THE ROLE OF FEEDBACK WHEN LEARNING WITH A DIGITAL ARTIFACT: A THEORY NETWORKING CASE ON MULTIMODAL ALGEBRA LEARNING

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Abstract

How digital feedback supports teaching/learning with a digital tool is not yet well understood. In a networking of theories approach, Activity Theory and the Instrumental Approach are combined to investigate the role of digital feedback for the teaching/learning of integers with the MAL-system, a multimodal algebra learning system. The MAL-system is designed as a multi-touch tangible user interface with feedback functions allowing students to mathematically operate with (negative) numbers represented as virtual tiles. We explore the role of digital feedback by a multi-case study at the grade five level, in which we conducted experimental task-based interviews of four student pairs, each supported by a tutor. Findings show that (digital) feedback mediates the teaching/learning activity in a supportive way. Reflection on the way the two theories are combined reveals that they can be regarded as locally integrated into a layered model enriching the describing of the transformation of teaching/learning mediated by digital feedback.

Keywords: Networking of theories, activity theory, instrumental genesis, digital feedback, negative numbers, transformation

INTRODUCTION

Hattie and Timperley (2007) emphasized feedback to be an essential part of teaching and learning. When learning is supported by digital tools, digital feedback is highly relevant (Fyfe, 2016; van der Kleij et al., 2015). Both immediate feedback as well as feedback variation have been shown to be effective for learning (Bockhove & Drijvers, 2012). A recent study on algebra learning shows that any digital feedback is better than no feedback and the most important digital feedback takes an explanatory form (Fyfe, 2016). However, "… the computer can serve as an effective problem-solving tool only if accompanied by more traditional forms of discourse between pupils and teacher" (Hillel et al., 1993, p. 38). But it is not so clear how feedback may foster processes of teaching/learning, and how teacher feedback and digital feedback differ and interact

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in the use of an artifact to support in-depth learning. Answering these questions depends on the background theory on learning (Evans, 2013). We use a networking of theories approach to investigate this problem through a multi-case study in which students learn about negative numbers with a digital tool specifically designed for this purpose. This paper aims at answering the following research question: *What is the function of digital feedback in the support of the teaching/learning of negative numbers with this specific digital artifact?* To answer this question we will conceptualize the main terms in our theoretical framework, using *Activity Theory* and the *Instrumental Approach* within a theory networking approach.

THEORETICAL FRAMEWORK

Our approach aims at networking two theoretical approaches in mathematics education (Bikner-Ahsbahs & Prediger, 2014), *Activity Theory* (Leontjew, 1987)[1] is used as a background theory (Mason & Waywood, 1996) for teaching/learning coordinated with the *Instrumental Approach* (Artigue, 2002; Trouche, 2020). We will conceptualize the *teaching/learning of negative numbers* using activity theory. The instrumental approach will allow us to embed the *use of the artifact* by the learners into the activity.

The networking of these two theoretical approaches builds on Radford's elaboration of the notion of theory (2008) and how theories can be related. Radford defines a theory as:

> a way of producing understandings and ways of action based on:
>
> - A system, P, of basic principles, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
> - A methodology, M, which includes techniques of data collection and data-interpretation as supported by P.
> - A set, Q, of paradigmatic research questions (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified) (2008, p. 320)

He abbreviates this definition by the triplet (P, M, Q). Radford (2012) proposes later to expand this notion of

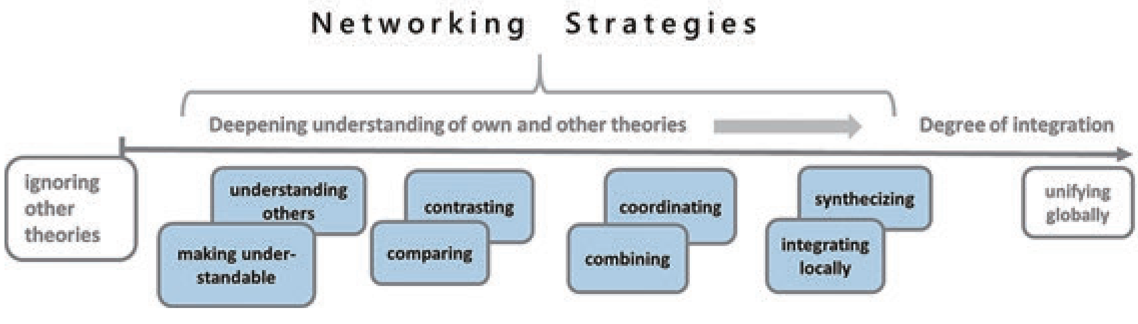

Figure 1: Networking Strategies
(CC BY 4.0, Bikner-Ahsbahs et al., 2016, p. 34, adapted from Prediger et al., 2008)

1 The author's name Леóнтьев is transliterated differently into Roman characters in different articles. Leontjew is the spelling in German literature and Leontjev, Leont'ev, Leontyev and Leontiev are spellings found in English literature.

theory by including results R as a relevant constituent because research results rebound on the theories used in research, thus pointing to the purpose of theories as tools for research. He represents this expanded understanding of theory as a dynamic structure by [(P, M, Q), R].

The networking of theories means to relate theories with respect to their constituents. Four pairs of networking strategies positioned on a scale of increasing integration (Figure 1) have shown to be useful to identify relationships. The two pairs of strategies that are relevant in this study are coordinating and local integration. While combining is a kind of triangulation, i.e., viewing an empirical phenomenon from different theoretical perspectives, coordinating can be considered, when "a conceptual framework is built by fitting together elements from different theory elements for making sense of an empirical phenomenon" (Prediger & Bikner-Ahsbahs, 2014, p. 120). The latter becomes relevant when we complement an already existing way of theoretical understanding with an additional theory element to achieve a more comprehensive understanding of an empirical situation. Local integration means theorizing a phenomenon at the boundary of two or more theories so that the theories themselves are expanded in their local understanding of a specific phenomenon.

Activity Theory

The concept of activity in activity theory is transformative as it is "the specifically human form of activity, of interacting with the world in which man changes it and himself at the same time" (Giest & Lompscher, 2006, p. 27, our translation). According to Leontjew (1987), activity is driven by its *motive*, i.e., the object that initiates the activity on which it is built, in our research this is the activity of teaching/learning of negative numbers. An activity is embedded in the cultural-historical situation of society, its development changes the society as well as society changes the activity, and by "interacting with the world" (Giest & Lompscher, 2006, p. 27, our translation) learners change it and themselves at the same time. Activity (Figure 2) is made and renewed by *actions* directed towards *goals,* which are related to the motive, here for example to add positive and negative numbers. Each goal is pursued by a task, e.g., to manipulate one side of the equality 2 – 3 = 2 – 3 preserving the equality relation. This task could result in sub-goals with respective

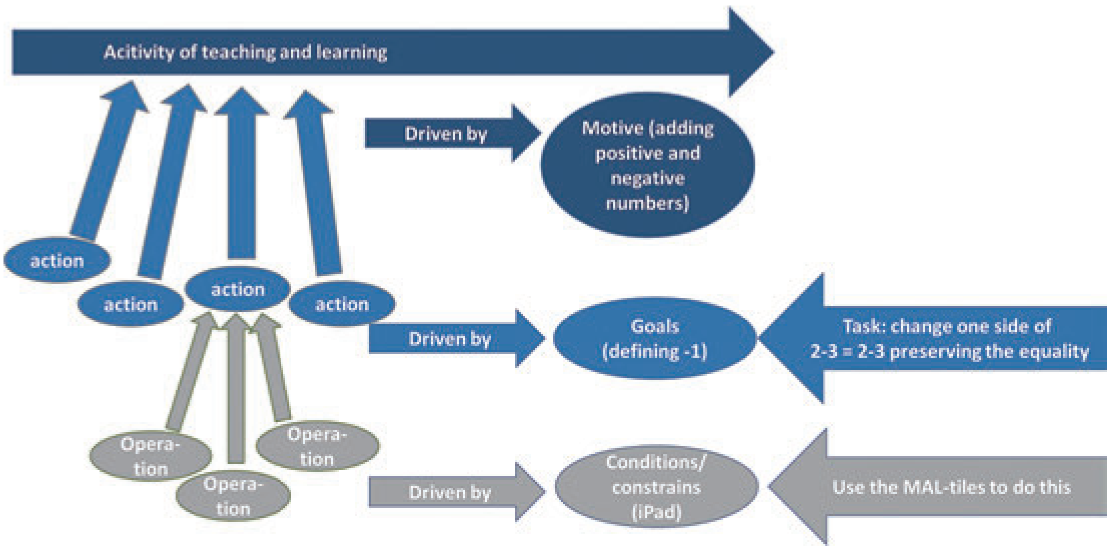

Figure 2: Activity system

sub-actions, e.g., to manipulate the numbers on the right side without changing its result (e.g., 2 – 3 = (2+1) – (3+1)).

The manner in which these actions are performed is determined by the *conditions* of the setting and the *constraints* the goals meet in the specific situation, e.g., by available artifacts (e.g., digital or non-digital tangibles), which allow specific *operations* to be performed but constrain others. For example, symbols used with paper and pencil differ from using a symbolic calculator or our MAL-system in which numbers are represented by virtual tiles, which can be placed, removed or re-arranged (Figure 3), as we will see. Activity can only be observed through its actions on the object that is the motive of the activity. Leontjew (1987) accentuates that activity is in constant transformation, it is (re)constituted by variations of actions developing over time. It even may change into an action, and when routinized it may be transformed into an operation (p. 109). While "[a]ctions becoming ever richer, outgrow the circle of activity that they realize, and enter into a contradiction with motives that engender them" (Leontyev, 2009, p. 175), then new motives may enter the scene, restructure actions and establish a new activity, for example solving algebraic equations including negative numbers after the latter is routinized.

Tools play an essential role in the transformative nature of an activity as they are cultural objects through which people act and interact with the world around them and so change it and themselves (see Leontjev, 2009). Tools mediate an activity and thus, its transformation. However, when new tools like the MAL-system are developed, it is not so clear how they contribute to these transformations as their use in culture is not yet established. This has yet to be explored.

In this paper, we focus on teaching/learning as an irreducible activity (see Shvarts & Abrahamson, 2019) with negative numbers shaping its motive. How students transform themselves and the teaching/learning, in which they interact with others and the tool, is mediated by the MAL-system, a digital artifact designed to support Multimodal Algebra Learning (MAL). For this setting, the MAL-system is elaborated as a digital tool for teaching/learning algebra. Here we address negative numbers (Figure 3) with a specific focus on the role of digital feedback. When the students use the MAL-system, they may transform the artifact into an instrument for a specific situation e.g., for calculating with negative numbers. To grasp this process, we adopt the Instrumental Approach (Artigue, 2002) as a theoretical frame that complements Activity Theory through conceptualizing the use of the tool. As it is rooted in activity theory (Vérillon & Rabardel, 1995) we expect

Figure 3: Teaching/learning setting (left figure), the MAL-system (right figure) with a task (1, text on top), the subtraction zone (2, frame in red), zero-pair (3, red & blue tile), unit-tile (4, blue tile), minus-one-tile (5, red tile), grouping (6, tiles surrounded by a yellow frame), symbolic expression (7) of the tiles equation 2 – 3 = 1 + (-1) – 1 (see Figure 5)

to be able to coordinate the two theories empirically towards a theoretical local integration.

The Instrumental Approach

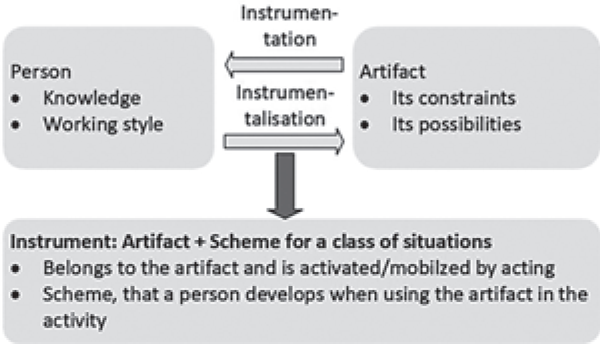

Figure 4: Instrumental genesis (© Springer Nature, permission received, see Abb. 2.12 in Bikner-Ahsbahs, 2022, p. 25; adapted from Fig. 3.1 in Trouche, 2020, p. 395)

In line with activity theory, we consider tools as mediating the dynamic of an activity in the way people use the artifact to achieve a goal. According to this view, instrumental genesis is the process of how a (digital) artifact becomes an individual instrument (Artigue, 2002), where an instrument is the hybrid collection of an artifact with individually and/or socially developed schemes of using the artifact in a specific situation (Trouche, 2020). Instrumental genesis is shaped by two interrelated dialectic processes (Figure 4); by *instrumentation* ("directed towards the subject") and by *instrumentalization* ("directed towards the artifact") (Trouche, 2020, p. 408). These are

> two intrinsically intertwined processes constituting each instrumental genesis, leading a subject to develop, from the artifact, an instrument for performing a particular task; the instrumentation process is the tracer of the artifact on the subject's activity, while the instrumentalization process is the tracer of the subject's activity on the task (Trouche, 2020, p. 409).

Rabardel (2002) emphasizes that "[t]he two processes jointly contribute to the emergence and evolution of instruments, even though, depending on the situations, one of them may be more developed, dominant or even the only one implemented." (p. 103) He further explains that artifact functions "are a characteristic property of the instrumental entity, and because in our perspective this entity is born of both subject and artifact, functions are also mixed in nature. They are rooted in both the artifact and scheme components of the instrument." (p. 104) There are two levels of the evolution of instrumentalization by the attribution of functions to an artifact enriching it with new extrinsic properties, that are acquired momentarily (local level) or durably (global level) (p. 106). This research only addresses the local level of instrumentalization. Figure 3 shows our teaching/learning setting in which the MAL-system is used as the artifact, with which students learn about negative number.

The MAL-system designed for learning about negative numbers

Activity Theory supported the design of tasks for the MAL-system (Reinschluessel et al., 2018), which provides virtual multi-touch tiles that can be arranged on a virtual "mat" to represent key structures of the arithmetic symbol system. In addition to the tiles, there are various features included in the system. An unlimited stock of tiles (Figure 5, 1) and a waste basket (2) on the left and the right side of the mat to allow two learners to work at the same time. The white area on the top (3) is for task descriptions. Pressing an arrow button to the top right (4) either moves on to the next task or restarts the current task. The pen-button on the top left (5) switches the interaction mode between either direct interaction with the tiles or 'painting' mode that allows, drawing or deleting a 'Subtraction Zone' (SZ, the red frame in Figure 5). The general area to place the tiles is an Addition Zone' (AZ).

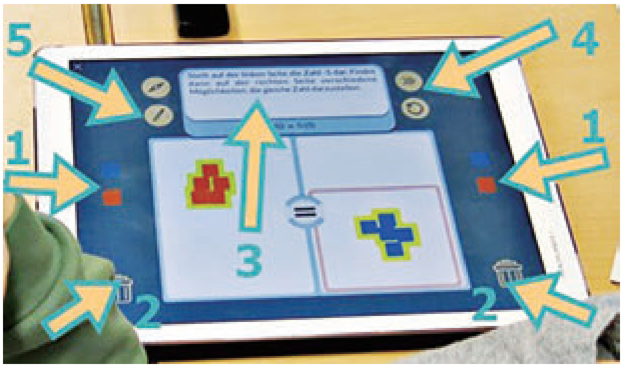

Figure 5: Functionalities for interacting with the MAL-system artifact.

The main representational part of the MAL-system is the mat with two sides (left and right), representing an equation, and an equal or unequal sign in the middle giving feedback on the correctness of the equation represented by the tile arrangements on both sides. Numbers are represented by square unit-tiles. Two different colors indicate the sign of the numbers (red for minus, blue for plus) and mathematical operations

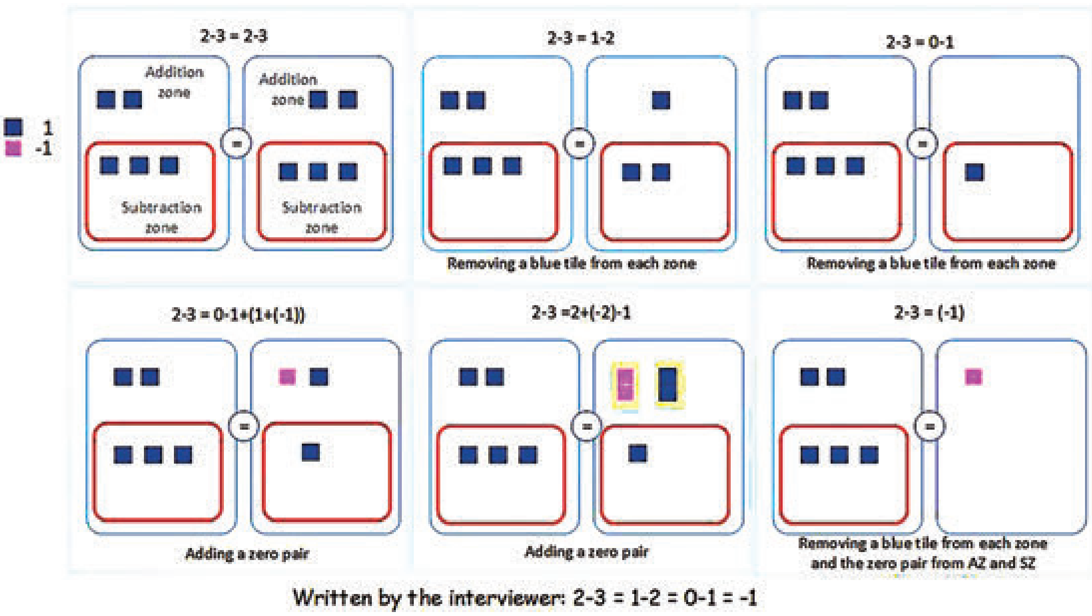

Figure 6: Key features of the MAL-system – defining (-1) as 0 – 1 by 2 – 3 = 0 – 1 = -1 + 1 – 1 = -1

with numbers are represented by practical actions with the tiles to achieve specific goals. The design starts from three fundamental actions: *placing*, *removing*, and *re-arranging* (Reid & Vallejo-Vargas, 2019), which students spontaneously conduct. Placing tiles in the 'Addition Zone' (AZ) (Figure 6) means adding them, removing means subtracting them, and re-arranging produces visual patterns without changing the results but expressed in the symbolic feedback e.g., if the tiles are pushed together in certain ways. A red "blob" drawn by a finger on the mat signals to subtract; it is called a 'Subtraction Zone' (SZ). A SZ shows what is to be subtracted from the AZ that contains the SZ. It is also possible to insert a SZ within a SZ. When tiles are pushed together they are grouped and this is shown by a yellow surround (Figures 5, 6), meaning that the tiles are considered to represent a single number by the system.

The *basic action rule* 'perform a calculation by acting on tiles on one side of the equation while keeping the equal sign' allows for legitimate action schemes e.g., removing two unit-tiles, one from the AZ and one from the SZ (Figure 6), or placing or removing zero-pairs made of a blue and a red tile. Through preserving the equality in this setting, the MAL-system emphasizes the relational notion of the equal sign rather than the operational notion.

Five kinds of digital feedback functions are implemented into the system (Figure 3). The equal or unequal sign produces "balance feedback" showing if the equation is correct or not. In the "symbolic feedback" the system automatically converts the tiles collections into a symbolic arithmetic expression above the mat. When tiles are grouped, this is shown by a yellow frame. If a zero-pair is grouped it vanishes from the mat and if moves are illegal the tiles wiggle or return to their original place.

A layered model and feedback

Learning with an artifact that models operations on negative numbers encompasses both learning to use it and letting it fade away from awareness. We assume that the activity of teaching/learning is layered (see Swidan et al., 2020) and conceptualize this layered process by focusing on four interrelated layers of acting and operating, as Leontjew's conceptualizing of an activity system considers (Figure 2). Layer 1 consists of learning to operate and handle the MAL-system as an artifact, Layer 2 is learning with the MAL-system e.g., how to handle equations based on operations with tiles. Layer 3 addresses learning to link MAL-expressions with their respective arithmetic expressions, and Layer 4 addresses the emancipation from the MAL-system. The third and fourth layers may capture transforming processes of the students as indicated by Leontjew. In Layer 3, the students' learning of negative numbers requires converting and expanding MAL-expressions to more general mathematical expressions by the use of the symbolic system of arithmetic. In Layer 4, the students emancipate themselves from the artifact in that the students' actions with negative numbers do not depend on the artifact anymore but rather are used for other activities. In the latter case, the students become aware of the instrument's limits although they also may use the tool on purpose if necessary. When the activity of learning negative numbers is enriched, handling negative numbers may be transformed from a motive for an activity to an action towards further learning goals, for example for learning to solve algebraic equations with negative numbers. As Figure 2 indicates, the four layers of handling the MAL-system are not separated but rather interrelated with Layer 2 building on Layer 1, and Layer 3 building on both. We conjecture that the layered model can be empirically coordinated with the process of instrumental genesis of the MAL-system when the students pursue goals provided by tasks or by the tutor.

Feedback is an essential part of our digital tool. It is defined as a *response that comes back to an eliciting action of an initial situation* (Bikner-Ahsbahs et al., 2020; Reid et al., 2022), either *by feedback of the MAL-system* or *in speech* (Figure 7). Using the MAL-system, digital feedback is always present as the MAL-system shows the result of the student's moves on the mat. It becomes relevant only if someone in the group, tutor or learner, reacts to it, bringing the feedback into the group's discussion e.g., by identifying a mistake. Only then it may initiate further reactions and contribute to learning. In this way the digital feedback becomes part of a feedback loop, which starts with an *initial situation* from which a feedback *eliciting action* emerges e.g., making a mistake on one side of the mat shown by an unequal sign. The reaction to feedback initiates *retroactive actions* directed to the initial situation, for example reflecting on a mistake and correcting it (Bikner-Ahsbahs et al., 2020). In the case of *tutor feedback,* the tutor orients his/her feedback to a goal to help the students solve a task, thus, tutor feedback is an action, provoking retroactive actions directly, without intermediate reactions.

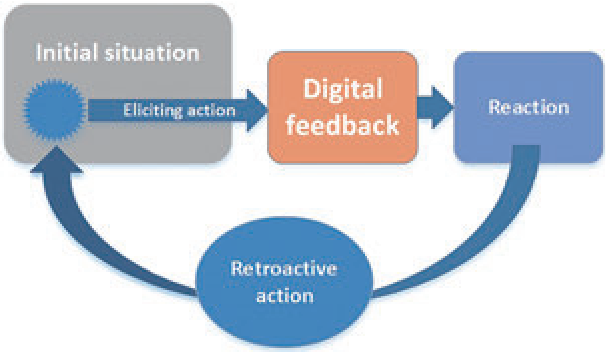

Figure 7: Feedback loop (© Springer Nature, permission received, see Abb. 2.14, Bikner-Ahsbahs, 2022, p. 30; see Bikner-Ahsbahs et al., 2020).

Fyfe (2016) distinguishes between four kinds of feedback, which in our case are not all digital. The MAL-system may offer *verification* feedback (e.g., through balance or symbolic feedback), (indirect) *correct answer feedback* (translating MAL-expressions into arithmetic expressions) and the tutor or the students may offer *explanatory feedback* and *try-again* feedback, and we add *giving-approval* feedback that appeared repeatedly. All these kinds of human feedback are directed to goals in the teaching/learning activity.

METHODOLOGICAL CONSIDERATIONS

We undertook an empirical study to investigate *the function of digital feedback in the support of the teaching/learning of negative numbers with the MAL-system.* The study consisted of four cases of experimental task-based interviews of a task sequence (Maher & Sigley, 2020) of about 110 minutes each. A tutor acting as a teacher conducted the interview that was shaped as a teaching/learning arrangement with two students (Figure 3, left). The student pairs worked collaboratively with one iPad, but distributed their common responsibility: each student was responsible for one side of the equation on mat while collaborating. The interview followed a sequence of 25 tasks (see appendix) that aimed at building knowledge for handling negative numbers. The first five tasks introduced the main functionalities of the MAL-system including the SZ through well-known calculation tasks with natural numbers. Tasks 6–9 introduced negative numbers based on zero-pairs and the SZ. Tasks 10–23 addressed adding and subtracting with negative numbers and reflecting and comparing on these more complex tasks. Tasks 24 and 25 were oral tasks to get an impression of the students' imagery of handling negative numbers.

When a tile is moved from the SZ to the AZ (respectively from AZ to SZ) the sign of the tile changes. This can be modeled by an exchange action (placing a zero-pair in the AZ (SZ) and then removing matching tiles from the AZ and the SZ) or an automatic color change shortcut (when the tile passes the boundary of the SZ). Such a color change happens also to a tile that changes from one side of the equation to the other (not introduced here). Two interviews were conducted with automatic color change and two without it to investigate if shortcuts foster or hinder handling negative numbers. In this paper, we will focus mainly on the two cases without color change. Color change is not the focus here but turned out to be an interesting feedback function in itself.

We conducted all the interviews in one Grade 5 class of a German gymnasium, a school in the two-tier school system with the higher achievement requirements. At this time of the school year, the students were familiar with natural numbers, they had met fractions but negative numbers were not systematically taught before the study, but rather a next step in the mathematics curriculum.

The interviews were video recorded, then paraphrased and images from the videos were extracted. These preliminary analyses showed that one case stood out as it provided specifically rich data because the students worked on all the tasks, addressed all layers with minimal guidance of the tutor and, thus, maximal activation of the students. This case served as a focus case, with which we compared the other cases. For the focus case, we transcribed the video recording until Task 19 verbatim and analyzed this case for each task episode. Tasks 20–25 were analyzed video based. The other cases were included by theoretical sampling i.e., if the analysis of the focus case yielded a conjecture, this was checked by analyzing respective episodes in the other cases for evidence.

In our analyses of the focus case, we investigated turn-by-turn the teaching/learning of negative numbers by reconstructing the reactions and retroactions to digital feedback, identifying the goals and related actions and operations. The findings of the analysis of the focus group are presented. Then they are used to coordinate the Activity Theory with a focus on the activity system with the Instrumental Approach towards local integration.

FINDINGS

Feedback loops of digital and tutor feedback

We first illustrate the loop of digital feedback for task 6.2 (*Place a blue and a red tile on the side where you are performing the calculation.*). Before, the students have experienced what is represented in Figure 6, upper row, where the symbolic feedback says: 2 – 3 = 0 – 1.

1 Timo: (places a blue tile in the AZ, right side) keeps the same I think (it appears ≠, he adds a red tile,

then = appears, see Figure 3, right) yes!

2 Tutor: (translates symbolic feedback into words) mhm yeah, together a red and a blue tile equals zero

3 Timo: because a red is always negative, so -1

4 Simon: that is (refers to the blue tile) 1 and this then (refers to the red one) I think -1

Timo solves the task (#1), where he first conjectures that equality will be kept. His *eliciting actions* are placing the two tiles one by one. Digital feedback shows an *unequal sign* first and then an *equal sign*. His *reaction* is "yes!" directly taking this digital feedback into the group. A second reaction comes from the tutor who refers to the symbolic feedback. *Retroactive actions* follow in two steps by Timo and Simon, who together explain why a red and a blue tile add up to 0 (#3, #4). (Figure 3)

Directly after this scene we can observe a loop for tutor feedback in task 6.3 (*Could you now further calculate as you just have done?*). Timo misunderstands the goal and puts into the same AZ an additional zero-pair (the feedback = confirms). This serves as an *eliciting action* for a tutor feedback, pointing to the symbolic feedback area and saying that just one number is expected to appear on the right side. The tutor says, "*Now at the moment you have* 2 + (–2) – 1" (Figure 6, lower row), providing explanatory feedback to clarify the goal to yield one number.

5 Timo: Then I would push away (refers to the zero-pairs) the reds, and the other tiles, then we would have only –1, hence only one number.

6 Simon: (pushes the blue and red tiles together, symbolic feedback is 0 – 1) this would be two numbers

7 Tutor: Before you have always tried this tile (taps on the blue tile in the SZ, right side) could you do it here, too (puts one zero-pair back)

8 Timo: I have an idea (takes a blue tile from the AZ and the SZ away, = appears), this is still equal, my idea then is to do so (takes the zero-pair away) and then one can do this (makes the SZ disappear), then we only have one number (refers to the red tile)

Tutor feedback indicates what is expected, the goal (one number) initiating directly retroactive actions (#5, #6) by Timo and Simon. Timo does not take 0 as a number, but Simon corrects him, pointing to the two numbers in 0 – 1. The tutor takes #6 as an eliciting action (Figure 6) for the next tutor feedback (#7) and now re-directs the task goal. Timo uses a previously developed instrumented actions to solve the task by showing that taking a blue tile away from AZ as well as from SZ and then pushing a zero-pair together to delete it leaves one red tile only. After deleting the SZ tapping three times on it, (–1) is the only number left (#8). (Figure 6).

What we observe here is a feedback chain of two feedback loops (Figure 7) without reaction. Such a reaction is not necessary because a tutor feedback is already part of the social interaction in the group as tutor feedback is normally directed towards the goal of the task to ensure the task will be solved.

The role of feedback in teaching/learning

By using the MAL-system, Timo and Simon went through different learning experiences. Learning the MAL-system itself (Layer 1, see Figure 9) means acquiring *instrumented action schemes* to meet the affordances of the system (e.g., pushing together tiles to group them standing for adding them). The first phase of acting addressed handling well-known natural numbers, which helped the students to learn to use the MAL-system for performing operations, for example how to place five unit-tiles on the left and the right side and to decompose them to learn the grouping function. Here knowledge about natural numbers is used to explain what grouping means, that a yellow frame around the tiles indicates grouping and that the translation into the symbols serves as symbolic feedback. While the beginning of teaching/learning mainly addresses the instrumentation of acting with the features of the artifact, instrumentalization is locally present, too, through enriching the artifact with attributing to it external knowledge about natural numbers and their decomposition as additional properties.

Layer 1 is also addressed when going beyond natural numbers, for instance, when solving task 6.1 (*Set the expression 2 – 3 on both sides and try to find the result of the calculation. How far do you get with your approach so far?*), and task 6.2 (*Place a blue and a red tile on the addition zone on the side where you are performing the calculation*). The (balance and symbolic) digital feedback supports defining a zero-pair as 'the red and the blue tile making together 0'. Likewise, task 8.1 (*Place together a unit-tile and a minus-one-tile and see what happens*) allows the students to learn how the MAL-system "behaves" after these tiles are pushed together; as Simon reacts with *"they disappear",* he points to the action of 'making a zero-pair disappear', which is being instrumented by the feedback from the MAL-system.

The students also learned with the MAL-system (Layer 2) e.g., Timo explains how this "vanishing" happens: *"the result is also zero and therefore they are only additional tiles which are not needed"*. Timo's explanation is based on how "the result" is experienced on Layer 1 whereas previously it was associated with the idea of having only one number (the result) represented by symbolic feedback. His claim, *"the result is also zero"*, points at least to two different situations, first, it can be read that 0 = 0, and second, after the blue and the red tiles were pushed together on the left side the result is also 0. In this case, Timo's retroaction to the feedback plays the role of an explanation initiated by the tutor. Thus, the fact that the zero-pairs vanished, initiated learning on Layer 1 (when we push a zero-pair together both tiles vanish) and mediated the transition to Layer 2, which resulted in an explanation (because they make together zero they are not needed) and was further used as an idea for treating tiles arrangements shown next.

Making generalizations based on examples are also supported by the MAL-system's feedback and as such constitutes learning with the MAL-system. For example, when solving task 8.2 (*Place additional tiles on the left side only so that the equal sign is still displayed, where the other side shows zero*). Unexpectedly, Timo offers an idea saying "well, simply place zero-pairs" while Simon places zero-pairs on the left side of the mat as eliciting actions. As the equal sign is kept, Timo explains *"it always works with as many zero-pairs as we want",* and the tutor reinforces him (*"when you have the same number of minus-one-tiles and one-tiles then the result is always zero, right?"*), which serves as a *giving-approval* tutor feedback. So balance feedback mediated teaching/learning on Layer 2. In addition, the students' expressing the rule "*it always works with as many zero-pairs as we want"* shows that they have developed an instrument for deleting zero-pairs to solve MAL-tasks. In this instrumental genesis, the following dialectic of instrumentation and instrumentalization is at play: the instrumented action of pushing pairs of tiles together enriched by knowledge about zero-pairs and simultaneously it is Timo's independent choice to place zero-pairs on the mat, check an implicit conjecture and finally infer a rule, thus, expressing an enriched understanding of the task situation by instrumentalization.

Learning to link MAL-expressions and arithmetic expressions (Layer 3) is also supported by the MAL-system's digital feedback. For instance, Timo solves Task 7 (*Place a minus-one-tile on the left and no tiles on the right. Represent a math task by placing tiles on the right side that has the result –1. You may start with red tiles, but in the end you should only see blue tiles on the right side*) by representing 1 – 2 with tiles, placing a unit-tile in the AZ and two unit-tiles in the SZ on the right side. When doing so, balance feedback shows that this is correct and Timo elaborates symbolically saying *"because one minus two is exactly the same as zero minus one*". Additionally, Simon suggests that it is always possible to add two blue tiles, placing one on the AZ and one on the SZ to keep the equal sign. Thus, Timo and Simon learned to consider the number –1 as the result of a mathematical operation e.g., 1 – 2, as well as a set of equivalent subtractions (–1 = 1 – 2 = 2 – 3 = 0 – 1 = …). This generalization of *treatments* of –1 represented in the arithmetic symbol system (see Duval, 2008) is a consequence of the learning with the MAL-system and its feedback on Layer 2 that mediates the transition to Layer 3 in Timo's explanation.

Figure 8: Making sense of the negative number -3.

In Task 8.3 (*Place minus-one-tiles together. How does this affect the symbolic representation above?*), they place three blue and three red tiles, representing 0, on the left and none on the right side. The symbolic feedback shows 3 + (–1) + (–1) + (–1) and Simon reads it loudly (Figure 8). This symbolic feedback makes Timo act on Layer 3 by identifying "*So there appears a double bracket because otherwise it would calculate plus minus three and that would not make sense and therefore there appears the bracket because it is another calculation"* (Figure 8). Simon asks "*will they disappear if they all are placed together?*" His conjecture in this question comes from the perception of how single zero-pairs vanished, an instrumented action learned before, which he now generalizes. Simon tried it out but nothing happened.

As the MAL-system is not programmed to fulfill this conjecture, the expected feedback is not shown resulting in a feedback by non-feedback, which points to Layer 1. However, pushing single zero-pairs together is successful. They disappear as Simon and Timo had experienced earlier. Simon's expectation indicates a step towards emancipating from the MAL-system (Layer 4) considering its limits and overcoming them by another way of acting. This situation shows that, as the students proceed in their process of instrumental genesis their flexibility in working at the three layers increases, interconnecting the layers, continuously supported by feedback.

Reflective summary

Taken together, digital feedback mediates teaching/learning between the four layers (Figure 9). We have shown shifts from Layer 1 to Layer 2 and from Layer 2 to Layer 3, but also to Layer 4. However, it may

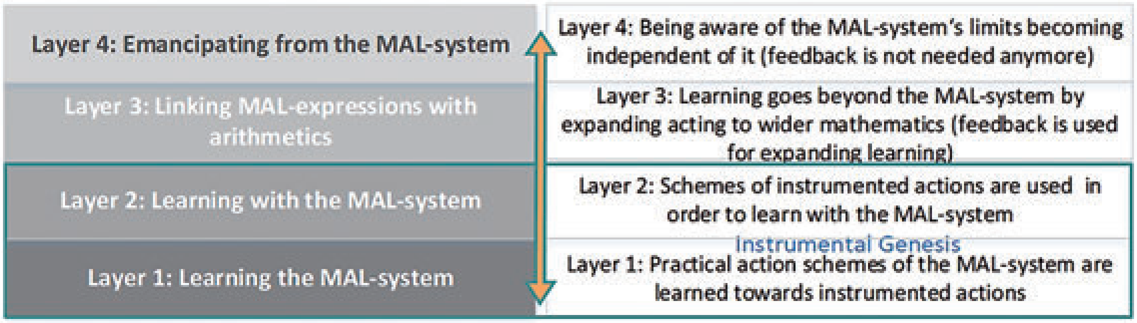

Figure 9: A layered model of teaching/learning mediated by the MAL-system and its feedback (double arrow), to be coordinated with the process of its instrumental genesis (green rectangle)

happen that symbolic feedback or symbolic tasks initiate translating the task into a MAL-expression moving tiles on the mat (shift from Layer 3 to Layer 2) and then coming back to the symbolic expression (Layer 3), thus mediating acting forth and back between Layer 3 and Layer 2. Balance feedback may keep the students acting on Layer 2, for example when they try to solve a problem with the tiles. An example is Task 9 (*Represent the number –5 on the left side. Then, on the right side, find different possibilities for representing the same number*.) In this task, the students discover the exchange-strategy i.e., Timo places a red tile on the AZ and removes a blue tile from the SZ. A bit later, Simon varies this strategy, he removes a blue tile of the SZ first and then places a red tile on the AZ instead and finally Simon removes a red tile from the AZ and places a blue tile on the SZ. In each case, balance feedback confirms the correctness of these exchange strategies. The students provide several explanations addressing Layer 2 by referring to their experience with tiles, such as Timo: "*Yes that would work too because it results in zero, and then we can exchange them.*" However, finally Timo converts tile expressions into symbolic language (see Duval, 2008) to explain: "*That is because a blue one in the subtraction zone is also minus 1 and the red one is already minus 1 and that is why one can simply exchange them because they are both the same.*" Here, first the balance feedback mediates the students' actions on Layer 2, but then an explanation is produced with symbolic terms linking Layer 2 and Layer 3 *without support by symbolic feedback*. Although the students cannot manipulate symbols as these only appear as symbolic feedback, they expand their symbolic knowledge.

Working through the 25 tasks (see appendix) the students become more and more fluent in changing and bridging layers, translating between them, and using them in entwined ways. Thus, their actions become richer and richer, but they also struggled when subtracting a negative number, for example in the task 2 – (–5). The students in the end choose tile expressions and symbolic expressions depending on their own needs, adding verbal explanations when necessary. Symbolic language was more often used for actions of conjecturing and explaining whereas operations with tiles of the MAL-system were often used for actions of checking, showing, testing and exploring. Within this single session, we only found one indication for emancipating from the MAL-system by going beyond its limits. However, the fact that the students became able to decide, which representation system they wanted to use as an instrument for a specific situation shows that the students have passed through a transformation process as described by Leontjew (1987).

THE TEACHING/LEARNING ACTIVITY PASSING THROUGH THE LAYERS

For teaching/learning negative numbers the MAL-system is used as an artifact with digital feedback. Task 6 (*Set the expression 2 – 3 on both sides and try to find the result of the calculation*) addresses acting with the artifact practically on Layer 1. For this, students must draw a blob to get a SZ placing tiles in the AZ and SZ (Layer 1). *How far do you get with your approach so far?* is the subsequent task aiming at reflecting and explaining, hence, learning with the MAL-system in Layer 2 about the current limit of calculating 0 – 1, since the result is not just one number as in other cases where the difference is positive. This results in defining a new number, (–1), by 0 – 1, which means practically placing a red tile in the AZ (Layer 1). (–1) then substitutes 0 – 1, practically and symbolically (Layers 2, 3). Hence, in this process learning on Layer 1 and Layer 2 are intermingled by instrumental genesis. Specifically, when the students proceed in working with negative numbers, more and more features are attributed to the MAL-system on Layer 1, addressing learning with the system on Layer 2 that even goes over into Layer 3. This makes clear that Layer 1 and Layer 2 are neither separated nor can they be regarded as ordered stages during the teaching/learning. Through the dialectic duality of instrumentation and instrumentalization they become entwined impacting the students' further personal development.

Task 7 (*Place a minus-one-tile on the left and no tiles on the right. Represent a math task by placing tiles on the right side that has the result –1. You may start with red tiles, but in the end you should only see blue tiles on the right side*) addresses a goal on Layer 3, to substitute a whole class of tile pairs, resulting in 2 – 3 = 1 – 2 = 0 – 1 …, by (–1) or practically by placing a red tile, but also by other suitable tile pairs (Layer 1). Thus, *operations* on Layer 1 realize actions on Layer 2, on which epistemic insight on Layer 3 is built-up and expressed by generalizing, supported by the symbolic feedback and transforming the learners' abilities towards the use of symbolic expressions.

Feedback assists in bridging between and within the layers. When the students proceed to act in an additional layer, the activity of teaching/learning is transformed in that it changes its nature. If Layer 2 adds to Layer 1, then the students use the MAL-system to explore a situation in their own way. However, this already happens on Layer 1 when the students begin to make sense of the actions they conduct and use the tiles in new ways not expected by the researchers. Layer 3 is already at stake in the symbolic feedback, when tile representations are converted into symbolic expressions by the system. In this way, the students may enlarge their capacity to act by using the symbolic language to explain, whereas on Layer 2 they instead can show how a goal can be achieved or explain – based on tiles – why the goal is achieved or not achieved. When Layer 4 is added, the MAL-system does not constrain acting any longer; the MAL-system can even be left aside.

LINKING TWO THEORIES TOWARDS LOCAL INTEGRATION

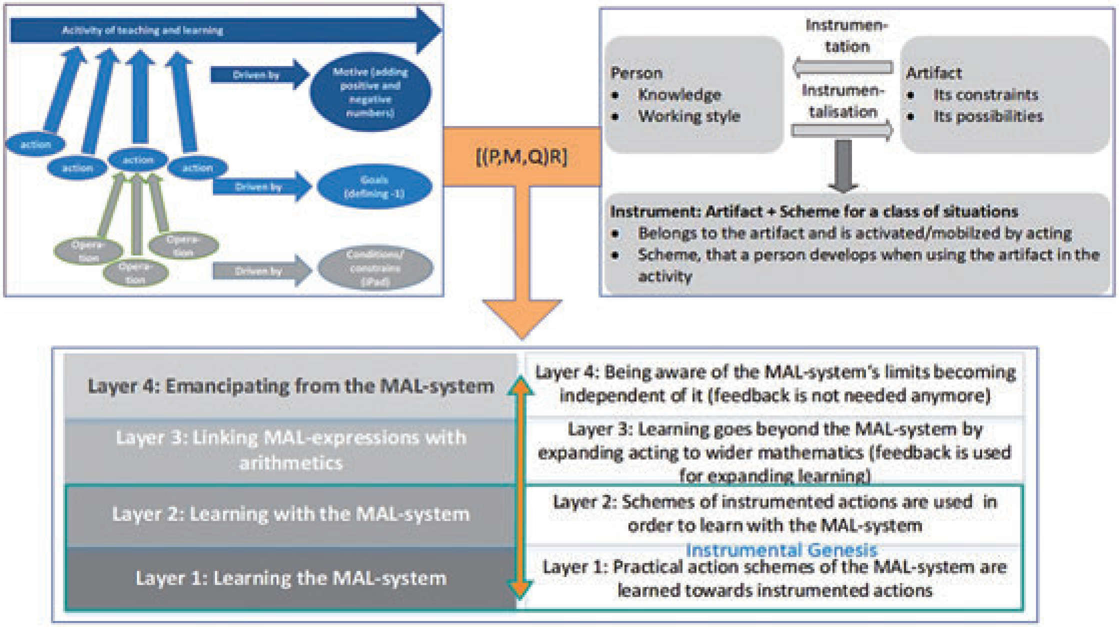

Figure 10: Local integration of activity theory and instrumental approach into a phenomenon of entwining the layers and thus transforming the activity (see Bikner-Ahsbahs, 2022, p. 25)

Activity Theory was empirically coordinated with the Instrumental Approach to explore the support function of digital feedback in the activity of teaching/learning negative numbers with the MAL-system (Figure 10). We assumed that this activity is layered and we elaborated theoretically a four-layer model related to the structure of the activity system. Our findings brought about the new phenomenon that *these four layers became continuously more entwined by digital as well as tutor feedback and the instrumental genesis of the MAL-system in the course of the teaching/learning process while the activity develops and through that the students develop.*

We could empirically confirm four layers of teaching/learning with the MAL-system (Layer 1: Learning the MAL-system, Layer 2: Learning with the MAL-system, Layer 3: Linking MAL-expression with arithmetic-algebraic symbols, Layer 4: Emancipating from the MAL-system) and thus confirming our assumption of a four-layer model. We showed that digital feedback and tutor feedback foster the accumulation of students' learning passing through these layers and thereby becoming more flexible while finally indicating emancipation from the MAL-system.

The MAL-system provides the condition for operations that shape the actions with negative numbers. These conditions are given but the students had to learn how to handle the MAL-system as well, while transforming this artifact into an instrument in a process of instrumental genesis. Analyzing the use of the artifact from the perspective of the Instrumental Approach made us identify instrumentation processes leading to various (sub-) instruments. These included instruments for:

- keeping the equality by *adding zero-pairs*, by simultaneously *taking the same tiles out of the AZ and SZ on the same side* or by *exchanging a blue tile by a red tile* (respectively a red tile by a blue tile) e.g., when the tile is shifted from the AZ to the SZ or vice versa,
- *transforming the action of taking a tile away into representing this by inserting the tile into a SZ and vice versa*
- expressing *–1 by n blue and (n+1) red tiles (while n is a natural number) or placing n blue tiles in the AZ and (n+1) blue tiles in the SZ.*

These processes involved Layers 1 and 2, but also Layer 3 when symbolic expressions were used to explain these processes.

After coordinating the two theories, Activity Theory and the Instrumental Approach, we will now justify that they can be considered being locally integrated by the new phenomenon in that we base our argumentation on Radford's theory concept [(P, M, Q), R] (Figure 11).

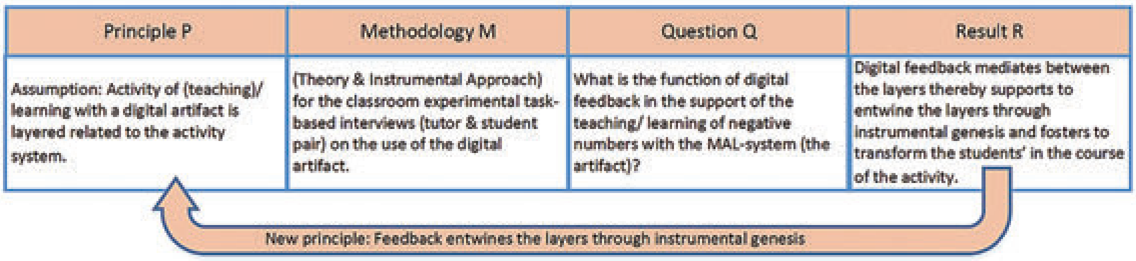

Figure 11: [(P, M, Q), R] for local integration

We have added the principle P of assuming the four-layer model for the activity of teaching/learning. We have elaborated that feedback mediates between the layers and that the instrumental genesis of the MAL-system entwines these layers. This is empirically shown by the methodology M of the dual-theory approach for experimental task-based interviews, conducted in the students' school. The goals of the tasks were exactly those that could come next according to the curriculum. Thereby, we could study an experimental situation of using the artifact in the natural setting for which the MAL-system was developed. The paradigmatic research question Q we have explored was: *What is the function of digital feedback in the support of the activity of teaching/learning of negative numbers with the MAL-system?* This question links the activity with the digital artifact that was not provided by the natural cultural class environment but that is a new tool. Our main results are that the digital feedback of this artifact mediates between the layers, thereby supporting the process of instrumental genesis that – as shown empirically – entwines the layers and assists the students' passing through the four layers in their transformation to emancipate from the MAL-system.

Figure 10 summarizes the local integration we have achieved by adding a principle P, a methodology M and a research question Q (see Figure 11). With the results a new linking principle on the role of feedback becomes apparent, that feedback entwines the layers by instrumental genesis. Thus, the results R rebound on the principles by adding to Activity Theory a better understanding of Leontjew's observation that activity is in constant transformations becoming "richer and richer" (1987, p. 200, our translation) thus pointing to the accumulative nature of teaching/learning.

DISCUSSION

As previously investigated in another setting (Bikner-Ahsbahs et al., 2020) also in this study we found evidence that digital feedback is embedded in a feedback loop shaped by the tutor and the students. Feedback does not speak by itself, the effect of feedback is determined by the way students retroact to the feedback and this may happen in a pragmatic way or in an epistemic way (Artigue, 2002; Reid et al., 2022). Feedback is considered to play the pragmatic role if the student reacts by (re)performing a practical action, for example drawing the blob on Layer 1 to subtract or tipping three times with two fingers on the SZ to make it disappear. It is epistemic when students feel invited to express or approach understanding e.g., with an explanation or a generalization expressed with symbols on Layer 3. The MAL-system's feedback normally does not show any preferred function. The function arises through the students' reactions and retroactive actions, when they just act practically, reflect about their actions or try to explain the feedback. Tutor feedback however is an action, directed to a teaching goal. It may show a pragmatic or epistemic function related to the goal that indicates what is expected from the students.

Digital feedback functions, like the wiggling of tiles in cases of illegal moves, are artifacts themselves. They need to be instrumented by the students. It is interesting to note that the students also used digital feedback for their own purposes. For example, some students used the balance feedback to explore legal moves on both sides of the mat as for an algebraic equation. Others provoked feedback to test conjectures and pushed three red and three blue tiles together to see if they vanish. As this did not happen, they learned to take this missing feedback as a new kind of feedback for what the system is not programmed to do. These and other explorations the students undertook are clear indicators for instrumentalization processes.

Applying Leontjew's (1987) view on the transformative nature of activity to the teaching/learning with the MAL-system the students' personal transformation happened within this activity through feedback that mediates between layers of learning. Instrumental genesis of the MAL-system is the basic process nested in the more comprehensive learning process in which tiles of the artifact are transformed into arithmetic symbols to be used as instruments in future. This transformation in learning advances the students' human ability to act. More generally, Leontjev (2009) explains:

> For man a tool is not only a certain object with some external shape and certain mechanical properties; he sees it as an object embodying socially developed ways of acting with it … An adequate relation between man and tool is therefore primarily expressed in his appropriating … the operations fixed in it, by developing his own human abilities to be developed and used as instruments. (p. 266).

CONCLUDING REMARKS

As the Instrumental Approach is rooted in Activity Theory, our local integration is not a surprise. What is new, however, is the fact that the data provide evidence that the transformations of the activity can be considered as layered and that the digital feedback of the artifact impacts on this transformation by mediating between the layers in the teaching/learning where instrumental genesis plays a key role. The networking of the two theories allowed us to observe the process of intertwining of the layers in the instrumental genesis and how the students developed in this process. A similar result – but not based on a layered model – was attained in our study on dividing algebraic equations with the MAL-system when students transcended integers (Janßen et al., 2020, 5.1.3).

Leontjew (1987) has emphasized that transformations may entail actions that are transformed into operations and activities that are transformed into actions. We could not identify these kinds of transformations. The reason is probably that the transformations we have observed within an activity consist of incremental transformative steps whereas we can expect that a transformation going beyond an activity consists of a stronger shift of view, for example, the shift that is necessary from arithmetic equations to algebra equations. Such transformations go beyond just one session, they must be considered as long-term developments.

A glance into the layered model draws our attention to a third theory: Duval's cognitive theory of representations (2008) warrants the relevance of transitions, within the same register (treatments) as within the MAL-system and transitions between different registers (conversion) here between the registers of the arithmetic symbol system and the MAL-system. Our results indicate that students use these registers for different things. For example they *explore* how to generalize with the MAL-system within Layer 2, but they *explain* perceiving generalizations with tiles using arithmetic symbols in Layer 3. The problem we are faced with in the networking of theories when we consider a cognitive approach is its incompatibility with the cultural-historical approach. Due to the contrasting principles and the lack of appropriate data, Duval's cognitive theory of representations could not be integrated into our theorizing process. But the ReMath project has shown that networking of it with an activity theory perspective could be conducted in the design studies (Artigue & Mariotti, 2014).

As transitions between registers were essential in our study, our layered model calls for further research on transitions between registers and the role of digital feedback there, in the MAL-system and with other artifacts. Our results raise the question whether such transitions may also lead to an emancipation from the artifacts by becoming more flexible in its use and how it is possible to simultaneously improve proficiency in the use of symbolic systems when going beyond the artifact.

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Note This article is an extended version of the paper "Role of feedback when learning with an artifact" presented by Angelika Bikner-Ahsbahs, Estela Vallejo-Vargas and Steffen Rohde in the Topic Study Group 57on the Diversity of Theories of the 14th International Congress on Mathematics Instruction in Shanghai (2020, postponed to 2021). It encompasses more information about the digital resource (the MAL-learning system) and more in-depth elaboration from the perspective of the networking of theories.

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APPENDIX

Table: The list of tasks

Task 1	1.1	Move individual tiles to one side of the mat and observe what appears at the top.
	1.2	What does each tile stand for? What happens when you push tiles together?
Task 2	2.1.	Represent the number 5 on the left side of the mat.
	2.2.	So far, you see an unequal sign in the middle. Move tiles from the stock to the right side and observe what happens. What do you have to do to make the unequal sign an equal sign?
	2.3.	Find as many different representations as possible for the number 5 on the right.
Task 3	3.1.	So far, everything on the mat has been added up. If we want to subtract, we can draw a circle with our fingers.
	3.2.	A subtraction zone is created. Everything that is placed here is subtracted from what is outside. Try it!
Task 4	4.1.	Set the expression 3 – 2 on both sides.
	4.2.	Make a guess and justify it: What will happen if you remove a tile from each zone from one side of the mat?
	4.3.	Try it: Remove a tile from each of the two zones on one of the two sides of the mat.
	4.4.	Use the procedure to get the result of the task.
Task 5	Determine in the same way the result of the arithmetic task 5 – 3. Explain your approach.	
Task 6	6.1.	Set the expression 2 – 3 on both sides and try to find the result of the calculation. How far do you get with your approach?
	6.2.	Place a blue and a red tile on the addition zone on the side where you are performing the calculation. When added together, they give 0, so an equal sign is still displayed.
	6.3.	Find out how the red tile is represented in the symbolic equation.
	6.4.	Now you can complete the task.
Task 7	7.1.	Place a minus-one tile on the left, no tiles on the right.
	7.2.	Represent a math task by placing tiles on the right side that has the result -1. You may start with red tiles, but in the end you should only see blue tiles on the right side.
Task 8	8.1.	Place a unit-tile and a minus-one tile together and see what happens.
	8.2.	Place additional tiles on the left side only so that the equal sign is still displayed.
	8.3.	Place minus-one tiles together. How does this affect the symbolic representation above?
Task 9	Represent the number -5 on the left side. Then, on the right side, find different possibilities for representing the same number.	
Task 10	Calculate the result of 2 + 5 with the tiles.	
Task 11	Calculate the result of –2 + (–5) with the tiles	
Task 12	Calculate the result of 3 + (–2) with the tiles	
Task 13	Calculate the result of –2 + 3 with the tiles	
Task 14	Calculate the result of 3 + (–5) with the tiles	
Task 15	Calculate the result of –5 + 3 with the tiles	
Task 16	Calculate the result of 3 – 2 with the tiles	
Task 17	Calculate the result of 3 – 5 with the tiles	
Task 18	Calculate the result of –2 – 5 with the tiles	
Task 19	Calculate the result of 2 – (–5) with the tiles	

Task 20 Link all tasks with the same solution

Task 21 How would you calculate 3 – (2 – 1) without tiles?

Task 22 22.1. Represent the solution of 3 – (2 – 1) with tiles and calculate the result.

22.2. Carlos believes that you can just calculate 3 – 2 – 1 = 0. Explain by comparing with your solution why this is wrong.

Task 23 Four students have solved the task 2 – (–3 – 1) in the subsequent way:

a. Alex: 2 – 3 – 1 = –2
b. Kim: 2 + 3 – 1 = 4
c. Karina: 2 – (–3 – 1) = 2 – (–4) = 2 + 4 = 6
d. John: 2– (–4) = 2 – 4 = –2

23.1. Explain with tiles the correct solution path.

23.2. Explain in each step what you have done.

Task 24 24.1. What does –5 mean to you?

24.2. How would you explain the number –5 to your classmates?

24.3. How would you explain –5 to a classmate, who only knows about positive numbers (such as the number 5)?

Task 25 25.1. Do you have an idea of what –4 – 2 means and how you can calculate it?

25.2. Do you have an idea of what –4 + (–2) means and how you can calculate it?

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