & Helenius, 2022, 2021a, 2021b), we have combined Vergnaud's characterization with the observation that representations can take two forms: iconic, where relevant invariants are directly observable in the elements that make up the sign representing the concept, and symbolic. By a symbol, we mean a token of human construction that acts as a sign for something but has no causal relationship with what it signifies. There is no way of obtaining information about 4 by looking at the form of the numeral token. In contrast, the iconic sign IIII can be separated into III and I, signifying that four can be understood as made up of three and one. Understanding symbolic mathematical information typically is about dealing with symbol systems, not just individual symbols. A symbol system is a finite set of physical tokens that can be manipulated through a rule-governed, purely syntactic system by combining and recombining atomic tokens and token strings. The system can be handled without any interpretation of meaning. Still, standard ways of understanding symbol systems say they should also allow for *semantic interpretability*, e.g., to represent objects or describe the state of affairs (Harnad, 1990). Symbol systems are essential in mathematics. The base-10 position system for representing numbers is typically the first one that children learn to use. With the help of the iconic-symbolic distinction, we conclude that the invariants that make up the structural part of a concept, in an initial phase of the conceptualization process, can be bootstrapped in three ways: (1) from situations, (2) from iconic imagery, or (3) from introducing invariants explicitly as relations in symbol systems. This distinction also connects to how Bruner (1966) describes three systems by which a developing child acquires knowledge. The enactive system, where a child uses action to manipulate objects, which is similar to how we think of situations, the iconic system, where a child employs mental images which are primarily visual or otherwise sensorily-based, and the symbol system, where the child acts by means of language, reasoning, and other systems of meaning. Our claim that there are no other ways of bootstrapping a mathematical concept should be understood as a theoretical/scientific claim in the sense of Popper. That is, there is probably no way of giving a direct proof of the claim, but anyone can suggest a different way of bootstrapping a mathematical concept, falsifying our claim or forcing us to adjust our formulation of it.

3.3 Conceptual polysemy

Conceptual polysemy refers to the recurrent many-to-one convergence by which distinct situations, S , or iconic imagery R_{iconic} , classes feed into a common symbolic symbol system R_{sym} , which demands a contradiction-free consolidation of the invariants, I . The somewhat paradoxical state of affairs here is that, on one hand, a specific symbol system representation, R_{sym} , of a concept is a subset of all representations, R . On the other hand, due to mathematical symbol systems necessarily being contradiction free and fully formal, R_{sym} , will act as an arbiter in the sense that any invariants relevant in some S or R_{iconic} related to some concept C that contradicts (i.e. are no longer invariants) in R_{sym} will have to be dismissed. For example, equal sharing of objects between persons and equal grouping of objects are two distinct classes of situations, that have their individual set of invariants and representations, of which one is the symbol system a/b . This forms two distinct but quite similar division concepts, C_1 and C_2 . with the corresponding representation sets R_1 and R_2

both including a/b . However, the symbol system description: a/b is a number c such that $a=bc$, creates a third division concept C_3 that acts as a super concept over the two previous ones, making them subconcepts. Invariants that contradict C_3 , like that equal sharing of objects does not support division by numbers smaller than 1 will have to be ignored or modified¹. In general when a superconcept C subsumes subconcepts C_i (i) a union and possibly refinement S of the respective situations classes S_i of C_i is created, (ii) invariants I_i of C_i must be consolidated to preserve a common core while pruning incompatible theorems-in-action, and (iii) all C_i as well as C get access to a symbolic common symbol system R_{sym} .

The value of introducing the concept of conceptual polysemy into Vergnaud's theoretical network is that it puts the light on the complexity involved when structurally diverse roots, such as equal-sharing situations and part-whole imagery are reconciled in a single symbol system (e.g., the fraction system, or quotient system², a/b) or superconcept. Despite the roots sharing most invariants they will tend to involve some incompatible theorems-in-action. The reconciliation process is not explicitly captured in earlier frameworks. While Vergnaud's notion of a *conceptual field* already accounts for conceptual diversity across related classes of situations, our notion of *conceptual polysemy* pinpoints the specific case where several S -classes or R_{iconic} -classes converge on the same R_{sym} ³. This convergence necessitates a reconciliation of invariants (I), as the symbol system requires internal consistency.

Conceptual polysemy thus serves not only to describe multiple coexisting meanings but also to explain how learning must, or at least should, progress over time. As learners encounter related models labeled by the same symbol system, meanings must gradually shift from being tied to situations or imagery toward being consolidated in the symbol system itself, supporting long-term epistemological development. Obviously, this will not always happen, and the result will instead be misconceptions.

Having established these definitions, we can now clarify where polysemy "lives." We continue to define a concept as $C=(S, I, R)$. *Conceptual polysemy* arises when different S or R_{iconic} roots, with partly incompatible theorems-in-action, converge within the same R_{sym} . We use *meaning* to denote the current distribution of semantic weight across S , I , and R . Progression is then modeled as an *epistemological shift*: over time, meaning migrates from situations and imagery to invariants consolidated within symbol systems.

To illustrate, we now examine quotient constructions (fractions and division), showing how different roots contribute to polysemy in the symbol system a/b .

1 To accommodate division with numbers smaller than 1, we can expand the situation to sharing some food in portions, allowing for half portions, for example. But this is just the point: the introduction of the formal division concept will force the situations used for defining the earlier concepts to be adjusted or specified to still serve the new concept.

2 Our standard way of representing decimal numbers is routinely called a symbol system, more specifically a base-10 position system. We have not seen the a/b representation called a symbol system, but we think it is helpful to do this. We most often call it the fraction system, but the quotient system is an alternative term.

3 The same concept can also be defined formally through different symbolic definitions, like our previous example with the exponential function shows. Within formal expositions, a concept normally has one unique definition, but the same concept can have different definitions across expositions. Symbolically defined concepts can also become subsumed by superconcepts without changing the notational system, like a^b first being defined in an elementary way with positive whole numbers and later gets expanded to real numbers.

4. POLYSEMY IN THE CASE OF QUOTIENT CONSTRUCTIONS

To illustrate how polysemy operates in primary mathematics education, we will use the case of fraction constructions, or more generally, quotient constructions, and their designated symbol system, which we refer to as the fraction symbol system. We will first describe how the fraction system formally is constructed from a whole number based system. We begin with the base-10 system, along with its associated operations of addition and multiplication. We construct our new symbol system by only introducing one atomic symbol $/$ and the rule that the composite symbol a/b should have the same role in the symbol system as a symbol c for which $a=bc$ holds. Note that this is a rule expressed without $/$, that is, in the symbol system we already had. This definition requires us to define the substitution rule $a/b=c/d$ if $ad=bc$ and the transformation rules $(a/b)(c/d)=(ac)/(bd)$ and $(a/b)+(c/d)=(ad+cb)/(bd)$, where b and d zero are forbidden cases. It takes a few pages in a standard book on abstract algebra to ensure the syntax is well defined. Note that the fraction symbol system denotes both fractions and division, and that the symbol a/b will operate the same regardless of whether it is conceived of as a fraction (number) or a division (operation).

The formal introduction of a/b as a symbol system is obviously not how the fraction system is typically introduced in school. Instead, several other models are used, commonly sequenced in some particular way (Ahl & Helenius, 2024). For example, division is often introduced as partitive division, which in our language refers to the class of situations involving equal sharing. That is, a/b is introduced as a signifier for the situation where a things are supposed to be divided equally among b persons. The concept of fractions is commonly introduced by referring to a part-whole idea, often using circular part-whole schematic imagery, i.e. through iconic representations. The notion a/b is then introduced as the signifier for a ‘thing’ divided into b equal parts, with a part marked.

The concepts derived from invariants in equal sharing situations and the concept derived from part-whole imagery are subsumed under the same symbol system a/b because the invariants in the two cases share sufficient structural similarity. Notably, though, there are invariants in the two model cases that are incompatible. For example, when fractions are first introduced, a is always smaller than or equal to b , which can be considered a perfectly valid theorem-in-action within the iconic imagery system. But when division as equal sharing is introduced, a is always bigger than b , which can be considered a perfectly valid theorem-in-action in the class of equal sharing situations. Yet, both the part-whole concept and the equal sharing concept are part of a more general quotient concept, symbolically represented by the fraction symbol system, where the two theorems-in-action are incompatible and no longer valid.

5. EXAMPLES FROM THE MULTIPLICATIVE CONCEPTUAL FIELD

As promised, we will now illustrate how we have worked with multiple models for multiplication in a large-scale intervention for grades 1-3 in Sweden. The point of the example is to show that even though polysemy in many textbook presentations create many conceptual conflicts in the multiplicative conceptual field (Helenius & Ahl, 2024) a well-chosen set of models can both allow some quite advanced multiplicative concepts and properties to be experienced early, in an embodied way and while still not introducing many

conflicts. We use the term *model* to denote a structured representational system, either situation-based, iconic or symbolic, that mediates between *S*, *I*, and *R*; models are not concepts, but rather scaffolds for constructing concepts. This intervention, TRR—thinking, reasoning, and reckoning from which the examples below are taken, builds on a specific teaching model previously evaluated in a randomized controlled trial (Sterner et al., 2020; Sterner & Helenius, 2015). The teaching model was then transferred to Grade 2, where concepts related to multiplication were introduced (Säfström et al., 2019). We cannot describe our full analysis of the multiplicative conceptual field here and we are not claiming any type of superiority. Rather, we offer the below examples to ground our theoretical discussion in teaching designs that we know can be carried out in real classrooms.

From a medium or long-term perspective, a few primary concepts that we think should become accessible through the models used to introduce multiplicative concepts right from the start are factorization, primes, commutativity of multiplication, the distributive property, multiplication with fractions, fractions as multiplicative operators, a grounding of multiplication in proportional reasoning, and the connection between multiplication and division. We also want all our models to be based on a structural approach, which can be easily generalized to an algebraic setting.

In line with a polysemic approach, we introduce multiplication in three different but related ways. We use two iconic models: a model based on rectangle arrangements, a model based on norming the number line, and a situation-based model based on children's intuitions for proportional reasoning. Below, all we present will look iconic, due to it being presented in a pictorial way. In our practical teaching though, everything is grounded in enactive/embodied activities, like working practically with cubes or physically walking on a room-scaled number line. From these enactive versions, we gradually move towards iconic paper-based versions. Describing this delicate process goes outside of the present paper though. Moreover, the description below is very schematic and goes into no detail of how the actual lessons are designed. The material presented covers something like 75 lessons.

The multiple models aim to avoid the notion that multiplication is the repeated addition of whole numbers, which gives rise to the whole-number bias (Ni & Zhou, 2005). Overall, we spend little energy on explaining how any of these models can be understood in terms of the other models. Instead, we use the same formal vocabulary and symbol system to label observable phenomena (invariants) in each model. In what follows, it should be noted that the explanation of what we do with children is given in mathematical language. The actual work is highly inquiry based and driven by children's reasoning (Ahl et al., 2022).

5.1 The rectangle model

The rectangle model is similar to what in the literature is called the area model, array model, grid model, or lattice model. Children use interlocking cubes (representing unit squares) built into rectangular forms and subsequently grid paper rectangles. The rectangle model is the first model we introduce when working with the multiplicative conceptual field. We simply call a rectangle built by, for example, 3 columns and 2 rows “three times two” and denote it by “ $3 \cdot 2$ ”. This means that the term and the symbolic representation are just labels for a particular geometric form, with no firm reliance on previously experienced arithmetic, except for what is intuitively experienced.

Several basic properties of multiplication become directly observable through physical manipulations with

the rectangle model. Commutativity follows from rotating a rectangle 90 degrees. There is also a direct relationship between multiplication and division because any quantity c formed into a rectangle $a \cdot b$ can be seen as a groups of b , as c grouped into a groups with b in each, or as b groups with a in each. Therefore, any rectangle represents one multiplication (two, not counting commutativity) and two divisions, $c=a \cdot b$, $b=c/a$, and $a=c/b$.

Factorization (of some number n) is represented by the question: “Which rectangles can be built with n unit squares?” as represented in Figure 1. This is the very first task in our multiplication sequence, in fact. In contrast to when a number is decomposed additively, the number of rectangles you can build from n units varies unpredictably with n . In this way, we discover prime numbers as the numbers that can only be built into one rectangle (counting rotations of the same rectangle as the same multiplication).

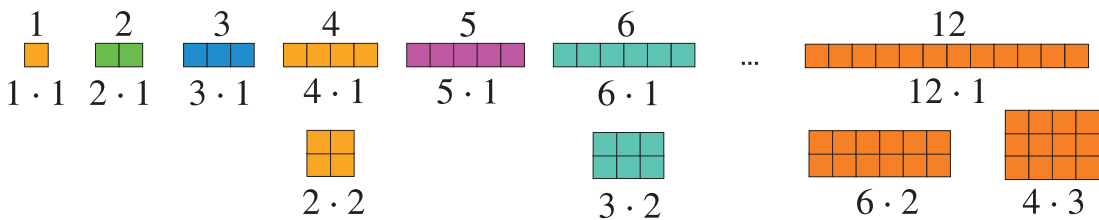


Figure 1. Building rectangles from unit squares

We also construct multiplication tables up to 10 using paper rectangles and elicit relations within and between the tables. For example, in the multiplication table for four, $4 \cdot 7$ can be obtained by adding $4 \cdot 5$, and $4 \cdot 2$, because putting the latter two rectangles together forms the former one. Similarly, you get any element in the table for seven by adding the corresponding multiplications from the table for five and the table for two.

The rectangle model makes it easy to regroup one multiplication into another, as illustrated in Figure 2. The multiplication $3 \cdot 4$ equals the multiplication $6 \cdot 2$ because you can cut the $3 \cdot 4$ rectangle in half across the 4-side and rearrange the pieces into a $6 \cdot 2$ -rectangle. We write $3 \cdot 4 = 6 \cdot 2$. Through the simple cutting activity, $3 \cdot 4$ and $6 \cdot 2$ become two different ways of labeling the same quantity. In similar ways, $3 \cdot 4 = 3 \cdot 2 \cdot 2$, because cutting the $3 \cdot 4$ rectangle in half along the 4-side yields two $3 \cdot 2$ rectangles. In fact, because we already worked with factorization, we can account for this phenomenon systematically. For any $b = m \cdot n$, $a \cdot b$ can be regrouped to $a \cdot m \cdot n$. Notice that this is no longer a matter of labeling actions on paper rectangles, but combining two previously derived arithmetical propositions into a new proposition. In grade 3, we take this to its limit and discover the existence part of the Fundamental Theorem of Arithmetic. A further advantage

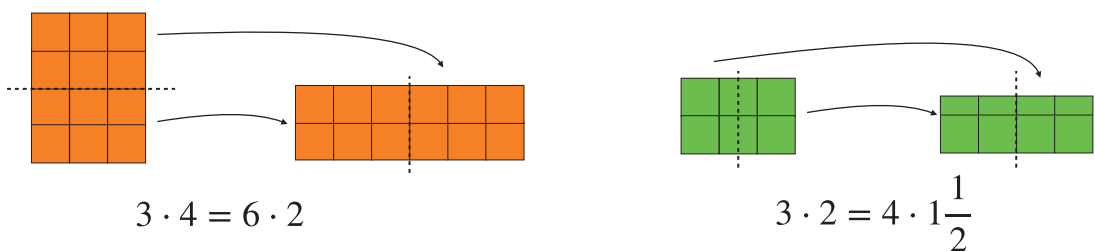


Figure 2. Cutting and regrouping rectangles

of the rectangle model is that as soon as you have moved to the paper rectangle version, you can cut across unit squares, splitting the unit in parts. Quite often, some children will discover this possibility already in the first lesson and suggest that 6 is not only $1 \cdot 6$ and $2 \cdot 3$ but also “four times one and a half”. A few weeks into the teaching sequence, after introducing half numbers with the help of a number line (see further below), we work systematically with such half numbers and multiplication. We label them as both mixed fractions and improper fractions. We then discover identities like $2 \cdot 2\frac{1}{2} = 1 \cdot 5 = \frac{1}{2} \cdot 10$ by cutting a $1 \cdot 5$ rectangle in half in two different ways and rearranging them. Working with numbers smaller than one early on means that we never allow children to form the common misconception “multiplication makes bigger” (Bell et al., 1981).

We introduce the distributive property by a similar technique. By cutting a $6 \cdot 7$ rectangle, you can create one $6 \cdot 5$ and one $6 \cdot 2$ rectangle and hence show that $6 \cdot 7 = 6 \cdot 5 + 6 \cdot 2$ (Figure 3). We account for this systematically by discovering that for any additive decomposition $b = m + n$, we have $a \cdot b = a \cdot (m + n) = a \cdot m + a \cdot n$. In third grade, we later use this identity, supported by rectangle models, to deal with the multiplication of multi-digit numbers by for example, by decomposing $23 \cdot 5$ as $20 \cdot 5 + 3 \cdot 5$ or $21 \cdot 34$ as $20 \cdot 30 + 1 \cdot 30 + 4 \cdot 20 + 1 \cdot 4$.

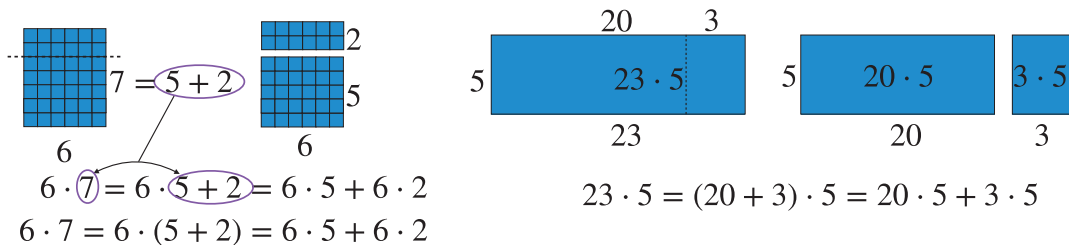


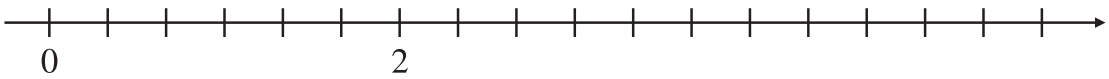
Figure 3. The distributive property and an application. The ellipse around $2+5$ denotes it should be treated as one element and that $6 \cdot$ before it should act on the whole element. We say that the parenthesis in $(5+2)$ works in the same way and that it can be thought of as the ellipse but where the roof and floor did not fit on the row

5.2 The number line model

In the teaching sequence we developed, children are used to working with number lines from previous work with the conceptual fields of number and addition. We deal with multiplicative phenomena on the number line by unitizing and norming (Lamon, 1994), i.e., choosing a reference unit and interpreting quantities in terms of this unit. By using a number line with tick marks, placing a chosen number on any tick mark will generate questions about where other numbers should be placed and what numbers particular tick marks represent. By placing 1 on the m -th mark after 0, we revisit the multiplication table for m by counting the number of steps (or jumps) required to arrive at 1, 2, 3, and so on. By asking: How many steps are there to n if 1 is on mark p ?, we introduce a proportional way of thinking on multiplication, that is, in $m \cdot n = p$, p is to m as n is to 1. Initially, we work with a large number-line on the floor, on which children can actually walk, which embodies the movements, positions and distances on the number line: The number line walk illustrates enactive engagements as situated S -instances that render invariants perceptible, before they are stabilized iconically and symbolically.

The number line model makes the introduction of fractions relatively straightforward because by placing 1 on the tick mark n (after 0), each tick mark represents $1/n$. In contrast to some well-known ways of using the number line to introduce fractions (e.g. Siegler et al., 2011), we explicitly refrain from starting with fractions

smaller than 1. This method of moving along the number line, combined with labeling it flexibly, also facilitates the introduction of multiplying fractions. Because $a \cdot b$ is a b -jumps, $a \cdot 1/n$ is a $1/n$ -jumps, which can easily be realized on a number line with 1 on tick mark n . But what is then $1/m \cdot 1/n$? Well, it is an m :th $1/n$ -jump. For example, $1/4 \cdot 1/3$ means one-fourth of one $1/3$ -jump. To quantify this distance, we place $1/3$ on the fourth tick mark after 0, which means $1/4$ of this distance will be on the first mark after 0. How much is that? Finding that out means finding out where 1 is on this number line. However, because $1/3$ is on the fourth mark, we need three such jumps to reach 1. That is, 1 is on the 12th mark, which means every mark is



If 2 is here, where is 5?

Figure 4. Unitizing and norming the number line

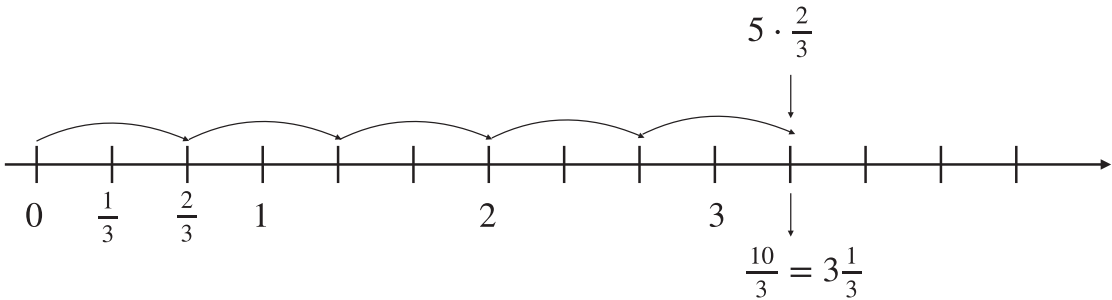


Figure 5. The multiplication $5 \cdot \frac{2}{3}$ as five two-thirds jumps from 0

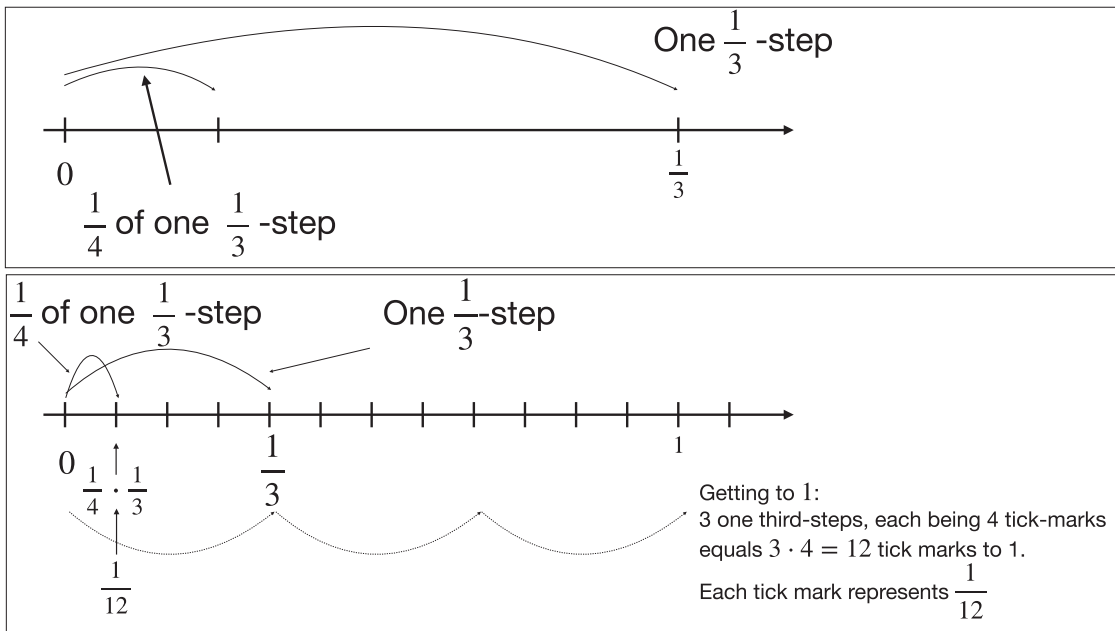


Figure 6. Multiplication of fractions on the number line

1/12. Therefore, the first tick mark, which represents $1/4 \cdot 1/3$, is 1/12. This is not a particularly useful method for thinking about multiplication, but it is a valuable method for introducing meaning to the standard multiplication rule in the fraction symbol system.

5.3 The proportion model

In contrast to the two previous models that are best understood as iconic, the proportion model we use builds on a class of situations. While it is disputed when children can formally reason with proportions (see Boyer et al., 2008, for an overview), it is well known that children have relatively clear intuitions about proportion from an early age. Vergnaud claimed that multiplication, usually described as a three-unit relation $a \cdot b = c$, is initially and intuitively best understood and analyzed as a four-unit relation c is to b as a is to 1 (Vergnaud, 1983, 1988). We want to emphasize that the type of task we will describe is not primarily intended to understand proportional reasoning. Instead, we use rudimentary intuitions about proportion as a basis for mathematizing certain multiplicative phenomena, particularly fractions as a scaling operator.

The model situation we start with is that Big Bear and Small Bear always have the same things in their bags, but Big Bear has twice as many, three times as many, and so on, as Small Bear. The basic task is then to complete partially packed bags, represented in the form of tables, and reason about the relations. Note that the tables are hence symbolic representations of proportional bag packing situation class. We then complement Mama Bear's bag with six times as many things as Little Bear's (Figure 7).

By introducing the two variants of the bag packing task in a table format and having different pupils work with one or the other, or both, we can, in a whole-class session, discuss the relationships between the tables as shown in Figure 7. We then introduce multiplicative operators in arrow form. As illustrated in the water bottle row, combining the two tasks introduces a scaling factor of 3. It can be discussed how consecutive

	Little Bear	Big Bear		Little Bear	Mama Bear
Water bottles	1		Water bottles	1	
almonds	5		almonds	5	
apples		4	apples		
bisquits		1	bisquits		
nuts		10	nuts	5	
grapes			grapes		18
buns			buns		3

Figure 7. Packing bags proportionally

scalings by 2 and 3 are equivalent to scaling by 6. As seen in the apple row, we introduce the notation $\cdot \frac{1}{2}$ as a label for the scaling that is the reverse (inverse) of scaling by 2. We don't spend any time explaining the relation between this polysemic version of $\frac{1}{2}$ and the $\frac{1}{2}$ that denotes a particular point on the number line or the $\frac{1}{2}$ that denotes half a unit square. We only use the same term "one half" and the same symbolic expression. We have so far found no explicit incompatibilities between these three concepts for $\frac{1}{2}$ though.

In the apple row in Figure 8, note the arithmetic symbolic expression in the Small Bear column. It reads, "One apple divided by two equals one half times one apple equals one half apple." This illustrates that the inverse scaling of $\cdot 2$ can be represented by either dividing by 2 or multiplying by $\frac{1}{2}$, and the result of applying either operation to 1 is the number $\frac{1}{2}$. The symbol $\frac{1}{2}$ is used in three different polysemic instances here. This is in grade 3. In grade 2, we work only with scaling by 2, 4, $\frac{1}{2}$, and $\frac{1}{4}$.

The idea of multiplicative scaling can later be extended to the property that if you have any numbers a and b , you can always scale a to 1 and then 1 to b . By concatenating these two operations, we can discover one of the most essential affordances of the fraction symbol system: the ability to provide an exact expression of a number that scales any number a to any other number b (except 0), namely b/a , as shown in Figure 9. So

	Little Bear	Big Bear	Mama Bear
waterbottles	1	$1 \cdot 2 = 2$	$1 \cdot 6 = 6$
almonds	5	10	30
bisquits	2	4	12
apples	$\frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2}$	1	$3 = 6 \cdot \frac{1}{2} = 3 \cdot 1$
nuts	5	10	$30 = 6 \cdot 5 = 3 \cdot 10$
grapes	3	$6 = 2 \cdot 3 = \frac{1}{3} \cdot 18$	18
buns	$\frac{1}{2}$	$1 = 2 \cdot \frac{1}{2} = \frac{1}{3} \cdot 3$	3

Figure 8. Using proportions to introduce multiplicative scaling and inverses

what we are essentially doing with the bag packing task is to build a symbol system representation for fractions as multiplicative operators on top of pupils' intuition for proportional reasoning.

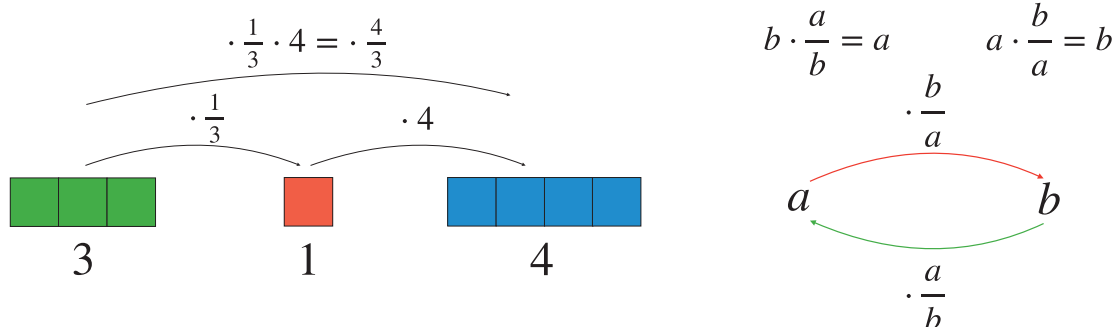


Figure 9. The general scaling operator

In the instructional sequences we have produced, the three models for thinking about multiplication described above are sometimes related to each other; however, it is essential to note that we do not intend to convey the idea that some of the models are designed to be subordinate to others. Instead, the models are connected by using the same words and verbal expressions when speaking of multiplicative relationships, and by representing all different related phenomena in the three models by the same mathematical symbol system. As mentioned, while our theoretical framing emphasizes iconic and symbolic representations, we note that many of the activities in our teaching design also have a distinctly enactive character in the sense of Bruner (1964). For example, students initially handle unit cubes to build rectangles, physically cut and rearrange paper models, or walk along large floor number lines to experience proportional jumps. These bodily actions are not simply preparatory steps but constitute meaningful engagements through which invariants can be perceived and later generalized in iconic and symbolic forms. In this way, the enactive, iconic, and symbolic modes operate in parallel rather than as a strictly sequential progression. We treat enactive engagements as situated *S*-instances that render invariants perceptible, before they are stabilized iconically and symbolically. Taken together, the three models illustrate how different constructions, grounded in situations (proportion) and iconic imagery (rectangles and number lines) can all contribute to the development of the same superconcept. From a polysemy perspective, each model foregrounds particular invariants while leaving others implicit, and their combined use enriches the conceptual field without requiring one model to be subordinated to another. This means that the coherence of the superconcept is not merely a pedagogical artifact of naming conventions, but rather a reflection of shared structural properties that can be expressed in the symbol system. In this sense, the role of models is to mediate between situations, representations, and invariants, thereby supporting conceptual progression across the multiplicative field.

DISCUSSION

We have introduced the concept of conceptual polysemy in this article to denote when a single mathematical concept has multiple related meanings. Our primary claim is that conceptual polysemy is prevalent in

mathematics and is beneficial to consider when planning long-term teaching. We illustrated that in elementary mathematics, conceptual polysemy can occur when a network of related concepts can be subsumed under a more general super-concept. An important instance of conceptual polysemy occurs when the same symbol system can denote several concepts, and the properties of these concepts can be expressed as relations within the symbol system, like with fractions.

While mathematics curricula aim to generalize and extend structures, our claim is that the polysemic character of mathematical concepts is often implicit rather than made explicit in curriculum design or teaching practice. Teachers and curriculum designers frequently present different models of a concept (e.g., fractions as part-whole, measure, or quotient), but rarely do they draw attention to the tensions, incompatibilities, and eventual reconciliation of these meanings within symbol systems. In this sense, polysemy is not sufficiently theorized as a resource for guiding long-term conceptual progression. Our contribution is to make polysemy explicit, to connect it with Vergnaud's framework systematically, and to demonstrate how this lens can inform the design of teaching sequences and anticipate conceptual difficulties. The phenomenon we have labeled conceptual polysemy has been studied before, although not under that specific term. The Rational Number Project (Behr et al., 1983), which analyzes different interpretations of rational numbers, can be considered an examination of the polysemic network related to the concept of fractions. Also, Vergnaud's work on analyzing and describing the additive and multiplicative conceptual fields can be seen as a form of polysemic analysis (Vergnaud, 1982, 1983), and so can Izsák, and Beckmann's (2019) analysis of multiplicative situations and expressions. Furthermore, one of the primary examples we used to illustrate polysemy was Lakoff and Núñez's (2000) description of whole numbers through four conceptual metaphors. Lakoff and Núñez are cognitivists, and for them, conceptual metaphor is a cognitive mechanism that, together with conceptual blending, creates the embodied mental object of whole numbers. While we also employed cognitive theory in our quest to characterize polysemy, we ultimately submit that the theory of conceptual polysemy is a material theory. We used Vergnaud's theory of conceptual fields and theory of representations (Vergnaud, 1998, 2009) as a steppingstone to conclude that concepts in elementary mathematics are typically bootstrapped from classes of situations or from iconic schematic imagery. While a class of situations is in itself a mental construct, it is still not far-fetched to consider situations as something that is directly experienced, hence material (Helenius & Ahl, 2025). Similarly, iconic schematic imagery is also material. As made explicit in the description of symbol systems by Harnad (1990), symbol systems are indeed material too, in the sense that even the rules for manipulating the system should be fully expressible within the system of symbols; that is, the manipulation rules can be made fully explicit and explicated by means of material symbols. While the semantic meaning a symbol system encapsulates may be abstract, the symbol system itself is not.

Thinking of mathematical concepts in terms of polysemy and considering the role that symbol systems play provides us with a theory of conceptual progression. As we described in the introduction, many theories or design frameworks, such as the concrete-representational-abstract framework (Witzel et al., 2003) or the concrete fading framework (Fyfe & Nathan, 2019), do not offer significant analytical power in distinguishing which concrete representations to choose. Understanding a conceptual field in terms of its polysemy structure is a worthwhile effort. As we have explained in this paper, concepts in elementary schooling must be introduced through situations, iconic schematic imagery, or both, because introducing concepts directly by

stipulating relations in symbol systems is almost always intellectually inaccessible. The initial role of the symbol system is then as labels for the conceptual content. Invariants in situations or iconic representations motivate the manipulation and evaluation rules of the symbol system. What takes on the role of abstraction in our account of progression is a gradual epistemological shift, in which the semantic meaning of concepts shifts from residing in situations and iconic imagery to living in the symbol system. The situations and iconic imagery that a concept was initially bootstrapped from more and more serves as examples of the concept, rather than representing its actual meaning (Ahl et al., 2022; Ahl & Helenius, 2021a, 2021b). Polysemy adds to this theory by explaining that two related but different concepts, c and c' , with roots in situations or iconic imagery, can lead to the same concept, C , expressed on the symbol system level, despite properties p of c and p' of c' being incommensurable. A complication in the progression process will then be to accept that such properties p or p' or both must be cleaned out or adjusted.

To illustrate the pedagogical choices that result from considering conceptual polysemy when planning instruction, we examined the concept of multiplication. We illustrated three different but related ways of introducing multiplication through three distinct models for thinking that we utilize in an ongoing elementary school intervention. In our account, models are not concepts themselves but structured representational systems that mediate between classes of situations (S), operational invariants (I), and the symbol system (R). A model highlights specific invariants, thereby facilitating the construction of concepts. While polysemy is a property of concepts, models chosen for teaching can instantiate polysemy by supporting different subconcepts that converge in the same symbol system. Importantly, our approach does not subordinate one model to another; rather, the superconcept (e.g., multiplication) is mathematically unified at the symbol system level, where invariants drawn from different models cohere. This explains why seemingly diverse activities can be connected: the convergence is not merely pedagogical, but grounded in the shared invariants captured by the symbol system.

In contrast to frameworks such as CRA (Witzel et al., 2003) or concreteness fading (Fyfe & Nathan, 2019), which emphasize sequencing from concrete objects to abstract representations, conceptual polysemy accounts for how multiple models can coexist and remain productive over time. Rather than assuming a linear fading of the concrete, polysemy highlights how different situated and iconic meanings can be reorganized and reinterpreted once anchored in a symbol system. This provides a theoretical mechanism for the epistemological shift from concrete to abstract: learners move from treating models as defining the concept to viewing them as examples subsumed under an abstract symbolic structure. In this way, polysemy enhances our ability to analyze and plan for long-term conceptual development by addressing not only how concepts are introduced, but also how their meanings are reorganized over the years of instruction.

While our account emphasizes the structural role of polysemy, we acknowledge two limitations. First, our analysis is primarily theoretical and illustrative; longitudinal empirical studies are still needed to test how polysemy functions in classroom practice across multiple years. Second, other theoretical perspectives may capture aspects of conceptual development that polysemy alone does not, and comparative studies could clarify complementarities. We view this work as a step toward developing a richer toolbox for analyzing and designing long-term instructional progressions.

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