PRAXEOLOGICAL ACCOUNTS OF RESEARCHERS' THEORISING WORK WITH TASK DESIGN PRINCIPLES IN A JAPANESE PROJECT

Yusuke Shinno[1], Takeshi Miyakawa[2]

[1]Hiroshima University,

[2]Waseda University

Abstracts

This study aims to deepen the understanding of researchers' theorising work concerning task design principles in mathematics education. To achieve this, we employ a praxeological analysis to explain and characterise the designing and theorising work of researchers and teachers within a Japanese project on task design. By utilising two distinct praxeologies—designing praxeology and research praxeology—along with the notion of theory elements, we illustrate the results of the case study conducted within the research project, revealing how design principles were produced and elaborated as outcomes of successive theorising steps. Furthermore, cultural issues related to the nature of theorising work are also discussed, providing insights into the subtle distinction between 'development' and 'research' within Japanese mathematics education research.

Keywords: Research praxeology, task design principles, theory elements, deign-based research

INTRODUCTION

In the field of mathematics education, design-based research has been widely adopted and developed across diverse national and cultural contexts. However, what is referred to as 'design-based research' varies according to scholarly traditions. Within the European tradition of didactics, for instance, *didactical design research* or *topic-specific design research* is well-documented in German contexts (e.g., Prediger & Zwetschler, 2013; Nührenbörger et al., 2019), while *didactical engineering* is a research methodology thoroughly developed in French and Spanish contexts (e.g., Artigue, 1994, 2015; García et al., 2019). As such, different researchers have assigned different names and meanings to design research, including *design experiments* and *educational design research* (Kieran et al., 2015; Fowler et al., 2022). This diversity stems partly from the inherent dual aim of design-based research: namely, to improve instructional designs by developing teaching and learning arrangements and to generate theoretical contributions by educational research for understanding the initiated processes as phenomena (Gravemeijer & Prediger, 2019; Prediger, 2019). For the theoretical contributions, researchers are often advised to generate domain-specific local theories about the processes for particular topics (Kieran et al., 2015; Komatsu et al., 2025). Furthermore, design-based research, including task design research, is strongly influenced by the cultural contexts and scholarly traditions in which researchers are working (García, 2019; Johnson & Otani, 2025; Watson &

Corresponding author: Yusuke Shinno

Otani, 2015).

In Japan, research in mathematics education is more or less influenced by Western theoretical and methodological cultures. Design-based research is also disseminated among the community of educational researchers in Japan. However, as our earlier studies have shown (Shinno & Miyakawa, 2025; Wang et al., 2023), 'research' in mathematics education in Japan has a strong orientation towards 'development' and 'improvement' of teaching practice and curricular materials. Accordingly, most design studies have primarily reported on aspects addressing their pragmatic aim. For these studies, developing tasks and improving designs are central to both their processes and outcomes. This is likely due to the close relationship between the mathematics education research community and the teachers' professional community (typically known as lesson study). This does not imply, however, that Japanese researchers are less interested in theoretical work in their research practice. Rather, it appears that Japanese and Western researchers hold differing views on what constitutes 'theory' (Shinno & Mizoguchi, 2023). How can such differences in researchers' views on theory be studied? Gascón and Nicolas (2017) have investigated this issue by focusing on the discourses of some European researchers. However, this is particularly challenging in the case of Japanese researchers, whose theorising work is often embedded within principles formulated to contribute to 'development' of tasks, lessons, or curricula in mathematics education.

In this paper, we address this challenge by presenting a way to understand and characterise researchers' theorising work in design-based research. Specifically, we focus on how researchers engage with design principles in a Japanese project on task design (Komatsu, 2023, 2024). We posit that design principles serve as boundary objects, co-created in both 'development' and 'research' contexts, as these principles can articulate both the pragmatic aim of improving designs and the theoretical aim of generating local theories. However, it is often unclear or implicit how researchers develop and elaborate design principles, and how such work leads to theorising. As some researchers also consider it relatively unexplored how local theories are generated and evolve during design research (e.g., Prediger, 2019, 2024; Prediger et al., 2024; Komatsu et al., 2025), we believe our study offers a novel approach to exploring this aspect, thereby contributing to the international body of knowledge.

In this paper, we retrospectively analyse a published article to illustrate researchers' work on task design principles from a praxeological perspective, based on the Anthropological Theory of the Didactic (ATD) (Chevallard, 2019; Chevallard & Bosch, 2020). As will be explained, ATD offers praxeology as one of its main constructs. This construct can be used to understand researchers' activities and knowledge (referred to as research praxeology; Artigue et al., 2011), as well as to characterise designing work as another type of praxeology. One of ATD's principles, the emancipatory principle (Bosch et al., 2019), allows us to detach from implicit assumptions within a given institution and to view an object of study from an external position. Although both authors are involved in the project, we strive to view the researchers' work as an object of study from a praxeological perspective. As such, ATD is a suitable framework for making the implicit explicit, focusing on researchers' theorising work with design principles.

THEORETICAL PERSPECTIVES

Praxeologies in ATD

Based on ATD, *praxeology* models any human activity in terms of four elements: *type of tasks* (**T**), *technique* (**τ**), *technology* (**θ**), and *theory* (**Θ**). The first two elements are called the *praxis block* (know-how) and the last two elements are called the *logos block* (know-why). **T** indicates a problem of a given type; **τ** is a way of performing **T**; **θ** represents a first level of discourse for explaining and justifying **τ**; and **Θ** provides a second level of discourse for explaining and justifying **θ**. Different types of praxeologies have been used to characterise different activities and knowledge in given institutions. Mathematical praxeology models the mathematical activity and knowledge, and didactic praxeology characterises the teaching practice and teaching knowledge of a certain mathematical praxeology.

Didactic praxeology is a notion that includes various types of tasks. We consider, in the Japanese context of teacher work, at least the following didactic types of tasks: (i) *kyōzai* analysis, (ii) *kyōzai* design, (iii) lesson design, (iv) classroom practice and (v) lesson reflection. *Kyōzai* is a Japanese professional term meaning a subject-matter or a teaching material in the target disciplines (in our case, mathematics). *Kyōzai* analysis is a didactic task for understanding mathematical content to be taught. for example, teachers might solve a problem or prove a theorem before teaching them in classrooms. In Japan, such work on *kyōzai* is often called *kyōzai kenkyū* (e.g., Fujii, 2016; Melville & Corey, 2022) and is frequently connected to design work on tasks and lessons (Shinno & Mizoguchi, 2021). *Kyōzai* design is a didactic task for connecting teaching and learning goals with the mathematical content and for designing a mathematical task appropriate for the classroom. Lesson design is a didactic task for planning the teaching and learning process in the classroom. Classroom practice involves teaching the designed tasks and the planned process, which can be described in terms of different didactic moments (Chevallard & Bosch, 2019, p. xxvi). Lesson reflection refers to a didactic task for reviewing the actual teaching or evaluating students' learning outcomes.

In this paper, we call a didactic praxeology with the second and third types of tasks a 'designing praxeology (DeP)', and focus on the second type for the analysis in this paper, because *kyōzai* design often refers to developing a mathematical task to be used for a classroom. As will mentioned later, the task design principles, the main focus of the project, are constructed for guiding the design of mathematical task itself, although lesson design and other didactic tasks are also relevant to implementing and undertaking the designed tasks. The work of teachers or researchers described in DeP is rather complex, as this praxeology consists of various elements that are often not organised systematically. Task design involves different techniques that are often implicit, and technological or theoretical discourses to explain the praxis block are often invisible. For instance, there are different reasons why (or why not) a teacher chooses the same task as one found in textbooks (this is probably due to curricular documents, teaching guidebooks, personal teaching experience, etc.). In the design research project, researchers also addressed task design work in collaboration with teachers. Although designing techniques and corresponding discourses include miscellaneous elements, this complexity is a characteristic aspect of DeP.

Theory elements and theorising steps in design-based research

The literature (Bakker & van Eerde, 2015; Gravemeijer & Cobb, 2006) acknowledges that design-based research aims to produce a local theory through cycles of design, implementation, and retrospective analysis[1]. Prediger (2019, 2024) and Komatsu et al. (2025) have proposed a framework to explore how different *theory elements* are produced, organised, and evolved through the process of design-based research. As shown in Table 1, Prediger (2019, 2024) lists five typical theory elements: *Categorical theory elements* are constructs that provide a language and thinking tool for perceiving and distinguishing phenomena. *Descriptive theory elements* describe certain phenomena qualitatively or quantitatively. *Explanatory theory elements* explain, provide causes, or identify the backgrounds of described phenomena. Normative theory elements specify and justify aims and rationales. Finally, *Prescriptive theory elements* serve to ground purposeful action or predict the effects of a design element[2]. In terms of the pragmatic aim of design research, these theory elements serve as the underpinnings for the designs of teaching and learning. In terms of the theoretical aim, these elements are crucially important for achieving the aim.

Prediger (2019) defines *theorising* as "a process of successively elaborating a web of intertwined theory elements with different functions" (Prediger, 2019, p. 7). For example, design principles are considered a set of prescriptive theory elements that are not produced in a single step of theorising. Instead, they are gradually developed from other theory elements through multiple steps, such as *identifying*, *connecting*, *transforming*, and *refining*.

Table 1. Theory elements and their functions and structures (Prediger, 2024, p. 6)

Theory elements	Function	Structure
Categorical	Providing a language and thinking tool for perceiving and distinguishing	Conceptual structures, i.e., categories, and relations
Descriptive	Describing a certain phenomenon qualitatively or quantitatively, focused by specific categories	Propositions stating existence, categorial hierarchies, or frequencies
Explanatory	Explaining, giving causes, or identifying backgrounds	Propositions with cause-effect structure or phenomenon-back- ground structure
Normative	Specifying and justifying aims and rationales (e.g., learning goals or process qualities)	Propositions with an aim reason structure
Prescriptive	Purposefully acting or predicting effects	Propositions in "in order to" structure or propositions in "if- then" structure

Research praxeologies and designing praxeologies

Research praxeology (RP) is a concept to understand researchers' activities and knowledge at the meta-level. This concept has been developed and used to better explain how researchers work on didactic theories (e.g.,

1 This is a type of design-based research that is well-documented in some European countries, such as Germany and the Netherlands. The concept of 'local theory' and the cycle of design and research are core to these research contexts. In contrast, in French and Spanish contexts of didactic engineering for research, there are distinct methodological phases, such as preliminary analysis, design and a priori analysis, implementation, and a posteriori analysis (Artigue, 2015; Barquero & Bosch, 2015).

2 By Prediger (2019), this theory element was called the 'predicate' theory element, and in a later paper (Prediger, 2024), it was rephrased as 'prescriptive'. We use the latter (newer) name to refer to this element.

Artigue et al., 2011; Artigue & Bosch, 2014; Shinno & Mizoguchi, 2023; Wang et al., 2023). Generally, a type of tasks in RP often refers to research questions and problems to be studied. Research technique comprises the research method that addresses these questions; research technology corresponds to methodological discourse that justifies the choice of the method and explains the results; and research theory includes a given theoretical framework and scholarly tradition behind it (see also Shinno & Mizoguchi, 2023). However, what constitutes an RP may vary according to the researchers and research institutions (Wang et al., 2023). Research articles can be seen as an empirical source for understanding the elements of RP, while the process of research practice is more complex and often difficult to identify from such articles.

Generating and refining different theory elements through successive theorising steps is essential for achieving the theoretical aim of design-based research. RP is used to describe and characterize how researchers work with them. For instance, identifying particular theory elements about the process of learning may correspond to a type of research task in RP, and theorising steps towards elaborating and structuring the theory elements leading to a local theory are concerned with research techniques. As mentioned earlier, different methodological assumptions on design-based research could be adopted according to the cultural contexts and scholarly traditions in which researchers are working. This may constitute the logos block of RP.

However, understanding the RP in design-based research is not straightforward, as it is a close relationship to the DeP. One could commonly observe the five types of tasks in the DeP in the Japanese context of ordinary teacher work, while corresponding techniques and discourses are rather pragmatic but often guided by official curricular documents and textbooks. Teachers sometimes explain their choice of tasks and the reasons of their appropriateness in their lesson plans (e.g., Shinno & Mizoguchi, 2021). In the context of design-based research, teachers' work on task design is interpreted as research-based, and some elements of the DeP, especially its logos block, could be relied on the constructs developed for the RP. For examples, design principles may contribute to technological discourse of the DeP to explain and justify the praxis block of the task design.

CONTEXTS AND METHODS

A Japanese project on task design

We refer to a Japanese research project on task design as an illustrative case study for our analysis. This project aimed to develop task design principles (TDP) enable researchers and teachers to design mathematical tasks that make proving activities explorative (Komatsu, 2023, 2024). The project is overseen by seven researchers, each of whom has been engaged in theoretical considerations and empirical case studies related to TDPs. Teachers are not considered official members of the project; however, they have been invited by researchers to implement and improve designs. Collaboration between researchers and practising teachers is not always a prerequisite, and there have been cases where undergraduate and graduate students have been asked to participate in some case studies.

In this project, TDP is viewed not only as a means of designing tasks to achieve specific learning goals in the initiated lessons but also as the main outcome of the research studies. Within the project, Komatsu (2023) defined TDP as follows: "theoretically-and-empirically grounded prescriptions that guide the design of tasks for specific learning contents or activities" (p. 5). Design-based research is adapted as a methodological underpinning of a cycle model of design, implementation, and analysis, which is reconceptualized in this project as comprising the five phases: (1) specifying the learning goal, (2) constructing TDPs, (3) designing tasks, (4) implementing the tasks, (5) analysing the implementation. In addition, this cycle mode includes two fields of study: "research" and "context (of implementation)". "Research" is a scientific field in which researchers refer to relevant theoretical references to provide the underpinnings of designs and contribute to the body of knowledge in mathematics education. "Context" is a practical field in which the designed tasks are implemented (e.g., problem-solving style lessons; see also Komatsu et al., 2025).

One characteristic of this project is that a model of the explorative proving activities (presented as Figure 1), based on a Lakatosian-style of teaching proofs and refutations (Miyazaki e al., 2019; Komatsu & Jones, 2019), is shared with researchers involved in this project before working on task designs and design principles. This model provides a basic assumption that guides learning goals and other elements in the aforementioned phases.

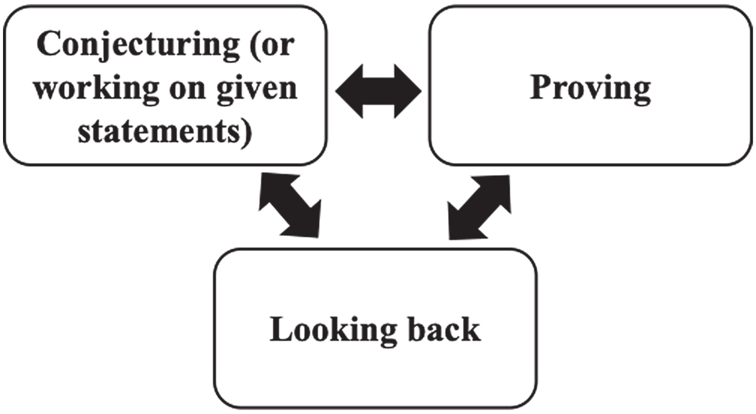


Figure 1. A model of explorative proving (adapted from Miyazaki et al., 2019, p. 2; see also Komatsu & Jones, 2019, p. 804)

Another characteristic is that the project offers a 'task design format' allowing researchers to work on learning goals, TDPs, and task designs. Figure 2 shows a template of this format. Using this format, researchers are advised to formulate the TDPs as an 'in order to' structure. In these statements, the 'focus' is written before 'in-order-to' and the 'aim' is stated after 'in-order-to'. Therefore, the work of formalising task design principles corresponds to the generation of prescriptive theoretical elements, and the process is not straightforward. We assumed that task design principles are constructed through various steps of theorising, and this could be explained and analysed from the perspective of RP in design-based research. On the other hand, in the format of Figure 2, the setting of learning objectives and task design itself are inseparable from the associated TDPs. Design principles are constructed to achieve learning objectives, and principles are realised through tasks and their implementation. These elements are thought to correspond not only to RP but also to DeP, and the distinction between them is not always clear. Through the analysis of case studies, our study examines the distinctions and relationships between each praxeology and better articulates the work of researchers on design principles.

1. Learning goal(s)	2. Task design principles				3. Task design	
	Principles			Theoretical references	Tasks	Expected activities
	#	Focus	Aim			
…	TDP1:…, TDP2:…, …				……	

Figure 2. Working format of the task design (Komatsu, 2023; our translation)

Methods

This study provides a retrospective analysis of one of the completed case studies within this project. The analysis focuses on researcher-and-teacher’s work with design principles, as documented in a published six-page conference paper written in Japanese (Tsujiyama & Kato, 2022), with one page devoted to the task design format. Although several studies have been undertaken as part of the project, there are two main reasons for choosing this particular one as an illustrative case study. Firstly, the study reports two cycles of task designs with two different implementations: the first utilising implicit or absent TDPs and the second based on more explicit TDPs. This allows us to observe how researchers address TDPs to make them more explicit. Secondly, the study was carried out in collaboration with a teacher (the second author of the article), and the authors provided descriptions of how they divided their work during the study. This will be helpful for us to identify the researcher’s individual work and their collaborative works through the analysis of the written article.

In order to share the objectives of this project, the model of the exploratory proving activities (Figure 1), and the task design format (Figure 2), a teacher attended several project meetings and discussed the planning and design of case studies with researchers. The division of roles between the researcher and the teacher in the task design research cycle is delineated in the paper (Tsujiyama & Kato, 2022). For example, in the first cycle, learning objectives and tasks were formulated by the teacher with the support of the researcher. The implementation of the tasks was conducted by the teacher, while the lessons were observed by the researcher. The researcher identified and formulated task design principles, and improvements to the principles and task designs were made collaboratively for the second cycle. In their article (Tsujiyama & Kato, 2022), they posit the significance of the researcher and the teacher assuming different roles throughout the cycles. This implies that the elements of DeP were developed with the involvement of both the teacher and the researcher. Conversely, the process of reflecting on the construction of TDPs and compiling them into an article was primarily handled by the researcher. We attempt to identify the theorising steps from that paper.

In the next section, we first describe the main outcomes of the case study, such as learning goals, TDPs, and designed tasks. We then illustrate how the researcher engages with producing and elaborating the TDPs in terms of praxeologies (DeP, RP), theory elements, and theorising steps (connecting, transforming and refining).

ILLUSTRATIVE ANALYSIS OF THE CASE STUDY

Approaches and outcomes of constructing TDPs

The design study by Tsujiyama and Kato (2022) reported the process and outcomes of developing TDPs for explorative proving in geometry in a lower secondary school. Their developmental process consisted of three key stages, as follows.

- Stage 1: Specifying the learning goal, designing tasks, implementing the tasks, and analysing the implementation (without explicit TDPs)
- Stage 2: Constructing TDPs based on the analysis of the implementation at the first stage, with reference to existing TDPs (Komatsu & Jones, 2019)
- Stage 3: Re-designing tasks, implementing the tasks, and analysing the implementation (with the TDPs constructed at the second stage)

Although both the researcher and the teacher collaborated across all stages, the teacher played a more prominent role in Stage 1, whereas the researcher was more extensively involved in the construction of the TDPs, task design, and the analysis of the implementation during the latter two stages. In the following section, we focus on Stage 2 to summarise the approaches and products of constructing TDPs.

Figure 3 illustrates the working format of Tsujiyama and Kato (2022), which includes some results. To construct TDPs, Tsujiyama and Kato recalled the existing TDPs (Komatsu & Jones, 2019, p. 807) that are relevant to their learning goal. The principles they referred to as prior knowledge are as follows:

- Using tasks whose conditions are purposefully implicit and thus allow the production of particular proofs and the occurrence of counterexamples.
- Providing tools that enhance the production of counterexamples, while making explicit the purpose of the tools' use (i.e. investigating the existence of counterexamples).
- Increasing students' recognition of contradictions that can help them revise conjectures/statements and/or proofs.

These prior principles were originally constructed in the teaching proof with a dynamic geometry environment. However, Tsujiyama and Kato (2022)'s task design relied on the teaching proof with diagrams in the paper-and-pencil environment. TDP1, as presented in the format (Figure 3), is adapted from the first principle in Komatsu and Jones (2019). In the first stage of the implementation, the task included the sentence "take the point D on the side AB"; however, Tsujiyama and Kato (2022) reported that this phrasing prevented students from becoming aware of the conditions of the statement. In reflecting on the implementation of the task at stage 1, many students were aware that their diagrams (e.g., a diagram of triangle ABC with $\angle$A=100°, $\angle$B=40°, and $\angle$C=40°) were *improper* but were unable to correct them[3], and the validity of proofs based on improper diagrams was not given much attention. To better meet the learning goals they set (see Figure 3), both the researcher and the teacher agreed to these reasons for revising tasks. In the revised tasks, the precise placement of point D (on the line AB or on the side AB) was left implicit. If students draw a figure with point D on side AB, it may be an improper, but if they place point D on the line extending side AB, it may be a

3 One of the key ideas underpinning the task design was what they termed an *improper* diagram which is "a diagram in which geometrical objects such as points and lines should not be drawn, are drawn in the same way as in the previous task" (Tsujiyama & Kato, 2024, p. 228).

proper figure. They decided that by modifying tasks that intentionally include ambiguous conditions in this way, the task design could promote exploratory proving activities among students.

They also referred to an earlier study (Komatsu et al., 2014) documenting students' difficulties related to proof problems with diagrams. To incorporate this idea into design principles, Tsujiyama and Kato (2022) reconstructed TDP3, as presented in the format (Figure 3), adapted from the third principle by Komatsu and Jones (2019). Additionally, another resource underpinning their TDP1, TDP2, and TDP3 was their experiences from the design implementation in the first stage. In the first implementation, only a few students were able to notice a contradiction between the (improper) diagram and the (valid) proof (Tsujiyama & Kato, 2022). However, such students' notice is crucial for explorative proving to enhance revising a statement and its proof as stated in the learning goal. This, then, led to the additional input to TDP3: "…the contradiction between the improper diagram and the valid proof…".

<table>
<tr><td rowspan="2">Learning goal</td><td colspan="2">Task design principles</td><td>Task design</td></tr>
<tr><td>Principles</td><td>Theoretical references</td><td>Tasks</td></tr>
<tr><td rowspan="3">General goal: Explorative proving activities that allow interactions between 'statement (conjecture)', 'proof', and 'looking back'

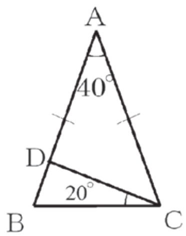
Concrete goal: Students revise diagrams and examine proofs, leading to the elaboration of statements and proofs</td><td colspan="2">TDP1: Use tasks whose conditions are purposefully implicit, in order to allow the production of a conjecture whose hypothesis is implicit (Komatsu & Jones, 2019; Kato, 2018; Kato & Tsujiyama, 2019)</td><td>Task 1: Given △ABC is an isosceles triangle, the point D is on AD. Find the size of ∠CDB in the diagram
</td></tr>
<tr><td colspan="2">TDP2: Provide a problem that considers if the conjectured statement is true, in order to allow the production of proofs with 'improper' diagrams (Komatsu & Jones, 2019)</td><td>Task 2: […] If point D is taken on AB such that ∠BCD = 1/2 ∠A, does ∠CDB = 90° for any degree of ∠A?</td></tr>
<tr><td colspan="2">TDP3: Increase students' recognition of the contradiction between the improper diagram and the valid proof, in order to allow the revision of the statement and the proof (Komatsu et al., 2014)</td><td>To ask what is strange and what needs correcting about the diagram and proof; and to help students focus on whether point D can be taken depends on what 'on AB' implies in the statement.</td></tr>
</table>

Figure 3. Task design format excerpted from Tsujiyama and Kato (2022)[4] (our translation)

4 References to Kato (2018) and Tsujiyama and Kato (2019) provide evidence for the first stage of implementation. The article by Komatsu et al. (2014) provides an empirical source in the context of a Lakatosian-style proof lesson. Regarding the language used in the three principles (TDP1–3), Tsujiyama and Kato (2024) have recently rewritten them in English as follows: TDP1' Using initial tasks of finding numerical values in specific cases whose conditions are purposefully ambiguous; TDP2' Using tasks of producing and examining a general conjecture; TDP3' Inducing students' focus on the relationship between improper diagrams and the corresponding arguments. They omitted the 'aim' parts ('in order to...') to shorten them.

Researchers' work from the perspectives of DeP and RP

As mentioned, the project adopted a five-phase cycle model of design-based research (Komatsu, 2023, 2024), which incorporates different types of tasks of didactic praxeology. Table 2 presents possible correspondences between the phases in a cycle and the types of tasks. As mentioned earlier, the five types of tasks refer to what is generally observed in teacher work in the Japanese context. For example, "specifying the learning goal" and "designing tasks" involve both teachers' and researchers' activities, corresponding to the task types of DeP. As various methods exist for performing these, identifying a single specific technique is not straightforward. However, *kyōzai* analysis, commonly known as *kyōzai kenkyū* in Japan, is considered one of the main techniques of DeP. For *kyōzai* analysis, teachers (and researchers) utilise various teaching resources such as curricular documents, textbooks, professional or research articles, and so forth. Tsujiyama and Kato (2022) mentioned that they referred to research articles related to the Lakatosian style of teaching proofs and refutations. "Implementing the tasks" is more closely related to teachers' work on lesson design, classroom practice, and lesson reflection. "Analysing the implementation" is also related to lesson reflection. However, it appears that "constructing TDPs" does not correspond well to any types of tasks of didactic praxeology, as this phase is rather "research-based" work and not usually observed in teacher practice. However, in the case study, there is a collaboration between the researcher and the teacher for sharing and agreeing TDPs, which could be seen as another type of task of didactic praxeology (DeP in particular) in the context of design-based research. In terms of RP, constructing and examining TDPs are considered research task types of constructing technological elements to explain and justify the praxis of DeP. Identifying and elaborating theory elements constitute researchers' work on theorising steps (Prediger, 2019, 2024), encompassing the research techniques addressed in these phases. We will illustrate these techniques in the case of Tsujiyama and Kato (2022).

Table 2. Relationship between the phases in the project and types of tasks of didactic praxeology

Phases in a cycle model of the project (Komatsu, 2023, 2024)	Types of tasks of didactic praxeology
(1) Specifying the learning goal	(ii) *Kyōzai* design
(2) Constructing TDPs	
(3) Designing tasks	(i) *Kyōzai* analysis, (ii) *Kyōzai* design
(4) Implementing the tasks	(iii) Lesson design, (iv) classroom practice, (v) lesson reflection
(5) Analysing the implementation	(v) lesson reflection

Researchers' work from the perspective of theorising steps

To characterise the research techniques identified in the case study, we use a framework according to what Prediger (2019) termed "typical theorizing processes for the successive intertwinement of theory elements that in the beginning are unconnected and vague" (p. 21). In Figure 4, the phrases included in 'typical steps of theorising' are adapted from the illustration by Prediger (2019), and the statements in "corresponding theory elements" are excerpted or reconstructed from the texts in the article by Tsujiyama and Kato (2022). Although the construction of the three principles is complex, we focused on the theory elements and the steps of theorising, particularily the development of TDP3. Thus, Figure 4 presents how the researcher approached the research task, producing the specific TDP, through successive steps of theorising.

Typical steps of theorising	Corresponding theory elements		
Identifying a phenomenon and developing categories for describing it	Students' activities to generate statements based on geometrical problem solving with calculations, and to validate a proof with a diagram that should not be drawn		
Refining categories in order to increase their explanatory power	A diagram that should not be drawn	→ *refined* into	"Improper diagram" which means that a diagram in which geometrical objects such as points and lines, which should not be drawn, are drawn in the same way as in the previous task (Ttsujiyama & Kato, 2024, p. 228)
Connecting descriptive elements to explanatory elements	● "Only few students revise their improper diagrams" (Ttsujiyama & Kato, 2022, p. 153) ● When validating proofs with improper diagrams, students focused on the validity of the diagrams rather than the validity of the proofs themselves (i.e. whether the same proof would apply if point D could be taken) (ibid., p. 153)	→ *connected* with	A finding in the literature (Komatsu et al., 2014): When teaching proof problems with diagrams through Lakatosian style, it is difficult for some students to distinguish if one can construct a diagram and if one can prove
Transforming descriptive elements into a normative element	The above two descriptive elements	→ *transformed into*	Learning goals to realise exploratory proving activities that allow students to revise diagrams and examine proofs, leading to the elaboration of statements and proofs
Transforming an explanatory element into a conjecture for a prescriptive element	"Based on the literature review [Komatsu et al., 2014], we need to design tasks that enable students to distinguish between the examination of a diagram and a proof" (ibid., p. 154)	→ *transformed* into	"Increase students' recognition of the contradiction between the improper diagram and the valid proof" (ibid., p. 154)
Refining a prescriptive element on design principles by adding a qualifier	"Increasing students' recognition of contradictions that can help them revise conjectures/statements and/ or proofs" (the prior principle; Komatsu & Jones, 2019)	→ *refined* into	"Increase students' recognition of the contradiction between the improper diagram and the valid proof, in order to allow the revision of the statement and the proof" (TDP3)

Figure 4. An illustration of typical steps of theorising in Tsujiyama and Kato (2022), adapted from Prediger (2019)

Figure 4 illustrates what theory elements were produced how those elements were elaborated into a design principle as outcomes of multiple theorising steps: *identifying*, *refining*, *connecting*, and *transforming*. For example, regarding the first two steps, “a diagram that should not be drawn” is an identified student’s production, and it became an “improper diagram” with a clearer definition. This allows them to describe and explain students’ proving activities with diagrams. Tsujiyama and Kato (2022) produced two descriptive elements, which were considered triggers for revising tasks, found in the first stage of their design study. These were then *connected* with findings in the literature (Komatsu et al., 2014) and subsequently *transformed* into a conjecture for a prescriptive element: “to provoke students’ awareness of a contradiction ‘while the diagram is improper, the proof is valid’” (Tsujiyama & Kato, 2022, p. 154). This transforming step was supported by the literature (Komatsu et al., 2014), which was then used to provide the underpinnings for the prescriptive element: the need for task design allowing students to distinguish a (proper or improper) diagram with a proof and the diagram. This element was subsequently *refined* and incorporated into one of the TDPs (see TDP3 in Figure 3). These successive steps of theorising contribute to increasing the prescriptive power of the TDP.

DISCUSSION AND CONCLUSION

Considering the dialectic between the pragmatic and theoretical aims of design-based research, researchers’ work can be described and characterised as both DeP and RP. As illustrated in this paper, researchers’ designing work is more evident in the case project in Japan, while their theorising work lies behind the construction of TDPs. This implicit work can be made explicit through the meta-level analysis in terms of RP with different theory elements. As the task design project intertwines design-based research and research-based design (Prediger, 2019), both kinds of praxeologies, DeP and RP, are inherently interrelated. RP models the work of the researcher, whereas DeP models the collaborative work of the researcher and the teacher in task design. On the one hand, for example, researchers develop a technological discourse of DeP to explain, justify, and guide their task designs, which can result in a set of TDPs. On the other hand, constructing TDPs also corresponds to a type of task in RP, as a set of theory elements can be organised to a local theory (Komatsu et al., 2025). The teacher makes use of principles to design tasks through various techniques. While the researcher support teacher’s work in this process, the researcher’s primary role is to refine the principles themselves. This is because, for the researcher, the act of generating principles constitutes an RP task in the project. As such, TDPs function as boundary objects for the researcher and the teacher, though each engages with them differently. To make the distinct features more explicit from a praxeological perspective, Table 3 summarises our illustration, including both general and particular statements from our analysis.

Table 3. Organisations of DeP and RP in the Japanese task design project

	Designing praxeology (DeP)	**Research praxeology (RP)**
T	Designing tasks presented in Figure 3	Generating a set of prescriptive theory elements
τ	A way of analysing and developing tasks, materials (*kyōzai kenkyū*)	Theorising steps: identifying, refining, connecting, transforming, transforming, and refining
θ	Task design principles: TDP1-3 presented in Figure 3	Design-based research: a five-phase cycle model (Komatsu, 2023, 2024)
Θ	Didactic model: a model of explorative proving (see Figure 1)	Research model (unidentified)

Although TDPs themselves may not appear as a fully-fledged local theory, generating and evolving different theory elements may lead to structuring those elements as a local theory (see Komatsu et al., 2025). In the previous section, we illustrated some research techniques and how the researcher works on producing prescriptive theory elements through different steps of theorising. What counts as research outcomes may vary according to the logos of RP. For instance, Prediger (2019, 2024) considers a local theory as such an outcome, aligning with a theoretical aim within their research paradigm. In our case study, the generation of TDPs as a set of prescriptive theory elements constitutes the main research outcome. Crucially, this also represents an explicit contribution to improving designs, thereby fulfilling a pragmatic aim. In the project, TDPs were first produced as the technology of DeP, and their potential contribution to the theoretical aim of design-based research was subsequently discussed.

Regarding the theoretical discourse of DeP, the model of explorative proving, adopted from Miyazaki et al. (2019), provides the rationale for the design principles employed in the project. We call this a *didactic model*, as it is the dominant model in this project. However, the theoretical discourse of RP is relatively underexplored in the present paper. We assume that there is *a research model* that explains and justifies the adaptation of design-based research in the project, potentially influenced by the scholarly tradition within which Japanese researchers work. Although the specific understanding of the research model in Japan remains underexplored in the present paper, it appears that mathematics education research in Japan has a strong orientation towards 'development' and 'improvement' of teaching practice and curricular materials (Wang et al., 2023). In terms of the cultural issues underlying design-based research in Japan, the distinction between 'development' and 'research' is rather subtle. As illustrated, the TDPs constructed in the cycles have a prescriptive nature, guiding and improving task designs. The construction of TDPs is considered both a research task of RP and a technological element of DeP. In the Japanese cultural context, collaboration between researchers and teachers comes naturally, and their collaborative work is effective in certain areas of the design-based research cycle. Accordingly, Japanese researchers in mathematics education often work with theories that lie at the boundary between scientific research and pragmatic development.. This is probably one of the characteristics of the research model that explains the adaptation of the five-phase cycle model from the literature (Bakker & van Eerde, 2015; Gravemeijer & Cobb, 2006; Gravemeijer & Prediger, 2019). For example, the nature of the five-phase cycle model could also be interpreted by what Gravemeijer and Prediger (2019) considered 'five common characteristics' among different design research approaches: a) interventionist, b) theory generative, c) prospective and reflective, d) iterative, and e) pragmatic roots and humble theories. Regarding the second point, they mentioned as follows: "the goal of design research is to generate theories about the process of learning and the means of supporting that learning […]; generating theories here means both developing and refining theories in terms of inventing categories and generating hypotheses" (p. 34). Although this aspect is often implicit in the Japanese case, as it is rather embedded in design principles, we attempted to make it explicit from the perspectives of praxeologies and theory elements. We also illustrated some aspects of researcher-teacher collaboration in the task design project. Although Table 3 presents the distinct features of DeP and RP, our case study showed that technological and theoretical elements of the DeP were shared among them and the construction of the technological element of DeP became new tasks or questions for RP. This case implies that researchers in the Japanese project are involved in both RP and DeP, and teachers also share some elements of RP as well as DeP. In Table 2, "constructing TDPs" does not correspond to any types of tasks of didactic praxeology, but this does not mean that teachers are not involved in the construction of TDP. The construction of TDP is produced as a technological element of DeP, shared with teachers, and agreed upon. It seems that the infrastructure of teacher education in Japan facilitates such collaboration between researchers and teachers, as it is well known in lesson study. Developing teaching materials and improving lessons often become common goals between teachers and researchers, and design principles function as boundary objects among them. We assume that such infrastructure is not necessarily natural in design-based research conducted in other countries and is influenced by the cultural context of Japan.

Within the community of researchers and teachers in Japan, there is a tendency to accept pragmatic or prescriptive theories that emerge from and develops alongside designing and teaching practice (Shinno & Mizoguchi, 2023; Wang et al., 2023). Consequently, our study also contributes to recent discussions on the role of theories that guide practitioners' actions (Herbst & Chazan, 2023; Prediger et al., 2024). For example, Prediger et al. (2024) discuss the difference between the discourses of researchers and practitioners, and how theory elements can be transformed into the discourses accessible to teachers and other stakeholders to guide their actions. In addition, our study showed a case in which transformations occur within the researchers' community itself, given that its members are involved in both designing and theorising work in the project.

In international contexts, however, there are diverse adaptations of design-based research or task design frameworks (García, 2019; Watson & Otani, 2015). For example, questions arise regarding how design principles are produced and shared with teachers in design-based research conducted in different traditions, what elements of praxeology design principles correspond to, and whether elements not called design principles fulfil similar roles. We can also assume that there are different research models underpinning researchers' technological discourse and the praxis of RP. For example, within the European tradition of didactics, researchers more often work with theories developed for scientific research (e.g., Gascón & Nicolás, 2017, 2019). Researchers working in international contexts may face challenges in communication due to differing scholarly traditions and cultural assumptions (Shinno & Miyakawa, 2025). We also discussed this from the perspective of the transposition of research knowledge from the West to Japan. When design-based research from the West is transposed into the Japanese research institution, it undergoes a process of adaptation with the existing Japanese research model. Further research is needed to elaborate methodological aspects of praxeological analysis, to develop approaches that make implicit research paradigms more explicit, and to advance our understanding of the cultural diversity of design-based research in mathematics education.

Acknowledgment

This work was supported by JSPS KAKENHI Grant Numbers JP19H01668 and 24K00410. We thank all members of the project for their help in making this study possible. Special thanks go to Kotaro Komatsu (University of Tsukuba) for his leadership of the project and Yosuke Tsujiyama (Gunma University) for his helpful comments on an earlier version of this paper.

REFERENCES

Artigue, M. (1994). Didactical engineering as a framework for the conception of teaching products. In R. Biehler, R. W. Scholz, R. Straesser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific disciplin*e (pp. 27-39), Kluwer Academic Publishers.

Artigue, M. (2015). Perspectives on design research: The case of didactical engineering. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), Approaches to qualitative research in mathematics education: Examples of methodology and methods (pp. 467–496). Springer. https://doi.org/10.1007/978-94-017-9181-6_17

Artigue, M., & Bosch, M. (2014). Reflection on networking through the praxeological lens. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 249–265). Springer International Publishing. 10.1007/978-3-319-05389-9_15

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak et al. (Eds.). *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). ERME.

Bakker, A., & van Eerde, D. (2015). An introduction to design-based research with an example from statistics education. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education: Examples of methodology and methods* (pp. 429–466). Springer. https://doi.org/10.1007/978-94-017-9181-6_16

Barquero, B., & Bosch, M. (2015). Didactic engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education* (New ICMI Study Series, pp. 101–120). Springer, Cham. https://doi.org/10.1007/978-3-319-09629-2_8

Bosch, M., Chevallard, Y., García, F. J., & Monaghan, J. (2019). Introduction: An invitation to the ATD. In M. Bosch, Y. Chevallard, F. J. García, & J. Monaghan (Eds.), *Working with the anthropological theory of the didactic: A comprehensive casebook* (pp. xii–xvii). Routledge.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, 12, 71–114. https://doi.org/10.24529/hjme.1205

Chevallard, Y. & Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In Lerman, S. (Ed.). *Encyclopedia of Mathematics Education* (pp.53–61). Springer.

Fowler, S., Cutting, C., Fiedler, S. H., & Leonard, S. N. (2023). Design-based research in mathematics education: Trends, challenges and potential. *Mathematics Education Research Journal*, *35*(3), 635–658. https://doi.org/10.1007/s13394-021-00407-5

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education*, *48*(4), 411–423. https://doi.org/10.1007/s11858-016-0770-3

García, F. J. (2019). Introduction to 'task design in mathematics education: a diversity of theoretical frameworks'. *Avances de Investigación en Educación Matemática*, *15*, 1–4. https://doi.org/10.35763/aiem.v0i15.264

García, F.J., Baquero, B., Florensa, I. & Bosch, M. (2019). Diseño de tareas en el marco de la Teoría Antropológica de lo Didáctico. *Avances de Investigación en Educación Matemática*, *15*, 75–94. https://doi.org/10.35763/aiem.

v0i15.267

Gascón, J., & Nicolás, P. (2017). Can didactics say how to teach? The beginning of a dialogue between the anthropological theory of the didactic and other approaches. *For the Learning of Mathematics*, *37*(3), 9–13. https://flm-journal.org/Articles/652706CCAD574E410A33668116D7F3.pdf

Gascón, J., & Nicolás, P. (2019). Research ends and teaching ends in the anthropological theory of the didactic. *For the Learning of Mathematics*, *39*(2), 42–47. https://flm-journal.org/Articles/12277582FF616CCF6BE231F64C6512.pdf

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen (Eds.), *Educational design research* (pp. 17–51). Routledge.

Gravemeijer, K., & Prediger, S. (2019). Topic-specific design research: An introduction. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 33–57). Springer. https://doi.org/10.1007/978-3-030-15636-7_2

Herbst, P., & Chazan, D. (2023). Keeping theorizing in touch with practice: Practical rationality as a middle-range theory of mathematics teaching. In A. K. Praetorius & C. Charalambous (Eds.), *Theorizing teaching: Current status and open issues* (pp. 189–224). Springer. https://doi.org/10. 1007/978-3-031-25613-4_7

Johnson, H.L., & Ohtani, M. (2025). Advances in task design in mathematics education. *ZDM Mathematics Education*. https://doi.org/10.1007/s11858-025-01690-3

Kato, K. (2018). Teaching proofs to develop an attitude of continuous questioning of the generality of propositions: locating special situations in the process of generalisation. *Journal of Japan Society of Mathematical Education, Special Issue of 100th conference.* 400. (In Japanese)

Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education. New ICMI study series* (pp. 19–81). Springer, Cham. https://doi.org/10.1007/978-3-319-09629-2_2

Komatsu, K. (2023). A framework for research on the development of task design principles in school mathematics. *Journal of Japan Society of Mathematical Education*, *105*(1), 2–13. (In Japanese) https://doi.org/10.32296/jjsme.105.1_2

Komatsu, K. (2024, July 7–14). *A framework for research on the development of task design principles in mathematics education* [Paper presentation]. 15th International Conference on Mathematics Education, Sydney, Australia.

Komatsu, K., & Jones, K. (2019). Task design principles for heuristic refutation in dynamic geometry environments. *International Journal of Science and Mathematics Education*, *17*(4), 801–824. https://doi.org/10.1007/s10763-018-9892-0

Komatsu, K., Shinno, Y., Stylianides, A.J., & Stylianides, G.J. (2025). Development of local theories in design-based research in mathematics education. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-025-10398-w

Komatsu, K., Tsujiyama, Y., Sakamaki, A., & Koike, N. (2014). Proof problems with diagrams: An opportunity for experiencing proofs and refutations. *For the Learning of Mathematics*, *34*(1), 36–42. https://flm-journal.org/Articles/14B6A5F36C5C0B5DD03869E2E94532.pdf

Melville, M.D., & Corey, D.L. (2022). Kyouzaikenkyuu: an exploration of Japanese mathematics teachers' daily planning practices. *Journal of Mathematics Teacher Education*, *25*, 371–383. https://doi.org/10.1007/s10857-021-09493-5

Miyazaki, M., Nagata, J., Chino, K., Sasa, H., Fujita, T., Komatsu, K., & Shimizu, S. (2019). Curriculum development for explorative proving in lower secondary school geometry: Focusing on the levels of planning and constructing a proof. *Frontiers in Education*, *4*, Article 31. https://doi.org/10.3389/feduc.2019.00031

Nührenbörger, M., Rösken-Winter, B., Link, M., Prediger, S., & Steinweg, A. S. (2019). Design science and design research: the significance of a subject-specific research approach. In H. N. Jahnke & L. Hefendehl-Hebeker

(Eds.). *Traditions in German-Speaking Mathematics Education Research* (pp. 61-89). Springer. https://doi.org/10.1007/978-3-030-11069-7_3

Prediger, S. (2019). Theorizing in design research: Methodological reflections on developing and connecting theory elements for language-responsive mathematics classrooms. *Avances de Investigación en Educación Matemática*, *15*, 5–27. https://doi.org/10.35763/aiem.v0i15.265

Prediger, S. (2024). Conjecturing is not all: Theorizing in design research by refining and connecting categorial, descriptive, and explanatory theory elements. *EDeR – Educational Design Research*, *8*(1), Article 60. https://doi.org/10.15460/eder.8.1.2120

Prediger, S., Jackson, K., & Koichu, B. (2024). When theory should guide action, what kind of theorizing do we need?. In M. Clements et al. (Eds.), *Fourth International Handbook of Mathematics Education*. Springer, Cham. https://doi.org/10.1007/978-3-031-51474-6_14

Shinno, Y., & Miyakawa, T. (2025, February 4–8). *Transposition of research knowledge: From the West to Far East Asia* [Paper presentation]. 14th Congress of the European Society for Research in Mathematics Education, Bolzano, Italy.

Shinno, Y. & Mizoguchi, T. (2021). Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics textbooks: two cases from *kyouzai kenkyuu* in Japan. *ZDM Mathematics Education*. *53*(6), 1387-1402. https://doi.org/10.1007/s11858-021-01269-8

Shinno, Y. & Mizoguchi, T. (2023). Networking praxeologies and theoretical grain sizes in mathematics education: Cultural issues illustrated by three examples from the Japanese research context. *Hiroshima Journal of Mathematics Education*. *16*, 77–94. https://doi.org/10.24529/hjme.1606

Kato, K., & Tsujiyama, Y. (2019). Developing teaching materials for mathematical inquiry based on geometrical problem solving: Through teaching practices in public schools. *Proceedings of 52nd Autumn Conference of Japan Society of Mathematical Education* (pp. 407-410).

Tsujiyama, Y. & Kato, K. (2022). Developing processes of task design principles in school mathematics: Focusing on mathematical inquiry based on geometrical problem solving. Nakamura, K., Koyama, M., Ikeda, T., Ninomiya, H., Shimizu, H., Hino, K., ..., Kawakami, T. (Eds.), *Proceedings of the 10th Spring Conference of Japan Society of Mathematical Education* (pp. 151–156). JSME. (In Japanese)

Tsujiyama, Y. & Kato, K. (2024). Task design principles for proofs and refutations focused on improper diagrams and corresponding arguments in paper-pencil environments. In T. Evans, O. Marmur, J. Hunter, G. Leach, & J. Jhagroo (Eds.). *Proceedings of the 47th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 228). PME.

Wang, C., Shinno, Y., Xu, B., & Miyakawa, T. (2023). An anthropological point of view: Exploring the Chinese and Japanese issues of translation about teaching resources. *ZDM Mathematics Education*. *55*(3), 705-717. https://doi.org/10.1007/s11858-023-01477-4

Watson, A., & Ohtani, M. (Eds.). (2015). *Task design in mathematics education: An ICMI study 22*. Springer. https://doi.org/10.1007/978-3-319-09629-2

Yusuke Shinno
Hiroshima University, Hiroshima, Japan
Email: shinno@hiroshima-u.ac.jp
https://orcid.org/0000-0001-6999-5075

Takeshi Miyakawa
Waseda University, Tokyo, Japan
Email: tmiyakawa@waseda.jp
https://orcid.org/0000-0002-1076-3592