A COMPARATIVE STUDY OF DIDACTIC MOMENTS IN A FIRST CHAPTER ON ALGEBRA IN DANISH AND JAPANESE MIDDLE SCHOOL TEXTBOOKS

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Abstract

Teachers often base their teaching on textbook material. Textbooks play a role as mediators between official guidelines and teachers' work. Therefore, it is interesting to study the mathematical organization of material and its connection to curricula. This paper presents a comparative analysis of Japanese and Danish textbook material based on the foundation of the anthropological theory of didactics. Our analysis focuses on how the very first introduction to algebra is organized in Danish and Japanese textbooks for the middle school, and in particular, how the distributive law is treated as a central, specific element of algebraic theory. We more generally look at the roles of algebraic theory that textbooks can facilitate. One result is that the stepwise, modular progression in the Japanese curriculum is indeed reflected in the Japanese textbook material, which treats one mathematical subject area after the other, in a clear progression. The Danish competence-based curriculum with its spiral structure is also reflected in the Danish textbook material, where the content areas are revisited and expanded over the grades.

Key words: Algebraic expressions, school algebra, textbooks, anthropological theory of didactics, first moment of encounter

INTRODUCTION

Typical curriculum resources in mathematics consist of textbooks, official guidelines and digital resources such as interactive worksheets. In their daily work, teachers interact with curriculum resources, which includes selecting and modifying, for example, textbook material (Trouche et al., 2020). The form and content of the textbook material have implications for teaching and the learned knowledge. According to the anthropological theory of didactics (ATD), this didactic transposition of knowledge to be taught (curriculum) to taught knowledge is of special interest (Chevallard & Bosch, 2014). In the transposition process, the textbook has a role as mediator between official guidelines and teachers' work, as a link between intention and implementation (Tesfamicael & Lundeby, 2019). Just as there are differences in the form and content of textbooks, there are also variations in teachers' and students' implementation of the curriculum. The way students 'use' the textbook depends on their mathematical knowledge and their knowledge of the material. For example, to find support for solving an exercise in the textbook, some students will look for help such as worked examples and the theoretical approach in the material (Pepin & Gueudet, 2020). In a systematic

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literature review of the potentials and limitations of the use of textbook materials in mathematics education, one of the findings was that there has been less emphasis on describing the textbook itself and the relationship between the textbook and the curriculum in relation to other themes such as teachers' use of textbooks and students' learning (Steen et al., 2020). The aim of this study is to describe and compare two different types of textbooks, Danish and Japanese, in terms of how they treat the first encounter with algebra and the connection between textbook and official objectives. The choise of algebra as a contenct area will be explained in more detail below.

To study the conditions and constraints of constructs of didactic phenomena, comparative studies are useful (Artigue & Winsløw, 2010), as they may highlight what depends on local contexts and what is more general. There is a certain variety of how and when algebra is introduced and operationalised in the school, according to different curriculum and teaching traditions (Eriksson, 2022). The purpose of this international comparison is to gain more knowledge about similarities and differences of curriculum, particularly the relationship between textbook content and national objectives. In this case, we are interested in how the transition from arithmetic to algebra is described in different textbook material, especially the first moment of encounter with algebraic expressions. An understanding of the diversity between curricula can assess the potential for transferring textbook material from one educational setting to another, to assist teachers in the teaching of school algebra.

School algebra

School algebra and hence the transition from arithmetic to algebra is one of the content areas where students and teachers in lower secondary school are particularly challenged (Kieran, 2007). This is also the case in Denmark, where Danish students continue to have major problems, throughout lower and upper secondary school, with numeracy and basic algebra (Education, 2022). This was also visible in the Trends in International Mathematics and Science Studies (TIMSS) 2019 International Results in Mathematics, which indicated a worrying decline in mathematical performance among Danish grade 4 students (Kjeldsen et al., 2019). In the TIMSS 2019, Japan was one of the top five performing countries (Mullis et al., 2019). The objective is to gain insight into underlying reasons for the observed differences in student performance. Therefore, this paper attempts to analyse and compare curriculum materials from this country with those of Denmark. We chose Japanese textbooks because they are based on systematic empirical research, have a strong theoretical foundation and are translated into English. In particular, we examine lower secondary school algebra textbooks, focusing on the introduction to algebra and the encounter with algebraic expressions, as this is a fundamental aspect of basic algebra.

This study considers algebra as a modelling tool that models intra-mathematical systems and also as a tool to study systems in other disciplines, such as biology and physics (Bolea et al., 2001). Two of the most fundamental concepts in algebra are equivalence and variables. Equivalence and the use of the equal sign as expressing an identity is central for the transition from arithmetic to algebra. Students need to be familiar with algebraic symbols in order to engage with the concepts and to prepare them for further study in mathematics. One of the most powerful tools in arithmetic, and an important foundation for school mathematics, is the distributive property, along with the commutative and associative properties. According to Bruner (1960), these three properties are fundamental for working with equations. These properties are a central part of school algebra because they provide a foundation for exploration and generalizations in arithmetic and for the justification of generalizations (Schifter et al., 2008). In this context, the distributive properties are central for the level of algebraization, especially modelling the relationship between calculation programmes (Ruiz-Munzón et al., 2013). The importance of these fundamental properties has been known for many years but still remains highlighted as a contributor to the challenges of school algebra (Jessen & Winsløw, 2017).

ANTHROPOLOGICAL THEORY OF DIDACTICS AS A THEORETICAL FRAMEWORK

Anthropological theories play an important role in understanding human societies, cultures, and behaviour. They offer frameworks and perspectives to analyse the complexities whining and across different institutions and cultural contexts. In this case, we use the anthropological theory of didactics (ATD), which has emerged as a theory of mathematics education, because we want to compare mathematical textbook material from two different cultures. In ATD, all human activities are considered as institutionally situated where human knowledge and practice are modelled by praxeologies. The notion of a praxeology was introduced as an fundamental implies of analyzing human activity (Chevallard, 2019). A praxeology is a general model that links the practical dimensions (the practice) and the theoretical dimensions (the theory) of any human activity (Barbé et al., 2005). A praxeology consists of types of tasks, techniques, technologies and theories (Bosch, 2015) and can be written as the quadruplet [$T/\tau/\theta/\Theta$] (Chevallard, 2019). The simplest praxeology in mathematics, as in other disciplines, consists of a task of some kind that is solved by a corresponding technique. This means the "practical block" or *praxis* is formed by the *type of task*, denoted by T, and the corresponding *technique*, denoted by τ, used to solve T (Barbé et al., 2005). The 'theoretical block' or *logos* consists of *technology*, denoted as θ, (the discourse on the techniques, such as how they work and what tasks they can solve), and *theory*, denoted by Θ (the general discourse that unifies and justifies technologies, both formally and informally). In other words, techniques for carrying out tasks are explained and justified by a 'discourse on the technique' called technology. The technology is the rationale or justification for the chosen technique – why does it work and where does its effectiveness come from? Taking this discourse to an abstract level yields mathematical theory, which validates the technological discourse and connects the entire praxeology (Bosch, 2015). The anthropological approach assume that any task, or the resolution of any problem, requires the existence of techniques, even though the techniques are hidden or difficult to describe (Barbé et al., 2005).

A mathematical praxeology can be conceptualized as a type of mathematical organisation (MO), where an MO consists of one type of task T and the corresponding technique τ (Bosch & Gascón, 2006). When a set of punctual MOs is explained by using the same technological discourse, they form a local mathematical organisation (LMO), characterised by its technology.

In ATD, we usually use the term 'didactic moments' to describe discernible moments in the study process (Chevallard, 1999, as cited in Barbé et al., 2005, p. 238f; Bosch et al., 2020): The *moment of the first encounter* with the type of task T is the *moment of exploration* of T, with the emergence of a first technique τ used to solve T; the moment of *constructing the technological and theoretical block* [Θ/Θ]; the moment *to* *work on* the praxeology; and the moment of refining the technique(s) and the *institutionalisation* of the entire praxeology produced [T, τ, Θ, Θ]; and lastly, the moment to *evaluate* the praxeology (Barbé et al., 2005). In this case, the notion of didactic moments is used to look into the potential of the moment of first encounter with T as the foundation for analyses of the textbook material.

It is a crucial principle for ATD researchers, when analysing any process of teaching or learning, to relate explicitly and critically to the mathematical content involved, in terms of its rationales in different institutional contexts. In line with Bolea et al. (2001) we define algebra as a tool to model intra- and extra mathematical systems through an algebraization process. Ruiz-Munzón et al. (2013), define school algebra as a process of algebraization, a practical and theoretical tool to carry out modelling activity related to any school mathematical praxeology. To detect what kind of school algebra the first moment of encounter offers, we can use the three-stage model of algebraization (Ruiz-Munzón et al., 2013). In the three-stage model of algebraization, arithmetic can be identified as the domain of calculation programmes (CP). The first stage of algebraization occurs as learners consider the CP as a whole and not only as a process. The second stage is introducing letters as parameters and unknowns, to model the relationship between CPs. The third and last stage of the algebraization process appears when the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated (Ruiz-Munzón et al., 2013). In this way, the three-stage model of algebraization can be used as a tool to detect and analyse general levels in the school algebra to be taught (Bosch, 2015).

Knowledge Taught by Immersion

Didactic processes can be organised in many other ways than by simply developing one praxeology at a time, following the order of the six moments. For instance, one could organise first encounters with several different types of tasks without pursuing any deeper technical work, and only later come back to a systematic approach. The meticulous pursuit of all moments for one praxeology would, by contrast, reflect a more structured progression, which in some cases could also be prescribed by official documents regulating the teaching in more or less detail. Similarly, textbooks could support the implementation of didactic moments corresponding to different praxeologies with more or less structured progression. We find it helpful to think of the different approaches using the analogy of teaching a foreign language: one can proceed systematically to introduce phrase structures, grammatical rules and so on, one by one, or, at the other extreme, one can follow an ‘immersion’ method, where the students are simply exposed to spontaneous language use in situations with native speakers. The same approaches could also be taken in textbooks – with language as well with mathematics. At the one extreme, one praxeology is developed at a time, through all six moments. At the other end of the scale, one would have a more unstructured meeting with types of tasks, techniques, etc., in different and possibly distant ‘natural’ situations – such as the immersion approach to language teaching. We can then talk of textbooks that are more or less strongly structured, and textbooks that are less structured and use an ‘immersion strategy’ for the organisation of the various moments.

RESEARCH QUESTIONS

In order to gain knowledge about the transposition from curriculum to textbooks and the organisation of algebra in Danish and Japanese textbooks for middle school students, it is necessary to begin by analyzing how the first encounter with algebra is presented. Furthermore, the manner in which this first encounter is developed, and the theory used is also central to a comparative analysis. Based on the above, the following research questions have been formulated:

- How is school algebra transposed from national objectives to textbooks in Japan and Denmark?
- How is the very first introduction to algebra organized in Danish and Japanese textbooks for the middle school – for instance, what tasks appear in the moment of first encounter?
- How do the three levels of algebraization appear in this first introduction?
- How does the distributive law, as a central, specific element of algebraic theory, appear?
- And what is the potential for achieving the moment of constructing the technological and theoretical block?

CONTEXTS OF THE CASES TO BE COMPARED

The Japanese Ministry of Education, Culture, Sports Science and Technology (MEXT) prepares the curriculum guidelines for Japanese primary and secondary school, with the outlines of objectives and content of mathematics at each level. Japanese curricula for primary school and junior high school consist of two levels of official programmes, a general course of study in mathematics, *Chugakko Gakushu Shido Yoryo*, and a teaching guide for the course of study in mathematics, *Chugakko Gakushu Shido Yoryo Kaisetsu Sansu-Hen* (MEXT, 2023). The official program and teaching guide includes the basic act and general goals of mathematics education, as well as an outline of the contents for teaching mathematics in a stepwise progression. All schools in Japan are required to use textbooks that have been evaluated and approved by the Ministry of Education. The textbooks used in public schools are selected by the local education council. The Japanese textbook *Junior High School Mathematics: 1* (Isoda & Tall, 2019) is one of these required textbooks for lower secondary school.

The Danish Ministry of Education publishes *Common objectives* (Education, 2019), which consists of the competence-based objectives for primary to lower secondary school. The common objectives is divided into three parts, primary school grades 1–3 (students aged 7–9), middle school grades 4–6 (students aged 10–12) and lower secondary school grades 7-9 (students aged 13–15). The competence-based learning goals are generally described in a spiral, integrated structure, where the mathematical content areas are introduced and re-introduced during primary, middle, and lower secondary school with increasing levels of depth and sophistication (Stein et al., 2007). The majority of Danish mathematical textbooks refer explicitly to the common objectives, but there is no systematic evaluation of textbook materials. Danish textbooks are primarily developed by mathematics teachers, based on their own didactic ideas and personal experience. One of the most commonly used textbook materials is *KonteXt+* (Alinea, 2023)

KonteXt+ is a series of materials for grades 0–9 (Alinea, 2023). For grades 4 to 9, *KonteXt+* is a set of materials consisting of a core book and a workbook, checklists for the core book and workbook, extra sheets for help, useful links, evaluation forms, and a description of how the content of each chapter contributes to the achievement of the national competence objectives for each level. Mathematics teachers can access all this material online. The students primarily use the core book and workbook.

METHODOLOGY

Our investigation is based on the Japanese *Junior High School Mathematics: 1* and the *KonteXt+* core books. First, we identify where the moment of first encounter with algebraic expressions appears in the books. This is done by analysing the structure of the material by reviewing the books' table of contents and locating chapters where the introduction to algebraic expressions is indicated or specifically stated. Then we analyse the selected chapter by exploring what type of task appears in this first encounter with algebraic expression, and the expected following moment of T emergence of a first technique τ used to solve T. We look at the potential to achieve the moment of constructing the technological and theoretical block [Θ/Θ], and finally the moment of refining the technique(s) and institutionalisation of the entire praxeology [$T/\tau/\theta/\Theta$], if possible.
Second, we use the three-stage model of algebraization to analyse the examples identified in the first part to detect the level of algebraization. Then we look at how the distributive property is applied in the selected chapters. In order to answer the question of how school algebra is transposed from the national objectives to the textbook, we will discuss how the first moment of encounter with algebraic expressions relates to the curriculum and to what extent the textbook supports the immersion approach.

ANALYSIS OF THE JAPANESE TEXTBOOK

Structure of the Japanese Textbook

The introduction to the textbook, *Junior High School Mathematics: 1*, provides an overview of how each chapter is structured, with specified descriptions of the content elements and task types used in the book. There is also a section for parents, a description of the overall format of the book consisting of the main text in the chapter, the end of the chapter, and the end of the volume as an overview of the LMO. After 'How to Use This Textbook', there is 'How to Use Your Notebook', with an explanation of how the student's personal notebook should be used for recording the student's learning. The student's notebook is a central part of the teaching practice in Japanese school institutions. The students' private work in the notebook is open for inspection during teaching, and the teacher might select some of the students' work to show and share different ideas and solutions on the blackboard (Shimizu, 1999). The last part of the introduction consists of 'Ways of Thinking Mathematically' and includes examples of analogical, inductive and deductive reasoning and review from elementary school.

In Japan, algebraic expressions with letters are taught in the sixth year of primary school (MEXT, 2023). Chapter 2 in *Junior High School Mathematics: 1 is* entitled 'Algebraic Expression' and is divided into three levels of subsections. The headings of the first level of subsections are 'Algebraic Expression' and

‘Simplifying Algebraic Expression’. The structure of the chapter and the relation between the subsections are presented in Table 1. The titles of the sections are similar to the content used in the textbook.

Table 1. The structure of Chapter 2 in *Junior High School Mathematics:1*

Chapter	Subsection Level 1	Subsection Level 2	Subsection Level 3
Chapter 2 Algebraic Expression	Algebraic Expression	Mathematical Expression Using Letters	
		How to Write Algebraic Expression	How to Express Products
			How to Express Exponentiation
			How to Express Quotients
			How to Express Quantities
			Expressing Quantities Using an Algebraic Expression
		Value of the Expression (substitution of symbols by numbers)	
	Simplifying Algebraic Expressions	Linear Expression	Terms and Coefficients
		Simplifying Linear Expression	Addition and Subtraction of Linear Expression
			Linear Expression and Multiplication of Numbers
			Division of Linear Expression by Numbers
			Various Simplifications
		Using Algebraic Expression with Letters	

The chapter begins with a mathematical problem, followed by fundamental questions for the problem as an introduction to the new content of the chapter. The Japanese term for such a ‘motivating problem’ is *hatsumon*, which means ‘asking a key question that provokes students’ and refers to the teacher’s act of teaching (Shimizu, 1999, p. 109). *Hatsumon* is not directly mentioned in the textbook, but there are key questions which, supports the development of central elements of the contents. The textbook does not provide the *hatsumon* itself, but present problems and questions that can be used as ‘material’ for the *hatsumon*. In this way, the first mathematical problem presented forms the foundation of the problem-solving process that leads to the subsequent moment of first encounter with algebraic expressions, which is described in more detail below. The chapter closes with ‘Summary Problems’, consisting of tasks for reviewing and consolidating the learned knowledge and also to support ‘deep learning’, which is content to extend the students’ understanding of the chapter’s contents.

Text Elements Supporting Didactic Moments

The opening problem of the chapter, which is in fact repeatedly returned to throughout the chapter, is called 'How many straws do we need?' The context is that a rectangular pattern is formed by joining straws of the same length side by side. For example, one can form two squares by using seven straws. Students are asked how many straws are needed to make four and 10 squares. The technique to solve these tasks could be to draw a model of the particular cases and then count the number of straws, or (as intended) to create a simple mathematical expression to model the situation. We are told that Yui used the math expression $1+3\times4$ to find the number of straws needed to make four squares. Then we are asked to explain her idea and apply her method to find the number of straws needed to make five, six, and 10 squares. Then Takumi's mathematical expression of $4+3\times(4-1)$ is presented to find the number of straws needed to make four squares, and we have to explain his idea. Next, we must suggest a method different from those of Yui and Takumi and explain the idea behind it (Isoda & Tall, 2019, p. 61).

This inductive work leads to the question: 'Using the same method as above, can you make a mathematical expression that can be used to find the number of straws needed to make any given number of squares?' (Isoda & Tall, 2019, p. 61). This is the moment of first encounter with algebraic expression in the Japanese textbook *Junior High School Mathematics :1*. This is the moment where algebra is introduced as a modelling tool to model a series of mathematical expressions in a general way by using an algebraic expression, which constitutes the transition from arithmetic to algebra.

This shows how to organize the moment of the first encounter with the task t: Express the relationship between the number of squares and the number of straws to build the squares, which is of course a more general type of task, in which some number pattern is described using algebra. The moment of exploration of this task leads to the emergence of a first technique τ used to solve T. In the case of t, the mathematical expressions from the previous introductory work are used to model the relationship by generalising arithmetic expressions, leading to the mathematical expression 1+3×(number of squares). Letting a represent the number of squares, we get $1+3\times a$.

The moment of constructing the technological and theoretical block $[\Theta/\Theta]$ begins with the written formula and the sentence 'Such mathematical expressions with letters are called algebraic expressions' (Isoda & Tall, 2019, p.62). Then the moment of refining the technique(s) appears when we have to write the other mathematical expression as an algebraic expression and get $4+3\times(a-1)$. The two equivalent expressions, $1+3\times a$ and $4+3\times(a-1)$, represent two different ways of 'seeing' and describing number patters and suggests a need to develop and compare the technique(s). In particular, we need ways to describe and recognize equivalent algebraic expressions and the insight that 'algebraic expressions using letters serve as both the method to find the number of straws, as well as representing the result we want to find' (Isoda & Tall, 2019, p. 63). This will also be central at the moment of institutionalizing the entire praxeology.

Level of Algebraization

The introduction to the algebraization process begins when the student is asked, 'How many straws are needed to make four squares?' The techniques to solve the task are based on repeated addition, with four straws to form the first square and then three straws to form the subsequently squares, written as $4+3+3+3$. This is a CP based on arithmetic, and when one changes the mathematical expression for the four squares to

$4+3\times3$, will see the CP in more condensed form. This efficient form of notation is the hallmark of algebra. It can help us see connections which were previously impossible to see. In this case, it is a step towards generalisation and preparation for the first level of algebraization.

The first level of algebraization appears at the moment of refining the techniques by introducing letters to model the relationship between the different CPs, which in this case are the various mathematical expressions for the number of straws to model the squares.

The last step of the algebraization process emerge with the statement 'Algebraic expressions using letters enable us to find the number of straws needed regardless of how many squares there are' (Isoda & Tall, 2019, p. 63). Equivalence is also introduced with the two expressions $(a+1)+2a$ and $4a-(a-1)$ on pages 82 and 83 in Isoda and Tall (2019) by using the unknown a. The algebraic expressions model the main problem with squares of straws from the beginning of the chapter, and initiates the algebraization process. Ideas such as substitution and solving equations are introduced when connecting these expressions with concrete tasks such as 'find the number of straws needed to make 50 squares'.

The Idea of Equivalence

In subsection 2, 'How to Write Algebraic Expressions' the aim is to learn how to express products and quotients as algebraic expressions by following the rules (Isoda & Tall, 2019, p. 65).

How to express products is highlighted in a box entitled 'important'. The two important rules are that in algebraic expression one must remove the multiplication sign, and when multiplying numbers and letters, one must write the number in front of the letter (Isoda & Tall, 2019, p. 65). In this case, the rules express a convention. This explicit description of the algebra discourse is followed by a series of examples, such as $x\times(-4)=-4x$. In addition, there is a note that when multiplying two letters, one must write them in alphabetical order, for example $b\times a$ must be written as ab. In this context, rewriting the letter is related to the algebraic notation form, which is 'legal' because of the commutative property of addition.

This is a situation where the construction of the technological block appears before the moment of the first encounter with the task T and the exploration of T through the selected examples. In that way, the explicit use of the algebraic condensed notation constitutes the institutionalization of the constructed praxeology, as the final moment.

The explicit introduction of the algebraic rules continues with the statement that 'instead of writing $1a$, remove the 1 and write a' (Isoda & Tall, 2019, p. 65). The explanation is followed by an additional frame with the equivalent expressions $1\times a=a, (-1)\times a=-a$. On the one hand, this can be perceived as the moment of refining the technique(s) as part of the institutionalisation of the praxeology. On the other hand, it is an exploration of T, with the emergence of the first technique τ used to solve T. These equivalent expressions are necessary to deduce that $a+a^2=(1+a)a$ and $ab+a=a(b-1)$, as an example. Then there is the exploration of T by solving the tasks $x\times1$, $a\times(-1)\times b$, and $y\times(-0,1)$ by the corresponding techniques τ. This is a textbook example focusing on the logos part of the praxeology.

Distributive Property

The first encounter with distributive property is part of previous teaching in arithmetic, namely the rules of calculation. In *Junior High School Mathematics: 1*, the calculation rules appear in 'Review – From Elementary School to Junior High School' – see Figure 1.

【Rules of Calculation ①】

Even if the order of the addend and the augend is changed, the sum is still the same.

$$\square + \triangle = \triangle + \square$$

When three numbers are added, even if we change the order of addition, the sum is still the same.

$$(\square + \triangle) + \bigcirc = \square + (\triangle + \bigcirc)$$

Even if the order of the multiplier and the multiplicand is changed, the product is still the same.

$$\square \times \triangle = \triangle \times \square$$

When three numbers are multiplied, even if the order of multiplication is changed, the product is still the same.

$$(\square \times \triangle) \times \bigcirc = \square \times (\triangle \times \bigcirc)$$

【Rules of Calculation ②】

$$(\square + \triangle) \times \bigcirc = \square \times \bigcirc + \triangle \times \bigcirc$$

$$(\square - \triangle) \times \bigcirc = \square \times \bigcirc - \triangle \times \bigcirc$$

Figure 1. Copy of Rules of calculation from Junior High School Mathematics: 1, p. 10 (Isoda & Tall, 2019)

The rules are added that when numbers and quantities are expressed: one can use letters such as *a* or *x* instead of the square, triangle and circle symbols. The first encounter with distributive property was part of earlier work with arithmetic. In the first chapter, 'Positive and Negative Numbers', in *Junior High School Mathematics: 1*, the section 'Addition' contains the subsection 'Commutative and Associative Properties of Addition' (Isoda & Tall, 2019, p. 25). The aim of this subsection is to investigate whether the commutative property of addition and the associative property of addition rules for addition, learned in elementary school, also apply to positive and negative numbers. In section 3, 'Multiplication and Division' the commutative property of multiplication and the associative property of multiplication are explored (Isoda & Tall, 2019, p. 40). In subsection 3, the four operations are combined through calculus, and the distributive property, which holds for both positive and negative numbers, is explored (Isoda & Tall, 2019, p. 48). Figure 2 shows how the distributive property is modelled with equivalent expressions and a geometric figure.

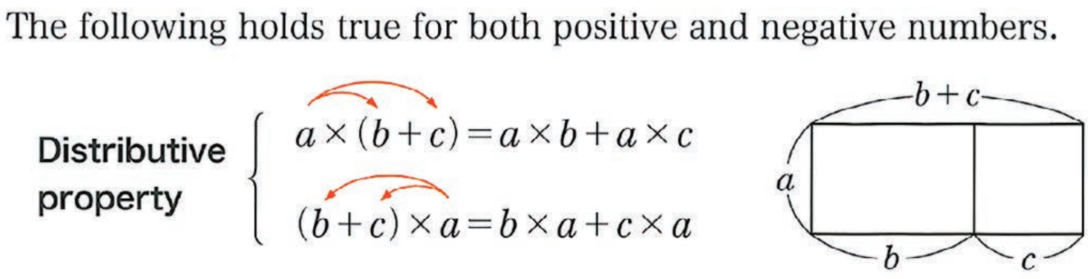


Figure 2. Copy of Distributive property in Junior High School Mathematics: 1 (Isoda & Tall, 2019, p. 79)

This is the first encounter with distributive property in respect of both positive and negative numbers.

In Chapter 2, 'Algebraic expression', the aim of the section 'Simplifying Algebraic Expressions' is to consider how to combine the terms of algebraic expressions (Isoda & Tall, 2019, p. 75). After an introduction to terms and coefficients, the moment of first encounter with the use of distributive property is to combine terms for the purpose of simply stating the algebraic expression. Then an exploration takes place with the example of $4x-6x=(4-6)x=-2x$, and the students must simplify the expressions as $5x+2x$ and $-y-4y$ (Isoda & Tall, 2019, p. 76). Next, the technique to rearrange the terms and combine them with letters and numbers is introduced. The technique is used to simplify the task as $2x-12-6x+15$, among others.

The section continues with the introduction to linear terms and expressions. In the subsection 'Linear Expression and Multiplication of Numbers', the method of removing the parentheses using the distributive property is presented by reviewing Figure 2 (Isoda & Tall, 2019, p. 79). Then there is an exploration through examples and tasks where the student must simplify linear expressions, for example, $-2(4x+5)$ and $(1-6x)\times 3$, explicitly using algebraic notation form. This explicit way of applying the distributive property is the moment of constructing the technological and theoretical block [Θ/Θ]. The continuous expansion of the distributive property contributes to refining the technique and leads to the institutionalization of the entire praxeology, as the final moment.

The moment of first encounter with the distributive property takes place in elementary school. In *Junior High School Mathematics: 1* there is an exploration of the property and the introduction of a corresponding technique by drawing knowledge from the distributive property learned in elementary school. Table 2 provides an overview of the stepwise progression of introducing distributive property in the Japanese textbook, *Junior High School Mathematics: 1* (Isoda & Tall, 2019).

Table 2. Overview of the organization of the distributive property in Chapters 1 and 2 in the Japanese textbook

Type of Task T	Technique τ	Technology θ	Theory Θ
Chapter 1			
Ex. p. 24 $(-1.2)+(-0.5)$ Ex. p. 24 $\left(+\frac{1}{2}\right)+\left(-\frac{2}{3}\right)$ Ex. p. 25 Calculate the following a) and b) and compare the results. a) $(+5)+(-7)$ b) $(-7)+(+5)$	$(-1.2)+(-0.5)$ $=-(1.2+0.5)$ $=-1.7$ $\left(+\frac{1}{2}\right)+\left(-\frac{2}{3}\right)$ $=\left(+\frac{3}{6}\right)+\left(-\frac{4}{6}\right)$ $=-\left(\frac{4}{6}-\frac{3}{6}\right)$ $=-\frac{1}{6}$	Addition of positive and negative numbers	Commutative Property of Addition $a+b=b+a$ $a,b \in Q$
T: Calculate a) and b) and compare the results. a) $a+b$ b) $b+a$	τ: $a+b=b+a$		
Ex. p. 25 Calculate $(+11)+(-5)+(+9)+(-7)$	$(+11)+(-5)+(+9)+(-7)$ $=(+11)+(+9)+(-5)+(-7)$ $=(+20)+(-12)$ $=+8$	Change the order of the numbers using the commutative property. Find the sum of positive and negative numbers using the associative property.	Commutative Property of Addition $a+b=b+a$ $a,b \in Q$ Associative Property of Addition $(a+b)+c=a+(b+c)$ $a,b,c \in Q$
T: Calculate $(+a)+(-b)+(+c)+(-d)$	τ: $(+a)+(-b)+(+c)+(-d)$ $=(a+c)+(-c-d)$		
Ex. p. 32 Calculate $7+(-8)-5-(-4)$	$7+(-8)-5-(-4)$ $=7+(-8)-5+(+4)$ $=7-8-5+4$ $=7+4-8-5$ $=11-13$ $=-2$	The subtraction of positive and negative numbers is changing the sign of the number being subtracted and then adding it.	The commutative and associative property cannot be used for subtraction. However, by changing subtraction into an addition-only math expression, both commutative and associative property can be used.
T: Calculate $(+a)+(-b)-c-(-d)$	τ: $(+a)+(-b)-c-(-d)$ $=a-b-c+d$		
Ex. p. 48 Calculate $(-5)\times\{(-4)+6\}$ Ex. p. 48 $12\times\left(\frac{1}{2}-\frac{1}{3}\right)$	$(-5)\times\{(-4)+6\}$ $=(-5)\times(-4)+(-5)\times 6$ $=-10$ $12\times\left(\frac{1}{2}-\frac{1}{3}\right)$ $=12\times\frac{1}{2}+12\times\left(-\frac{1}{3}\right)$ $=6-4$ $=2$	Calculate with positive and negative numbers, using the distributive property.	Distributive property of multiplication: $a(b+c)=ab+ac$ $a,b,c \in Q$
T: Calculate $(-a)\times\{(-b)+c\}$ a,b,c C-Q	τ: $(-a)\times\{(-b)+c\}$ $=(-a)\times(-b)+(-a)\times c$		
Chapter 2			
Ex. p. 79 Simplify $2(x+4)$	$2(x+4)$ $=2\times x+2\times 4$ $=2x+8$	Remove the parentheses, using the distributive property.	Distributive property of multiplication: $a(b+c)=ab+ac$ $a,b,c \in Q$
T: Simplify linear expression $a(x+b)$	τ:$a(x+b)$ $=a\times x+b\times x$ $=ax+b$		
Ex. p. 76 Simplify $4x+7+5x+8$	$4x+7+5x+8$ $=4x+5x+7+8$ $=(4+5)x+7+8$ $=9x+15$	Rearrange the terms using the commutative property. Combine the terms with same letters and the terms with numbers using the distributive property.	Commutative property of addition: $a+b=b+a$ $a,b \in Q$ Distributive property of multiplication: $a(b+c)=ab+ac$ $a,b,c \in Q$
T: Simplify algebraic expression of the form $ax+b+cx+d$	τ:$ax+b+cx+d$ $=ax+cx+b+d$ $=(a+c)x+b+d$		

Table 2 shows the step-by-step structure where techniques and theory from Chapter 1 are used in Chapter 2. In Chapter 2, letters are introduces to express the relationship between quantities by means of algebraic expressions. The explicit description in the textbook of the distributive property constitutes the moment of constructing the technological and theoretical block. By using the distributive property to simplify the algebraic expression, refining the technique is introduced. In this way, the praxeology of the distributive property forms a bridge between arithmetic and algebra, constituting the final moment of institutionalization.

ANALYSIS OF THE DANISH TEXTBOOK

Locating the First Encounter with Algebraic Expression

To identify where the moment of first encounter with algebraic expressions appears in the *KonteXt*+ book series, we look at the headings of the chapters for the LMOs. In *KonteXt*+5 for grade 5, 11–12-year-old students, one of the headings is 'Numbers and Letters', indicating the first encounter with algebra. We will take a closer look at this chapter to locate the moment of first encounter with algebraic expressions.

The chapter 'Numbers and Letters' is divided into sections related to various types of activities, and the subsections are in general named after their content. Table 3 presents the subsections of the chapter 'Numbers and Letters' in the Danish textbook *KonteXt*+5, according to the headlines and the online teacher's guide to the material. The headings 'Introduction', 'Knowledge of', 'Exercises' and 'Reflection' indicate what is central to each subsection. The sections 'Scenarios' and 'Activities' require more detailed description. A 'scenario' is a story or setting relating to the problem-based exercises. 'Activities' refer to mathematical problems, investigations and games which can be linked to the objective of the chapter. Below follows a more systematic review of the chapter.

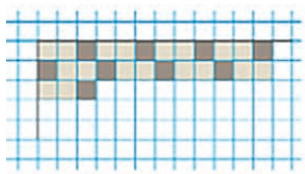
Table 3. The structure of the chapter 'Numbers and Letters' in *KonteXt*+5. The actual titles of the chapter and it sections are listed (except for the 'Scenario' subsections which do not have headings)

Chapter	Subsection Level 1	Subsection Level 2
Number and Letters	Introduction	Picture and Classroom Conversation Activity
		Learning Goals
	[Scenario]	New Square in the Pedestrian Zone Thomsen's Numbers Fencing
	Activities	Your Own Formula for Step Counting Figure Number The Angular Numbers Find Patters in the Numerical Table
	Knowledge of	Can you Calculate with Letters? Formulas and Arithmetic Expressions Formulas and Letters Number and Figure Patterns
	Exercises	
	Reflection	

In the introduction to the chapter, there is a picture with coloured balls and four questions related to the picture for class discussion. The first two questions are: ‘How many different coloured balls are in the photo?’ and ‘How would you name them (the balls) if you should use a letter?’ After the questions, there is a group activity, where four students are each given a card with information about a specific number; by using all four pieces of information, the students must determine the number. In the last part of the introduction to the chapter, a list of what the students will learn in the chapter are presented. The first four goals on the list are to learn that letters can represent different numeric values, to use formulas and arithmetic expressions, to calculate with letters as if they were numbers and to model simple everyday situations as equations (Andersen et al., 2019). The introduction provides insight into the diversity of task types and praxeologies included in the chapter; these will be further elaborated in the following sections.

Text Elements Supporting Didactic Moments

After the introduction, the chapter opens with a story, the scenario, about Anna who has a paving company and a job laying stones in a pattern.



Nyt torv i gågaden

Annas brolæggerfirma ALT I STEN har fået til opgave at lægge nye sten på Nytorv, som de har gjort det tidligere på Gammeltorv. Anna har valgt at blande lyse (L) sten og mørke (M) sten. Hun laver en model med 12 x 12 sten, som skal have sit eget mønster. Hun overvejer at lægge stenene på denne måde.

LLMLLMLLMLLM
MLLMLLMLL
LLM ...

Opgave 1

a. Tegn de fire første rækker af lyse og mørke sten på ternet papir.
b. Hvor mange mørke og lyse sten er der i hver af de 12 rækker?
c. Skriv rækkefølgen af lyse og mørke sten i første række med bogstaverne L og M fx LLMLL osv.
d. Skriv rækkefølgen af lyse og mørke sten i anden række med bogstaverne L og M.

Opgave 2

a. Hvorfor kan man skrive antallet af lyse og mørke sten i en række som 8L + 4M?
b. Hvis man kan skrive grundmønstret af lyse og mørke sten i første række som 2L + M, hvordan vil du så skrive anden række?
c. Hvis man kan skrive rækken af sten i første række som 2L + M + 2L + M + 2L + M + 2L + M, hvordan vil du så skrive anden række?
d. Hvis man kan skrive antallet af sten i første række som 4 · (2L + M), hvordan vil du så skrive anden række?

Figure 3. Copy of Model and collection of tasks linked to the story of the new pedestrian square (Andersen et al., 2019, p. 128)

L represents the light-coloured stones and M represents the dark-coloured stones. Anna's model of the pattern consists of 12 x 12 stones. To extend the initially drawn model item 1.a, ask the students: 'Draw on squared paper the first four rows of light and dark stones.' Using the model of the stone pattern created in 1.a, the students can answer 1.b: 'How many dark- and light-coloured stones are there in each of the 12 rows?' by counting. In 1.c, the students are instructed: 'Write the sequence of light- and dark-coloured stones in the first row, by using the letters L and M, e.g. LLMLL etc.' This is the first encounter with the type of task *T*: Model a sequence by using letters. The model of the sequence is noted in the box on the left. The introduction of a first technique to solve *T* could be copying the list of letters from the box. The moment of exploration of *T* takes place when answering 1.d: 'Write the sequence of light- and dark-coloured stones in the second row, by using the letters L and M.' In this case, the model of the sequence is not complete, and a technique to solve *T* is required.

The question in item 2.a is: 'Why can you write the number of light- and dark-coloured stones in one row as 8L+4M?' This question is consistent with the type of task *T*: determine the number of elements (different stones) in a sequence. The technique to solve *T* could be first τ_1, counting, and then τ_2, use algebraic notation to write down the sum of the elements. Another technique to solve *T* could be τ_3: write the sequence as addition, for example, L+L+M+L+L+M+L+L+M+L+L+M, or τ_4: reduce the terms by applying the distributive property. The combination of τ_1 and τ_2 is expected to be the dominant technique, but central is that the exploration of *T* could lead to the introduction of different techniques to solve *T*. The moments of constructing the technological block, containing the algebraic notation form, and the theoretical block, by applying distributive property, follow more implicitly. To answer the initial question, 'Why can you write the number of light- and dark-coloured stones in one row as 8L+4M', the students have the opportunity to evaluate the entire praxeology.

In item 2.b, the basic pattern of the rows is modelled. In the box on the left, the first-row pattern is written as LLMLLMLLMLLM. Applying algebraic discourse and notation form, the sequence of letters would normally be interpreted as multiplication, with the product L^8M^4. In this case, the sequence is also modelled by addition to 2L+M+2L+M+2L+M, where 2L+M is the basic pattern. This is the first encounter with the type of task *T*: Model the pattern by an algebraic expression. We therefore consider that this is also the introduction to algebraic expressions. In this situation, the two types of tasks, *T*: Model a sequence by using letters, and *T*: Model the pattern by an algebraic expression, relate to different discourses on the technique and therefore different technologies.

The moment of exploration of *T*: Model the pattern by an algebraic expression, is through working with item 2.c: 'If you can write the sequence of stones in the first row as 2L+M+2L+M+2L+M, how would you write the second row?' The answer to the question is: M+2L+M+2L+M+2L+M+2L. If we apply the commutative property for addition, we will get the same solution as for the first row and have the opportunity to construct the theoretical blocks, logos.

The last item is 2.d: 'If you write the number of stones in the first row as 4(2L+M), how would you write the second row?' To answer the question, the students must accept that L and M are not numeric variables, but a 'stone unit' representing the colour of the stones. The basic pattern of the stones is written in the form of 2L+M, and the algebraic expression models the basic pattern repeated four times in a row. These different aspects are central for constructing the technological and theoretical blocks. The next page consists of variations of the previously presented types of tasks.

The Idea of Equivalence and Variables

The next two pages in the chapter relate to a story about a grocer called Thomsen. A person Jacob is sent to the grocer to buy apples, which cost 5 cents each, and the grocer Thomsen writes an equation on a piece of paper: $a \cdot 5=40$. In the first three items of exercise 1, the students must explain what 5, *a* and 40 represent, but there is no explicit description of the form of notation in relation to current conventions. As noted previously, two of the most fundamental concepts in algebra are equivalence and variables. The verb ‘explain’ refers to the task to define the coefficient, the variable and the constant of the linear equation.

Item 1.d asks the question: ‘How many apples does Jacob buy?’ This is type of task *T*: solve linear equation of the form $ax=b$. This is the moment of first encounter with linear equations and the introduction of the exploration of *T*. The first technique to solve the task is expected to be τ: solve by substitution, according to the dominant epistemological model (Tonnesen, 2022). The exploration of *T* continues with the item 1.e: ‘What would the equation have looked like, if they had been bought for 1 dollar?’ (Andersen et al., 2019, p. 130). Here the student must change the constant from 40 to 100.

Exercise 2 is about a person named Inge who wants to buy pears. One pear cost 8 cents, and Inge has 48 cents in her purse. Item 2.a asks: ‘How many pears can Inge buy?’ This question can be solved by division. In item 2.b, the students is instructed: ‘Write the question as an equation’ (Andersen et al., 2019, p. 130). This is the moment of introducing the praxeology, despite the fact that constructing the technological and theoretical blocks remains. If we search for the moment of constructing the technological and theoretical blocks, we must go to the ‘Knowledge about’ section, where there is a description in Figure 4 of the notation form ‘when using letters’ (Andersen et al., 2019, p. 138).

1) Når man skriver 3a, mener man 3 · a – altså ”tre gange a”.
2) Når man skriver ab, mener a · b.
3) Når man skriver 2b + a, mener man 2b + 1a.
4) Når man skriver 2(a + b), mener man 2 · (a + b).

Figure 4. Copy of Overview of the notational form I (Andersen et al., 2019, p. 138)

Figure 4 presents several examples of correct algebraic notation. Next to the figure is written: ‘One of the differences between calculation with numbers and calculation with letters is that you cannot find the result until you know which numbers should replace the letters.’ This is followed by the statement: ‘Many of the rules that apply to calculation with numbers also apply to calculation with letters’ (Andersen et al., 2019, p. 138).

Bogstaveksempler	Taleksempler
a + a + a = 3a	5 + 5 + 5 = 3 · 5 = 15
a + a + b + b = 2a + 2b	4 + 4 + 7 + 7 = 2 · 4 + 2 · 7 = 22
2a + 5a = (a + a) + (a + a + a + a + a) = 7a	2 · 9 + 5 · 9 = (9 + 9) + (9 + 9 + 9 + 9 + 9) = 7 · 9 = 63
5a − a = (a + a + a + a + a) − a = 4a	5 · 6 − 6 = (6 + 6 + 6 + 6 + 6) − 6 = 4 · 6 = 24
3(a + b) = 3a + 3b	3(5 + 12) = 3 · 5 + 3 · 12 = 51

Figure 5. Copy of Examples of calculation rules (Andersen et al., 2019, p. 138)

On the left side, examples with letters are presented, and on the right side, there are examples with numbers. The five examples cover variations of multiplication as repeated addition, bracket rules, the convention about notation and distributive property. The first example in Figure 5 is the same as the first example in Figure 4 and is an example of the consistent repetition form used throughout the material. Figure 5 also contains the first description of distributive property in the textbook for grade 5, based on an example. This diverse range of tasks, list of conventions in Figure 4 and examples of calculation rules in Figure 5 form the praxeological foundation for students' further development of algebra. To see how this foundation is developed and get insight into the progression, we will look at the textbook *Kontext+7* for grade 7, students aged 13–14.

Level of Algebraization

We will analyse the chapter 'Formula and Equations' in the textbook *Kontex+7*, which is organized in almost the same way as 'Numbers and Letters', shown in Table 3, with the addition of supplementary exercises, 'Calculate with Letters' and 'Solve an Equation'. The first scenario is named 'An Evening in Paris". It starts with a story about the mathematician François Viète and his search for 'a simple way to solve difficult calculation tasks' (Hansen et al., 2015, p. 92). This scenario introduces algebra as a tool to model arithmetic. The problem in this scenario is as follows: 'Three brothers, Oliver, René and Cyrano, must share 1025 silver coins. Oliver must have 275 coins more than Rene. Cyrano must have 150 coins less than Rene' (Hansen et al., 2015, p. 92). Representing the share of René by x, the calculation becomes $1025=x+(275+x)+(x-150)$, which simplifies to $1025=3x+125$. According to the context, Francois has the idea to calculate backwards to get: 'Rene has 300 silver coins. Oliver has 575 silver coins and Cyrano has 150 silver coins. And "Viola [*sic*]! Francois invented the modern equation' (Hansen et al., 2015, p. 92).

The problem requires a process of calculation. In this case, the first step of algebraization appears when the story about the brothers is modelled by an equation representing a relationship between the CPs. The model of the story provides an opportunity to consider the CP as a whole and not only as a process.. In the modelling process, x is introduced as an unknown to model the relationship between the CP and then the equation is simplified to $1025=3x+125$ which is the first level of the algebraization process (Ruiz-Munzón et al., 2013). After this introduction, an example of how to solve an equation in three steps is presented, Figure 6, and the students must explain, in their own words, what happens in steps 1) to 3).

1) $1025=3x+125$
2) $900=3x$
3) $300=x$

Figure 6. Example of how to solve an algebraic expression in three steps (Hansen et al., 2015, p. 92)

To explain and defend each step, the technology of the coefficient, the variable and the constant of a linear equation is a part of the praxeology. Knowledge about equivalence and the use of the equal sign as expressing an identity represents the level of theory. Then, three linear equations, $87=12x+45$, $x+73{,}44=89{,}22$, $3x+175=238$, must be solved using the described technique and can be described as technical work.
Later in ‘An Evening in Paris’ it is stated that sometimes it is faster just to guess (by replacing the unknown by one or more numbers) to solve the equation. This is the moment of introducing substitution as a technique to solve linear equations. This is followed by eight equations to solve to consolidate the technique.

The Distributive Property

As shown in Figure 5, there is a description of distributive property in *KonteXt+5*. An almost identical Figure 7 is found in the section ‘Knowledge about’ in *KonteXt+7*. The first thing on the ‘Knowledge about’ page is the word ‘algebra’, which is defined as calculation with letters. In addition, we learn that ‘The calculation rules known from numbers can also be used when calculating with letters’ (Hansen et al., 2015, p. 102). This abstract theoretical statement is supported by the examples in Figure 7.

$a + a + a = 3 \cdot a$	$2 + 2 + 2 = 3 \cdot 2$
$a + b = b + a$ og $a \cdot b = b \cdot a$	$2 + 5 = 5 + 2$ og $2 \cdot 5 = 5 \cdot 2$
$a \cdot (b + c) = ab + ac$	$2 \cdot (5 + 7) = 2 \cdot 5 + 2 \cdot 7$
$a + (b + c) = a + b + c$	$2 + (5 + 7) = 2 + 5 + 7$
$a - (b + c) = a - b - c$	$2 - (5 + 7) = 2 - 5 - 7$
$1a = a$	$1 \cdot 7 = 7$

Figure 7. Copy of Examples of calculation rules from KonteXt+7

These six examples in Figure 7 cover a somewhat unstructured variety of algebraic identities: multiplication by 3 as addition by a number with itself three times, commutative properties for addition and multiplication, the distributive law, bracket rules (including a form of the associative law for addition) and a notational convention [1a means $1 \cdot a$ which is a]. The right column illustrates that these identities hold with numbers replacing the letters, and thus provides examples of the theoretical statement cited above (when reading from right to left). This is the first time the textbook presents a general form of distributive property.
In *KonteXt+8* (grade 8), another example of distributive property is presented. In the chapter called ‘Formula and Equations’, under the section ‘Knowledge about’, algebra is described as the language of mathematics: ‘Working with letters as symbols for unknowns and variables is central in algebra’ (Hansen et al., 2016, p. 94). No further explanation of what defines unknowns and variables is given. A more general description

follows: 'The rules for calculation in algebra are often similar to the rules for calculation with numbers' (Hansen et al., 2016, p. 94). Then a table with examples is presented, similar to the examples in Figure 5 and Figure 7. This demonstrates the repeated use of similarity between 'rules' in arithmetic and algebra and the presentation of distributive property exemplified in the same, but not exhaustive, way, throughout the grades. To describe the rules for calculation, which include distributive property, in a more general way, geometric models are used in *KonteXt+8*.

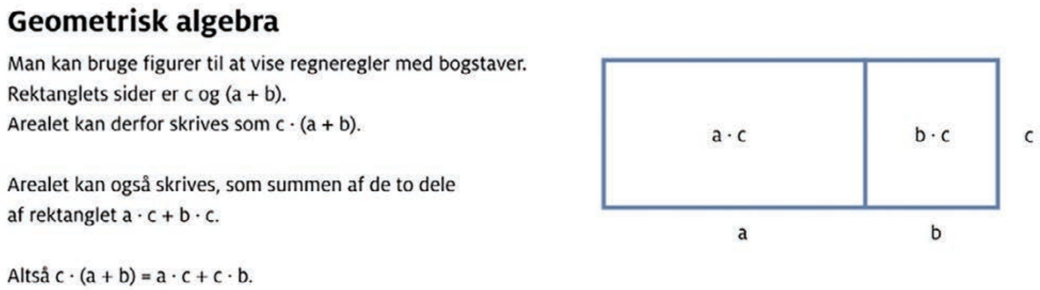


Figure 8. Copy of Geometry model of distributive property (Hansen et al., 2016, p. 94)

'Geometrical Algebra' is part of the section 'Knowledge about' and includes the following statement: 'You can use geometric figures to show rules for calculating with letters' (Hansen et al., 2016, p. 95). It is not pointed out that this is in a way a special example, since we have to assume that $a,b,c>0$. It is implicit that the area of the boxes is the product of the side lengths, and we make use of the fundamental property that areas are additive. Usually this is shown by using distributive property, not the other way around. From an algebraic point of view, it could be more correct to say that the distributive law agrees with the algebraic model of a geometric identity (of areas, also involving lengths). This method of modelling distributive property but also commutative and associative property by geometric models is repeated in the *KonteXt+* series.

COMPARISON AND DISCUSSION

The analysis of the Japanese and Danish textbooks can be organized according to three themes:

1. Identifying the moment of first encounter with algebraic expressions in the Japanese and Danish textbooks, and the relation between the LMO and curricula;
2. The mathematical praxeologies found in the material, which include the respective levels of algebraization and explicitness; and
3. The didactic approaches that seem to be suggested and supported by the textbooks.

Curriculum Organization and the Introduction of Algebra

In the Japanese textbook, chapters and sections are consistently named with reference to their content and objectives. The analysed chapter, 'Algebraic Expressions', shows a systematic structure with clear mathematical sub-objectives, as presented in Table 1. In the Danish textbook, the name of the chapter also relates to the mathematical content, but the sections are divided according to a large variety of task types and activities, as presented in Table 3.

The differences in the structures of Tables 1 and 3, and the differences between the Japanese and Danish textbooks in general, reflect the structures of the respective national common objectives. The large variety of task types and activities presented in different contexts found in the Danish textbook represents an 'immersion strategy', where praxeologies are developed over time. This is in line with the spiral and integrated Danish curriculum, where mathematical competences and praxeologies are developed over several years while revisiting the same content repeatedly. The stepwise and structured approach in the Japanese textbook reflects the stepwise structure of the Japanese curriculum, with a clear progression with the outlines of the objectives and content of each mathematical level.

This connection between the organization of textbooks and the respective curricula can also be illustrated by our analysis of the introduction to distributive property. In the Japanese textbook, distributive property in arithmetic forms the basis for distributive property in algebra and constitutes an extension of theory. This means there is a theoretical progression and connection from arithmetic praxeologies to algebraic praxeologies. This stepwise modular approach has, according to Stein and Kim (2006), the implication that subsections cannot be separated and reconstructed into other configurations without losing efficiency in goal achievement. In this case, distributive property in arithmetic must be well established praxeology with $[T/\tau/\theta/\Theta]$ before distributive property in algebra is presented. A reordering of the modular form could therefore lead to loss of theoretical coherence, by not developing praxeologies in their logical order.

The spiral structure of the Danish curriculum is reflected in the *KonteXt+* series, where distributive property appears several times, in somewhat different forms, in the various grades. The distributive law is the only field axiom that links addition and multiplication, and consequently it is crucial in many ways in school arithmetic and algebra. In the Danish textbook, the introduction to distributive property is difficult to locate precisely, because it emerges in a variety of special cases or 'rules', which are listed and exemplified in several different sections and in different grades. In this way, the distributive law for arithmetic and that for algebra are intertwined, if not almost a merged praxeology. This gradual and, to some extent, repetitive approach is consistent with the spiral philosophy of the curriculum, according to which students are expected to acquire general principles such as the distributive law over time, as they appear in special cases and in task types of increasing difficulty. Stein and Kim (2006) argue that in spiral and integrated curricula, knowledge and skills (more or less, theory and techniques) are linked together, and because they are difficult to separate, they must be taught in similar ways over the years.

Level of Algebraization and Explicitness

In terms of the three-stage model of algebraization (Bosch, 2015), our analysis above demonstrates that only the first level of algebraization appear in the Danish and Japanese textbooks. A main difference is that *hatsumon* connects the algebraization process in high school mathematics, while the first level of algebraization in *KonteXt* are developed through different examples and exercises. This can also be linked to the difference between the curricula, as explained above.

In the Danish textbook, the term 'rules' is used to refer to both substantial properties and notational conventions. For instance, the fundamental commutative law $ab=ba$ for multiplication is a level of theory Θ, where the convention to write $a \cdot x$ rather as ax is technology θ. In the Danish textbook, both are presented as 'rules', with no distinction made between technology and theory. By contrast, in the Japanese textbook, there are explicit distinctions between tasks, techniques, technologies and theories with a clear description of algebraic assumptions, conventions and results. As an example, there is a clear connection between distributive property in arithmetic and in algebra and the explicit description of the notation form as convention in 'algebra discourse'. The Danish textbook material has a more implicit approach to central algebraic principles, such as distributive property. As illustrated above in Figures 5, 7 and 8, distributive property is applied in various examples, but its theoretical description and status remain implicit. This implicit approach can also be explained by the spiral and integrated structure of the curriculum, where the mathematical content is revisited and integrated over the years. This process is in line with Gravemeijer and Terwel (2000), who state that central algebraic assumptions on commutative, associative and distributive property might emerge as a part of mathematizing and the process of organizing the subject matter (Gravemeijer & Terwel, 2000). The same assumption about the mathematizing process may also apply to the idiosyncratic use of symbols in *KonteXt+*, where a repeated disposition for the conventional compact notation form might entail adaption by students, over time, to acquire important conventions.

The Japanese explicitness can also be seen in the headings of sections. This explicitness of content and learning goals is evident for the student during the learning process. The Danish textbook also contains learning objectives in the introduction to a chapter, but the connection between the type of tasks and corresponding techniques, and the level of theory, are present in a more implicit form.

It is also worth noting the structured focus on language in the Japanese textbook. There is an explicit description of 'coefficient' and 'linear term' before introducing 'linear expression'; this is an example of the explicit stepwise development of the praxeology. The explicit use of the mathematical terms can be considered analogous to language learning by grammatical accuracy. The Danish textbook makes more use of common-sense terms, for example, 'calculation rules' in Figures 5 and 7, which are used to refer to both mathematical properties and conventions for algebraic notation.

The Didactic Approach

The didactic processes are organized in different ways in the Japanese and Danish textbooks. In the Japanese textbook, the introduction of algebraic expression is clearly located due to the LMO. In *KonteXt+*, the didactic moments include work on several types of tasks, where the (*hatsumon*) work with the initial problem has the potential to generate all six moments in the study process.

The use of the metaphor 'Algebra – the language of mathematics' in *KonteXt+* describes the textbook's different approach to the acquisition of algebra. Models for language learning can roughly be placed on a continuum, with content-driven models at one end and language-driven models at the other end (Snow, 2001). The prototypical content-based approach is the immersion model of foreign language education. If we use the same continuum for the Danish and Japanese textbooks, we could place *KonteXt+* close to a pure immersion model, with 'immersion through examples', where the *Japanese High School Mathematics: 1* is more like a pure 'language-driven' approach.

CONCLUSION

The analysis of Japanese and Danish textbook material shows clear links to the respective curricula. The Japanese curriculum uses a stepwise modular progression, where the content areas are built on previous mathematical foundation. The Danish curriculum is based on competences and has a spiral structure, where the content areas are revisited and expanded over the grades. A detailed praxeological analysis of the Danish curriculum is not possible, given that it provides only broad and vague guidance on the content that should (or rather can) be taught. The Japanese stepwise modular form of curriculum is also evident in the introduction to algebra in the Japanese textbook. The first introduction of algebraic expressions can be located quite precisely and consists of a task type where the students must model arithmetic relations using an algebraic expression. In the Danish textbook, the first encounter with algebraic expression is more difficult to locate. The transition from arithmetic to algebra is more fluid, and the introduction of letters in algebra is first linked to units and then to terms and variables in the *KonteXt+* material.

Both textbooks include work with distributive property. The Japanese material has an explicit and theory-based approach, whereas the Danish textbook material has a more implicit and example-based approach. This difference between the explicit and implicit approaches is also visible in the work with the algebraic notation form. The Japanese material has an explicit description of how to use the algebraic notation form properly, whereas the implementation of the algebraic notation form in the Danish textbook material is more implicit and presented at the same time as the work on tasks.

In their introduction to algebra, both textbook materials include exposure to the first level of algebraization. When applying the theory of didactic moments to look at the potential of the textbooks to support technico-technological moments, we see that the Japanese textbook material primarily builds one praxeology at a time, while the Danish textbook material – through its numerous activities – leads to work on several praxis-blocks simultaneously, possible long before construction of the logos blocks. In this way, the Danish textbook material's approach to the introduction of algebra can be compared to the approach to language learning described as 'teaching through immersion', with exposure to a large variety of task types which can later be consolidated at the level of theory. The connection between curriculum structure and textbook material is clear when we look at the LMO, and it is therefore interesting to compare textbook material from countries with different curriculum structures.

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