NETWORKING PRAXEOLOGIES AND THEORETICAL GRAIN SIZES IN MATHEMATICS EDUCATION: CULTURAL ISSUES ILLUSTRATED BY THREE EXAMPLES FROM THE JAPANESE RESEARCH CONTEXT

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Abstract

This study aims to identify the theoretical underpinnings of researchers' work on networking theories in terms of the *Anthropological Theory of the Didactic*. The study is based on the concept of *research praxeology*, which allows us to characterise researchers' practice and knowledge of networking. Three research examples, conducted in the Japanese educational context, are used to illustrate a means to characterise the four praxeological elements (*type of tasks*, *technique*, *technology*, and *theory*). The results imply that the notion of theoretical grain sizes (*grand*, *intermediate*, and *local levels*) can be used to deepen our understanding of researchers' work on networking theories. Based on our results, different characteristics of networking theories are discussed according to the cultural issues and specificities of the illustrative examples.

Keywords: Networking theories, networking strategies, research praxeology, logos block, theoretical grain sizes

INTRODUCTION

The growth and success of mathematics education as a scientific research field are marked by the existence of diverse theories. However, such growth also presents a challenge. As Prediger et al. (2008) stated:

> [T]he more the number of theories grows, the more difficult it will be to get an overview of all of them and of small theoretical bricks using different languages with slightly different meanings. (p. 169)

This motivates networking between different theories wherein the researchers work towards creating a dialogue between different theoretical approaches while respecting the identity of the different approaches in mathematics education (Prediger & Bikner-Ahsbahs, 2014). Several studies have worked within the context of networking theories for over a decade, and some have reviewed and reflected researchers' networking endeavours (Bikner-Ahsbahs & Prediger, 2014; Kidron et al., 2018). It appears that the latter is a meta-theoretical study of the former. For instance, Artigue et al. (2011) use the notion of *research praxeology* based on the *Anthropological Theory of the Didactic* (ATD); and Radford (2008) conceives a dynamic cultural semiotic space as a semiosphere introducing a triplet comprising a system of basic principles, methodology, and a set of research questions. In this space, networking of theories takes place and hence may also inform the dynamic development of the semiosphere. More recently, Tabach et al. (2020) elaborated

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on an argumentative grammar for networking theories by adopting Toulmin's model of argument.

While such meta-level studies clarify the nature of networking theories, attempting them can be difficult because characterising researchers' practices and knowledge in networking requires a meta-theoretical framework; however, this is still under development. Research praxeology has been adopted as a meta-theoretical framework in this paper, along with some research examples of networking theories that have taken place in the Japanese educational context, to contribute to the development of a framework for researchers' activities. Many researchers in mathematics education in Japan have worked on different theoretical approaches developed in Western countries for their studies in the local context. However, only a small number has explicitly described their networking endeavours. In this study, we attempt a retrospective analysis, which aims to make the implicit explicit in terms of praxeologies. Thus, the research question in this study is as follows: *How can we characterise researchers' theoretical work on networking endeavours in terms of praxeology?* Through this research question we expect to formulate a method to characterise the praxeologies of networking theories in different types of studies. Additionally, this paper discusses some cultural issues related to the Japanese educational context.

RESEARCH PRAXEOLOGY AND NETWORKING THEORIES

Elements of research praxeology

The concept of praxeology is one of the main constructs of the ATD (Chevallard, 2019). Using this concept, any human activity can be understood as consisting of two blocks: praxis and logos. Each of these blocks has two elements: *type of tasks* (**T**) and *technique* (**τ**) for the praxis block, and *technology* (**θ**) and *theory* (**Θ**) for the logos block. A task (**T**) indicates a problem of a given type; a technique (**τ**) is a way of performing tasks of this type; a technology (**θ**) is a way of explaining and justifying the technique; and theory (**Θ**) explains or justifies the technology. Artigue et al. (2011) extended the concept of praxeology to researchers' practices and knowledge (see also Artigue & Bosch, 2014; Artigue & Mariotti, 2014; Artigue, 2019). Research praxeologies of researchers' activities also comprise four elements (**T/τ/θ/Θ**) (Artigue & Bosch, 2014). The type of tasks (**T**) of a research praxeology often refers to research questions and problems to be studied; the technique (**τ**) comprises the research method that addresses the research questions; the technology (**θ**) corresponds to methodological discourse that justifies the choice of the method and explains the results; and the theory (**Θ**) includes the main principle and notions of a given theoretical framework.

Networking praxeologies

Networking praxeologies are used to understand the research practices of networking theories (Artigue et al., 2011). The praxis and logos blocks of networking praxeologies can be identified through retrospective analysis of studies (Artigue & Bosch, 2014; Artigue & Mariotti, 2014; Artigue, 2019). However, such analysis is sometimes difficult to practice, because:

> [T]he theoretical block of networking praxeologies is also in the process of emerging, perhaps at the moment it is in a less-developed state. (Artigue & Mariotti, 2014, p. 351)

Our study attempts to make the logos block of networking praxeologies more visible, based on our retrospective analysis of networking strategies and theoretical grain sizes. For this analysis, let us explain here what we can locate as the four elements (**T/τ/θ/Θ**) of networking praxeologies.

The praxis block of a networking praxeology also corresponds to the networking questions or problems (**T**), and method (**τ**). Unlike research practice in general, the networking endeavour refers to two or more theoretical approaches. The choice of theoretical frameworks and their usage are part of the type of tasks and the techniques. The logos block of networking praxeologies is quite different from that of research praxeologies in general. The methodological discourse (**θ**) of networking theories describes, explains, and justifies networking strategies. Prediger et al. (2008) propose a formulation in terms of: *understanding* and *making understandable*, comparing and contrasting, *combining* and *coordinating*, and *integrating locally* and *synthesising*. However, the proposal of Artigue and Bosch (2014) includes other ingredients based on the praxeological structure of research strategies. Finally, the theoretical discourse (**Θ**) explains and justifies the networking strategies. Identifying the elements that qualify as a theory (**Θ**) of networking praxeologies is difficult as "the technological and theoretical discourses are not fully articulated" (Artigue & Bosch, 2014, p. 260) in the current state of networking praxeologies. This is probably because of the scientific challenge of sharing a common meaning of theory and the role of theory in mathematics education and why it is important – or even necessary for researchers – to network theories. Therefore, we attempted to identify technological and theoretical discourses behind the networking endeavour. The theoretical grain sizes can be used to describe the praxis block of networking praxeologies as well as to interpret the implicit logos block for characterizing researchers' discourses.

Theoretical grain sizes

The notion of *grain sizes*[1] has been used to describe, review, and categorize different levels and magnitudes of theories in mathematics education (e.g., Kieran, 2019; Shinno & Mizoguchi, 2021; Silver & Herbst, 2007; Watson & Ohtani, 2015). For examples, Silver and Herbst (2007) distinguished three theoretical levels: *grand*, *middle-range*, and *local* theories. Kieran (2019) also distinguished between three theoretical levels in the context of task design research: *grand*, *intermediate*, and *domain-specific* theories (or theoretical frames). These categories allow us to understand and describe the nature and spectrum of a given theory in terms of different theoretical levels and magnitudes. However, it is not easy to determine the grain sizes of theories in a general way, because one could consider different categorizations from different perspectives.

For Silver and Herbst (2007), grand theories respond "to a need for broad schemes of thought that can help us organise the field and relate our field to other fields" (p. 60); and middle-range theories grow "from the need to inform a discrete variety of practices, including individual mathematical thinking, teaching, and learning in classrooms, or mathematics teacher education" (p. 61). Local theories help mediate connections between *problems*, *research*, and *practices*, in a particular study's context. However, for Kieran (2019), grand theories include the cognitive-psychological, the constructivist, the socio-constructivist, the sociocultural, and other general educational theories. Intermediate theories have a more specialised focus

[1] Someone might interpret the meaning of the word *grain sizes* differently. In this paper, we used this term as a metonymy for describing and understanding different levels and magnitudes of theories in mathematics education. The use of this term is similar to what Watson and Ohtani (2015) mentioned in the reviews of task design research: "Grain size descriptions are intended to be *descriptive tools* [emphasis is added] for thinking in a structural way about task design, rather than being prescriptive" (p. 5)

than grand theories and, as such, can contribute in a more refined way to the design of curricular areas (pp. 271–272). Domain-specific theories "deal with distinct mathematical concepts, procedures, or processes of mathematical reasoning" (p. 272). The list below summarises Kieran's (2019) categorisations with examples of the theoretical grain sizes[2].

- Grand theoretical frames shape the background understanding of research in mathematics education (e.g., the cognitive-psychological, the constructivist, the socio-constructivist, the sociocultural)
- Intermediate level frames[3] are located between the grand theories and the domain-specific frames (e.g., realistic mathematics education theory (RME), theory of didactical situations (TDS), ATD, lesson study, variation theory, conceptual change theory)
- Domain-specific frames specify distinct mathematical concepts, reasoning processes, or tools (e.g., a frame for fostering mathematical argumentation within problem-solving, frame for proof problems with diagrams, frame for learning algebra using technological tools)

Silver and Herbst's (2007) categories are more general than Kieran's (2019), despite both considering three levels of grain size. This is because Kieran aimed to review and discuss different theoretical frameworks which have been used in task design research, but Silver and Herbst argued for the role of theories in mathematics education research in general. More recently, Shinno and Mizoguchi (2021) distinguished three theoretical grain sizes in the context of mathematics teacher education as follows:

- Grand theories are well-established frames for research inside and outside mathematics education that have been developed in a broader context of human activity (e.g., ATD, CHAT)
- Intermediate theories are frames that do not specify a feature, but rather a general aspect of teachers' knowledge or activities (e.g., knowledge for teaching, professional growth, documentational approach)
- Local theories are frames that specify a particular feature of teachers' knowledge or activities (e.g., teacher noticing, teacher design).

In this way, the method of distinguishing the grain sizes of theories cannot be absolute but is rather relative to the research area of the study being undertaken. For the analysis in this paper, we re-categorize Kieran's (2019) distinctions of grain sizes into three levels (*grand*, *intermediate*, and *local*) by generalising the above descriptions of the three levels of Shinno and Mizoguchi (2021). In Table 1, there are a few critical points when re-categorizing the earlier version by Kieran (2019). Despite the *local-level* being almost identical to *domain-specific frames* in Kieran (2019), we reconceptualised the other two levels to distinguish the degree of generality of a given theory in mathematics education. For instance, while both ATD and TDS were categorised as *intermediate-level frames* by Kieran (2019), we acknowledge the differences between the two theories in terms of grain size. Nonetheless, our categorisations cannot distinguish theories for research inside and outside mathematics education (e.g., ATD and variation theory) as we do not consider theories developed in other disciplines in this study. In addition, we excluded "the background understanding of

[2] In Kieran (2019), what she called the grand theoretical frame is based on Cobb (2007). Examples for the intermediate and the domain-specific frames includes several frames used in the task design research. While Kieran (2019) includes references to each frame or study, we omitted the references in this table.

[3] Kieran (2019) provided an additional explanation of the intermediate-level frames referring to their roots as follows: "In addition, intermediate level frames can also be characterized according to whether their roots are primarily theoretical or whether they are based to a large extent on deep craft knowledge. An example of the former is the Theory of Didactical Situations and the latter, Lesson Study" (Kieran, 2019, p. 272).

research in mathematics education", which might refer to a general philosophical standpoint, from the description of *grand-level* theories. It is because we think background theories (such as constructivism) providing general principles that shape the universe of what can be a researchable object are relevant to all levels of grain sizes [4].

Table 1. (Re)categorizations of different theoretical grain sizes

Grain sizes of theories	Descriptions	Examples
Grand-level	well-established frames for conducting research inside and outside mathematics education, that have been developed in the broader context of human activity	ATD, commognitive framework, variation theory, etc.
Intermediate-level	frames that do not specify any particular knowledge or activities but specify a context rather than a general aspect of the research object to be studied	TDS[5], theory of realistic mathematics education (RME), etc.
Local-level	frames that specify a particular mathematical or didactic knowledge or activity	A model of problem solving, van Hiele levels, etc.

METHOD

Seeking an interface between the logos and praxis blocks is an appropriate way to develop the researchers' discourse of networking praxeologies (Artigue & Bosch, 2014). Three research examples, utilizing multiple theoretical frameworks for different purposes, are used to illustrate the praxis and logos blocks of networking praxeologies. The three case studies involved different types of research (comparative study, empirical study, and curriculum development). While these studies can be considered as cases of networking theory research, the networking strategies used are not explicitly mentioned in these papers. The theoretical grain sizes were also implicit. Therefore, we must interpret these implicit aspects. Thus, we selected research examples from studies that have been conducted in the Japanese educational context (such as lesson study, classroom teaching, and curriculum development). One study (Example 1) was selected because it involved a type of study (lesson study) that is culturally situated in Japanese educational research. The other two studies (Examples 2 and 3) were conducted by the author(s), for which we can analyse their theoretical aspects

4 For example, Shvarts and Bakker (2021) considered the grain sizes of levels in terms of the philosophical and historical roots of theories under considerations. They distinguished six different levels by means of *vertical analysis* for networking theories: such as epistemic presumption, ontological presumption, grand theory, local theory in mathematics education, application for teaching, and application for educational design. From this vertical perspective, *background theories* refer to higher levels of a certain level of theories.

5 Some researchers would also call TDS a grand-level theory as it can be applied to any situation wherein teaching and learning take place. Therefore, it should be noted that the distinctions or categorisations of different theoretical grain sizes are not reserved for the ones mentioned in Table 1.

retrospectively. While the three research studies are analysed as illustrative examples, they cannot be conceived as typical or representative, and the results of the analysis cannot be generalised. Our attention is drawn to the different praxeological models of the research on networking theories. Thus, the primary aim of this analysis is to make implicit discourses in the logos block explicit by focusing on the theoretical grain sizes of each theoretical framework to be networked. It allows us to understand the interface between the logos and praxis blocks of networking praxeologies.

Regarding the praxis block of networking praxeologies, we identify two elements (**T**/**τ**) by analysing the aims, questions, and methods of networking in the papers. For the types of networking tasks (**T**), we especially focus on the theoretical frameworks and/or models being used for networking, rather than practical methods of procedures or analyses in implemented studies (such as data collection or data analysis). Then, the networking strategies used in each study are identified as techniques (**τ**). Concerning the logos block of networking praxeologies, the technological discourses (**θ**) to explain and justify the networking strategies are interpreted by referring to some relevant quotations from each paper. To better understand the theoretical discourses (**Θ**) underlying the technologies in each study, we discuss the role of cultural issues in networking praxeologies; for example, what is a theory in different educational research traditions? What is the role of theory regarding educational practice? Why is it necessary to connect or reduce the number of theories? For such praxeological analyses of research on networking theories, the theoretical grain sizes categorized in Table 1 may play a determinant role for both the praxis and logos blocks.

ILLUSTRATIVE RESEARCH EXAMPLES

Three research examples

Example 1 – Comparing and contrasting

Miyakawa and Winsløw (2009) compared two didactical designs to introduce primary school students to proportional reasoning. The designs were based on two different approaches: *didactical engineering* in France, and *lesson study* in Japan. The comparison is informative as both approaches are used for planning a lesson, but the method of designing is very different. In the former, the lesson was designed according to the theoretical principles proposed in a specific theory; however, in the latter, the lesson was created according to the teachers' experiences in the classroom. Didactical engineering is a methodology for TDS introduced by Brousseau (1997) to gain theoretical insight into the functioning of a didactic system. In contrast, lesson study does not necessarily involve an explicit didactic theory, but often refers to a certain practical or pedagogical approach as a theoretical basis for teachers' practice. Miyakawa and Winsløw (2009) referred to a Japanese teaching approach called *open approach method* by Nohda (2000), which is relatively known as a teaching method to enhance multiple ways of thinking in the process of problem-solving, and contrasted it with the notion of *fundamental situation*[6] in TDS through their analysis and observations. Below, some elements in the process of designing a lesson to introduce students to proportional reasoning are summarised and contrasted according to theoretical basis, design formats, and realised lessons.

[6] A f*undamental situation* for a concept is a mathematical situation for which the concept constitutes a priori an optimal solution (Artigue et al., 2014, p. 49).

- A Japanese case
    - Theoretical basis: Open approach (Nohda, 2000)
    - Design formats: Lesson study
    - Realised lesson: *Hatsumon* (questioning)[7] to enquire about an *open problem* (Nohda, 2000)
- A French case
    - Theoretical basis: Fundamental situation (TDS) (Brousseau, 1997)
    - Design formats: Didactical engineering
    - Realised lesson: *adidactical* milieu[8] in the *puzzle* situation (Brousseau, 1997)

According to Miyakawa and Winsløw (2009), a comparison of the two didactical designs shows crucial similarities and differences in a lesson in the following ways.

> Both of the designs emphasize the social interaction and independent thinking of students. Both formats for design require quite similar kinds of analysis, including anticipating student strategies and revising the design in an experimental cycle. […] In a fundamental situation, they should lead to the personalization and institutionalization of a target mathematical knowledge (*savoir*), consistent with the 'official' mathematical knowledge. […] In the open approach, the aim is for students to apply and test their mathematical knowledge, through two main processes: the process in which some conditions and hypotheses from a 'real world' problem are formulated mathematically and the process of generalization and systematization after solving a problem. (Miyakawa & Winsløw, 2009, p. 216)

Miyakawa and Winsløw (2009) also discussed how didactical engineering and lesson study differ at the level of their objectives. Lesson study provides an opportunity for teachers to develop their teaching professions; the main objective is to improve lessons. In contrast, didactical engineering aims to establish scientific knowledge wherein a lesson confirms or rejects/questions the conditions for learning.

Example 2 – Combining and coordinating

Shinno (2018) aimed to characterise the development of mathematical discourses in a series of lessons in terms of the *semiotic chaining* model (triadic nested model) proposed by Presmeg (2006) and the *commognitive* framework introduced by Sfard (2008). There were two reasons for using two different theoretical frameworks in this study. The first is related to the content-specific aspect of students' difficulties of reification in the learning of square roots. Shinno (2018) argued that the cognitive account of reification can be used to explain this difficulty; however, conceptualising it differently from semiotic and discursive points of view may allow us to arrive at a deeper understanding of the reification phenomenon. The second reason is related to the setting of the mathematics classroom used in this study. Based on the observations of earlier studies on Japanese classroom culture (e.g., Emori & Winsløw, 2006), Shinno (2018) mentioned that "the social interaction between the teacher and the students in a Japanese mathematics classroom can be assumed to constitute a culturally unique discursive community" (p. 279), and that the semiotic and discursive approaches can be suitable for analysing the process through which students become familiar with the usage and meaning of square roots through classroom interactions. For example, Shinno (2018) analysed how a new expression

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[7] *Hatsumon* (in Japanese) is a general pedagogical term which refers to a teacher's key questioning in a lesson.

[8] Within the TDS, *milieu* is a component of a didactical triangle (teacher, students, and milieu), which constitutes a didactical situation. In an *adidactical situation*, "students accept to take the mathematical responsibility of solving a given problem, and the teacher refrains from interfering and suggesting the target mathematical knowledge" (Artigue et al., 2014, p. 51). The *adidactical* is referring to "the situation has been temporally freed from its didactical intentionality" (ibid.)

$\sqrt{2}+\sqrt{3}$ becomes a mathematical object rather than a computational process through classroom interactions to justify or reject a teacher's question 'is $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$ true?'

As shown in Table 1, Shinno (2018) attempts to coordinate the commognitive terms[9] with semiotic terms[10] in order to gain a multidimensional insight of the reification phenomenon through the networking of the two theoretical approaches. Shinno (2018) proposed two theoretical benefits of using two theoretical lenses in the strategy of combining and coordinating:

> From the viewpoint of the *commognitive framework*, the implicit meta-discursive rule may become explicitly identified as the third component of 'interpretant' by means of the triadic nested model. From the viewpoint of the triadic nested model, the first and second components, the signifier and signified, can be characterised as distinct features, such as keywords, visual mediators, and endorsed narratives. (Shinno, 2018, p. 302)

Table 2. Coordinating theoretical terms from the two different frameworks

Commognitive framework (Sfard, 2008)	*Triadic nested model* (Presmeg, 2006)
Word use (keywords)	Signifier or signified
Visual mediators	Signifier or signified
Endorsed narratives	Signifier or signified
Routines (meta-rules)	Interpretant

Note. Adapted from "Reification in the learning of square roots in a ninth-grade classroom: Combining semiotic and discursive approaches" by Y. Shinno, 2018, *International Journal of Science and Mathematics Education, 16*(2), p. 302.

In this empirical study, the semiotic chaining model was used to identify the three components (signifier, signified, and interpretant) in the classroom episodes and then characterised by their discursive features from a commoginive point of view. For example, when a signifier $\sqrt{2}+\sqrt{3}$ was reified (objectified), an interpretant 'treating $\sqrt{2}+\sqrt{3}$ as one irrational number' was also understood as a meta-discursive rule. In this way, Shinno (2018) sought to use theoretical terms from the two different frameworks (Table 2) interchangeably for the analysis. Thus, combining and coordinating the two theoretical frameworks allowed him to gain a multi-faceted insight into the reification phenomenon, meaning a change from process-oriented use to object-

[9] The meanings of the commognitive terms are as follows: *word use* refers to mathematical vocabulary, syntax, and ordinary words associated with mathematics; *visual mediators* include physical, diagrammatic, and symbolic mediators of mathematical objects; *endorsed narratives* mean a set of mathematical statements, proofs, rules of calculation, which are accepted within a given community; and *routines* refer to regularly employed and patterned repetitive activities (e.g., calculating, proving, generalising) as well as a set of meta-rules (e.g., how to calculate, how to prove, how to generalise).

[10] The model of semiotic chaining by Presmeg (2006) based on Lacan's inversion of Saussure's dyadic model (signifier and signified) as well as Peirce's triadic model (object/signified, representation/signifier, and interpretant). For this model, the relationship between *signifier* and *signified* is considered to have a nested structure, and meaning-making in the two components is understood by the third component *interpretant*.

oriented use of a signifier[11].

Example 3 – Locally integrating

Shinno et al. (2015, 2018) proposed a theoretical framework for curriculum development of proof in secondary schools in Japan. Unlike the other two research examples, the theoretical framework undertaken in their paper was aimed at being used for developing, designing, or improving a curriculum related to mathematical proof. For this aim, the theoretical framework was constructed by adapting and integrating different theoretical models and concepts as follows:

- The *Mathematical Theorem* as a system, consisting of *statement*, *proof*, and *theory* (Mariotti et al., 1997)[12]
- A model of mathematical proofs, comprising *knowledge*, *formulation*, and *validation* (Balacheff, 1987)[13]
- The concept of *local organisation* and *global organisation* (Freudenthal, 1971, 1973)[14]
- The concepts of *small theory* and *large theory* (Hanna & Jahnke, 2002)[15]

On the one hand, the theoretical framework proposed by Shinno et al. (2015, 2018) was a framework for curriculum development, describing and prescribing mathematical contents and sequences in the Japanese educational context. On the other hand, it seems that the theoretical models and concepts to be integrated have different theoretical natures and are not always considered the ones for educational development. For example, the notion of Mathematical Theorem (Mariotti et al., 1997) is a model to understand mathematical practice wherein a proof is carried out, but originally this model does not prescribe curricular contents and activities to be taught in schools. Therefore, when constructing the framework, Shinno et al. (2015, 2018) reconceptualized some concepts, adapting (or detaching from) their original senses, and integrated them into one framework so that it is relevant to the Japanese curriculum for proof. Let us briefly explain how they adapted and integrated different theoretical concepts.

In their framework, the three elements – *statement*, *proof*, and *theory* – were used as the foreground in shaping the framework. Balacheff's (1987) categorisation of *formulation* and *validation* was used to understand levels of *statement* and *proof*. While the original notion of theory in Mathematical Theorem

[11] Regarding the implications of using the two frameworks, Shinno (2018) concluded: "[This study] enables to characterize reification as the change in a set of meta-rules, which means the replacement of the process-oriented use with the object-oriented use of a new signifier. Although the model of semiotic chaining itself conceptualizes reification as the act of making signifier-signified couple in each node, as far as the reification in the discursive process is concerned, it may be important to detect a set of interpretants (meta-rules)" (Shinno, 2018, p. 311).

[12] A *Mathematical Theorem* consists of a system of relations between a *statement*, and its *proof*, and the *theory* in which the proofs make sense. "[The] existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense" (Mariotti et al., 1997, p. 182).

[13] Balacheff (1987) proposed a theoretical framework composed of *knowledge, formulation,* and *validation*, which allow us to analyse the complex nature of proof and proving. The four levels (naïve empiricism, crucial experiment, generic examples, and thought experiment) are regarded as the levels of validation.

[14] The local organisation by Freudenthal (1973) is a concept to explain short deduction chains consisting of a statement and its proof in geometry, wherein some properties can be accepted as taken for granted, while the global organisation is a concept to explain an axiomatized system (such as a system of Euclidean geometry).

[15] The distinction between *small theory* and *large theory* is similar to that between local and global organization. Hanna and Jahnke (2002) mention that "instead of building a large theory (namely, Euclidean geometry) in the course of the curriculum, it seems to be more appropriate to work in several small theories" (p. 3). Thus, *small theory* and *large theory* are considered concepts rather than part of an operational framework.

(Mariotti et al., 1997) mainly refers to mathematical theory (such as arithmetic, algebraic, or geometrical theories), Shinno et al. (2015, 2018) reconceptualised it as a component which includes different layers of the system (such as local and axiomatic theory) to consider the wide range of contents and levels in the curriculum. They adapted the notions of Freudenthal (1971, 1973) and Hanna and Jahnke (2002).

Their reconceptualization of the element (i.e., theory) of Mathematical Theorem was needed to account for the axiomatic aspect of geometry curriculum in secondary schools in Japan, as the Japanese geometry curriculum emphasizes Euclidean geometry (Miyakawa, 2017; Shinno et al., 2018). Miyakawa (2017) characterised this nature of the system in secondary school mathematics as quasi-axiomatized geometry with respect to some characteristics such as "the term axiom or postulate is not used; some properties are introduced after observation; some are admitted implicitly and there is no long list of axioms as in Euclid's Elements" (p. 49). This is also related to what Hanna and Jahnke (2002) distinguished as small theory and large theory.

Shinno et al. (2018) called a theoretical framework for curriculum development, the reference epistemological model (REM) and leaned on the elaboration of RME by Bosch and Gascón (2006), which constitutes the basic theoretical lens for researchers to analyse different types of mathematical knowledge among different institutions. Thus, Shinno et al. (2018) attempted to integrate different theoretical constructs regarding proof into one framework (REM), wherein an internal consistency within the framework is created to understand contents and levels in a curriculum, summarised as follows:

> Regarding the proposed REM, we conceptualized that the statements and proofs are interrelated according to the nature of theory and illustrated how this model helps to describe the evolution of each element. This model may help us understand the gaps in the evolution of the formulation of a statement and how different meanings of proof relate to the distinction between local and (quasi-) axiomatic levels of theory. (Shinno et al., 2018, p. 30)

For example, there are three levels of the formulation of a statement, i) drawing, diagrams, manipulation, and gesture; ii) ordinary language and words; iii) mathematical words and symbols. Considering a universal proposition (statement), a universal quantification can be formulated by ordinary words (e.g., *all*, *any*) in schools and by mathematical symbols (e.g., $\forall$) at advanced levels. Based on the proposed framework, Shinno et al. (2018) implied that there is a gap in the levels of formulation in the Japanese curriculum (especially between the ordinary and mathematical language of a universal proposition), and there is a crucial transition between the levels for development[16].

From the perspective of networking praxeologies

Our retrospective analysis of the three case studies in terms of networking praxeologies and theoretical grain sizes can shed light on the principal aspects of networking in each study (as summarised in Table 3), and thus enable the comparison and discussion of the praxis (**T/τ**) and the logos (**θ/Θ**). The types of tasks (**T**) were found in the choice of the theoretical frameworks in the papers. The techniques (**τ**) were identified by focusing on the networking strategies that were most relevant to the studies (based on our interpretations and retrospections). In Example 1, the strategy of comparing and contrasting was chosen and the similarities and

[16] Regarding linguistic issues, we also discussed how Japanese language may affect the difficulty and gap in the levels of the formulation of quantifications (Shinno et al., 2019).

differences between the French and Japanese approaches (TDS and open approach) to didactical designs are examined. In Example 2, the strategy of combining and coordinating is used to analyse classroom teaching with a commognitive framework, and with the semiotic chaining model. In Example 3, local integration is used to construct a theoretical framework for the curriculum development of proof in secondary schools in Japan. The strategy in this case also offers a coordination of theoretical concepts (such as the relationship between Freudenthal's local/global organizations and Hanna & Jahnke's small/large theories). We think this is natural since coordinating is a strategy that "can be a starting point for a process of theorizing" (Prediger et al., 2008, p. 173).

Table 3. Networking praxeologies and theoretical grain sizes in the three research examples

	Example 1	Example 2	Example 3
T	To compare *TDS* and *open approach method*	To combine *the model of semiotic chaining* and *the commognitive framework*	To construct a theoretical framework for curriculum development, using the *Mathematical Theorem*
τ	A strategy of *comparing and contrasting* in terms of the theoretical basis, design format, realized lesson	A strategy of *combining and coordinating* in terms of the theoretical terminologies between two frameworks (see Table 2 (Shinno, 2018, p. 302))	A strategy of '*coordinating* and *integrating locally*' in terms of *statement, proof,* and *theory*
θ	Discourses on similarities and differences on the didactic designs based on the two approaches (see the quotation above (Miyakawa & Winsløw, 2009, p. 216))	Discourses on the compatibilities and interchangeabilities between the semiotic and the commognitive terms	Discourses on how to integrate different theoretical elements, levels, and concepts into a common framework
GS[a]	*Intermediate-level* (TDS) and *local-level* (open approach)	*Grand-level* (commognition) and *intermediate-level* (semiotic chaining)	*Local-level* (all theoretical constructs)

[a]GS; grain sizes

To determine the elements of the logos block, we needed to interpret the descriptions explicitly or implicitly written in the papers. We considered the technological discourses (**θ**) by focusing on the descriptions to explain and justify how two or more theories were networked by a certain strategy and why the strategy was most relevant to the study undertaken. In the previous subsection, we tried to show such descriptions by the quotations or elaborations from the papers. Table 3 included summarised technological discourses. However, it is more difficult to determine the theoretical discourses (**Θ**) because they are mostly implicit in the papers.

We considered such theoretical underpinnings by focusing on cultural elements included in each study; for example, what is regarded as a theory in the Japanese educational and research context? What is the role of theories regarding educational practice? What is the influence of the educational system in attempting to answer these questions? We develop the discussion related to the cultural issues in the section titled 'Discussion and conclusion'.

From the perspective of theoretical grain sizes

To better understand the networking praxeologies in each study (Table 3), different theoretical grain sizes were considered. Based on our categorizations of grain sizes mentioned earlier, TDS from Example 1 is considered an intermediate-level theory, since it is often used to understand and analyse the epistemological processes in mathematical classrooms from a systemic perspective. TDS is not a domain-specific theory which specifies any mathematical knowledge or activity. Rather, it is primarily developed to model the functions and mechanisms between teachers, students, and the didactical milieu in a didactical/adidactical situation (in the classroom). In contrast, the open approach is seen as a local-level framework or instructional principle which does not specify any mathematical content but does specify teachers' teaching methods. For instance, using an open problem is a characteristic aspect of this approach which is comparable with the notion of fundamental situation from TDS (Miyakawa & Winsløw, 2009). However, TDS and the open approach are not fully comparable because the latter does not provide a tool to analyse teaching and learning of mathematics at a general level (e.g., the approach cannot be applied to a lesson with a closed problem). Therefore, the two frameworks are comparable only if one can focus on a specific notion of one approach that has a counterpart in the other (such as a fundamental situation and an open problem). Thus, the discourses on two didactical designs, as cited above from Miyakawa and Winsløw (2009, p. 216), have focused on some comparable constructs from each framework.

In Example 2, the commognitive framework is viewed as a grand-level theory which is a comprehensive theory of learning from a discursive point of view. The model of semiotic chaining originates from Peirce's semiotics. However, it was introduced by Presmeg (2006) into mathematics education for analysing the relations between signifier, signified, and interpretant in the teaching and learning of mathematics. This model was used as an intermediate-level framework in Example 2, although one could also consider it as a collection of theoretical concepts. Shinno (2018) combined commognitive and semiotic approaches to provide different accounts of reification in mathematical learning. Since the commognitive framework comprises a much broader range of theoretical constructs, coordination with semiotic chaining is related to only some semiotic parts of this grand theory. This implies that the strategy of coordinating between different grain sizes requires an explicit discourse on complementarity; for example, which part of one theory can be coordinated with another theory.

Networking in Example 3 offers local integration, but the connectivity of different theoretical concepts is rather complex. All theoretical models or concepts used in the study are regarded as local-level frames which have been developed in the context of research on proof and proving. However, this does not mean that all the theoretical constructs have the same theoretical grain size. For example, the triplet (statement, proof, theory) has been used as a foreground frame to integrate other theoretical concepts. There have been many discourses on argumentative connectivity about how to integrate different models or concepts since Shinno et al.'s (2015, 2018) attempt to propose a theoretical framework. A characteristic aspect of such a discussion is that it changes or extends the original meaning of the theoretical terms (statement, proof, theory) to those with a broader sense; this allows the inclusion of different contents and levels in the curriculum in secondary schools in Japan. Additionally, they created new terminologies (for the three different layers of the system; the logic of the real world, local theory, and quasi-axiomatic theory) through the adaptation of existing notions (such as small/large theory)[17].

DISCUSSION AND CONCLUSION

A deeper understanding of researchers' work on networking theories

The praxis block corresponds to how researchers work when they choose different theories and network them. The researchers may consider the grain sizes of theories according to the types of networking tasks (**T**) for the study undertaken. We then considered networking strategies as a technique (**τ**) and explored the technological discourse (**θ**) to explain and justify the strategy through a retrospective analysis of three illustrative research studies as examples. Artigue and Bosch (2014) argue that the characterisation of the logos block (**θ/Θ**) of a networking praxeology has not yet been established. In our study, we offered the notion of theoretical grain sizes to deepen our understanding of the networking praxeologies. As mentioned in the previous section, the networking techniques are affected by the grain sizes of the theories involved. While we have only described the grain sizes (e.g., grand, intermediate, or local levels) of the theoretical frameworks used in the three research examples shown in Table 3, attention was paid to the discourses on networking different theoretical frameworks, which may have different sizes or are of the same grain sizes.

The characteristics of such networking praxeologies may differ according to the study type, such as empirical studies, design studies, or theory development. For example, Bikner-Ahsbahs and Prediger (2014) discussed the notion of *perspective triangulation* (Denzin, 1970), meaning a research practice to increase the validity of empirical analysis. Although this notion does not refer to theories, increasing the validity of empirical studies can bring forth a benefit from such networking. Combining theories in/for empirical studies often provides deeper insights into complex phenomena (Prediger et al., 2008), wherein the results of the empirical analysis can respond to the original theories. Bikner-Ahsbahs and Prediger (2010, 2014) also mentioned other perspectives (explicitness, empirical scope, stability, and connectivity) to discuss the empirical and theoretical benefits of networking theories for different types of studies. Some of these perspectives are related to what we identified as technological discourse (a justification of the strategies involved in each case study). For instance, perspective triangulation or connectivity can support the strategy of combining and coordinating in Example 2; connectivity can explicate the strategy for locally integrating in Example 3; and *explicitness* can be the strategy for comparing and contrasting in Example 1, as well as the strategy for

[17] For example, a geometric statement "the sum of the interior angles of a triangle is 180°" can be explained by a physical approach (such as a measurement or experiment) which is acceptable within *the logic of the real world*. The same statement can be proved by previously accepted properties of parallel lines, based on *local theory* which is often regarding one particular proof. The *quasi-axiomatic theory* can be relevant when considering a system of statements and proofs in school geometry (even though the term axiom or postulate is not introduced). More descriptions of the three layers are mentioned in Shinno et al. (2018).

locally integrating in Example 3. In addition, it is also worth considering how theoretical terminologies may change or preserve their meaning before and after networking attempts when examining linguistic connectivity (Shinno, 2017). As shown in Example 2, by coordinating, the theoretical terms in one framework are interchangeable with those in another framework. In Example 3, some original concepts are locally elaborated and integrated into new terminologies in the new framework for curriculum development; however, such new terms do not preserve their meanings in the original sense.

Cultural issues to be considered

Networking praxeologies can provide a model for understanding researchers' practices and knowledge of networking theories. This implies that it is important to consider the diversity of research activities related to theories of mathematics education. One of the reasons for this diversity is *cultural issues,* which may affect the researchers' work. This is also the case in the three research examples involving Japanese educational contexts. If the theory (**Θ**) in the logos block of networking praxeologies is implicit, its description can only be based on the researchers' conception of a theory in research and practice.

For instance, Example 1 indicated the differences between TDS and the open approach; where the open approach was a practice-oriented theory or a prescriptive theory, which is often used to improve teachers' classroom teaching and facilitate students' problem-solving activities. In fact, both the descriptive and the prescriptive nature of a theory in mathematics education are often discussed in Japan (e.g., Koyama, 2006; Yamada, 2011). This is related to what has been discussed in the previous Topic Study Group in ICME-13 (Dreyfus et al., 2017); for example, "[i]n Japan, 'theory' is always a theory of some practice. A practice develops, somebody notices it, reflects upon it, and constructs a theory of practice" (p. 617). This implies two distinct aspects. One is that a theory emerges and develops from/for some practice. The other is that a theory is very close to practice and the theory often seems to be a pragmatic one[18]. However, in the international context, some researchers might not or cannot accept such a *pragmatic theory* as a theory in mathematics education (cf. Miyakawa & Shinno, 2022). Thus, there is miscommunication among international researchers regarding what is called theory in Japanese educational research (e.g., Zazkis & Zazkis, 2013). This may open a discussion that reveals the cultural specificity of Japanese (or East Asian) educational research, which contrasts with the French (or European) tradition of didactics (Blum et al., 2019). Here, we do not discuss the nature of theory in different research cultures (although such discussion is promising, it lies beyond the scope of the paper), but merely wish to point out that there is a cultural issue that may shape researchers' knowledge, even in the context of networking theories.

In addition, in Example 2, Shinno (2018) considered the cultural specificity of the Japanese mathematics classroom to be related to semiotic activities (Emori & Winsløw, 2006); however, this specificity of classroom is generally attributed to the Japanese teaching pattern (Stigler & Hiebert, 1999). Such a cultural factor may influence the researcher's choice of theory or his/her perspective of using a theory for classroom study. Recently, Funahashi and Hino (2014) have also investigated the interactive classroom pattern, called a guided focusing pattern, which is often observed in Japanese classroom lessons, in terms of a discursive or commognitive perspective for their theoretical and methodological framework. Although other theoretical

[18] Here we use the term *pragmatic* as an ordinary word without any theoretical underpinning.

approaches can be used to analyse a classroom culture, what we would imply is that the researchers' perspective of using a theory (or multiple ones) can be affected by the local educational context and by the researchers' conceptions of how the theory can be useful for investigating such a cultural specificity.

Furthermore, in Example 3, the specificity of the Japanese mathematics (geometry) curriculum in secondary schools was considered when adapting some theoretical concepts that were integrated into a framework for curriculum development. This also implies that the educational contexts (particularly the curriculum) in a country may affect researchers' work on theories. For example, given that the Japanese geometry curriculum is characterised as quasi-axiomatized geometry (Miyakawa, 2017), we could adopt or adapt the related concepts (such as, local/global organization, small/large theory) in their framework. In addition, developmental research, which is often aiming at improving educational practices (including task design, curriculum development, or lesson study) is a dominant type of research and/or study in Japan. Although a theoretical framework is often constructed for such developmental work, what is important for the framework is to be useful for development, rather than for scientific research (Miyakawa & Shinno, 2022). This is one reason why the proposed framework in Example 3 referred to different constructs (models, concepts, frameworks) to be integrated. For this reason, the researchers see all constructs as theoretical underpinnings of the framework for development, even though some constructs are not usually brought forth as theory for research. This may have the implication that one can distinguish between *theory for research* and *theory for practice*.

Finally, we attempted to translocate the cultural issues (such as research culture, classroom culture, and curriculum culture) into the theoretical discourses (**Θ**), as shown in Table 4, which are implicit in the networking praxeologies (Table 3). Furthermore, the logos block of networking praxeologies is also related to the researchers' conception of the field of mathematics education and the nature of mathematics education as a scientific discipline. This raises at least two questions: How can we make such implicit discourses more explicit in networking theories? Why is such a meta-level and a self-referential study important for researchers? While we believe that the cultural issues in different educational research traditions play a determinant role, further research is needed to address these questions and issues.

Table 4. Theoretical discourses (Θ) regarding cultural issues in the three research examples

	Example 1	Example 2	Example 3
Θ	Discourses on the nature of theories in different research traditions (research culture)	Discourses on a culturally unique discursive community in a Japanese mathematics classroom	Discourses on the necessity of a theoretical framework for curriculum development in Japanese mathematics education

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