DESIGNING ACTIVITIES FOR CAS-BASED STUDENT WORK REALISING THE LEVER POTENTIAL

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Abstract

This paper explores two types of lever potentials of CAS and a tool for designing such activities. In the first activity, CAS is used for students to study the concept of equations. In the second activity, CAS is used to strengthen the relation between traditional algebraic paper-and-pencil manipulations of equations and the theory that the solution of the equation must remain the same. As the design tool for CAS-based activities that capitalise on the potential of CAS, the notion of praxeology from the Anthropological Theory of Didactical is employed.

Keywords: Computer algebra system; school algebra; task design; lever potential; the notion of praxeology.

INTRODUCTION

In the early days of research on implementing CAS in mathematics education, the entrance of this digital tool was met with applause by the community of researchers, and the focus was on the potential that CAS could offer mathematics education (Dreyfus, 1994). The idea is, that, at the level of upper secondary school, ‘low level’ work, such as solving an equation, can be outsourced (Bang, Grønbæk, & Larsen, 2017) to the CAS. This would leave more time to focus on more complex issues such as the mathematical discourse. This potential is denoted the lever potential (Winsløw, 2003).

The lever potential has been documented in several articles since the 1990s (which we will further elaborate on in the background section). However, using a CAS on any problem in the subject of algebra does not automatically lift the students’ learning to a higher level, quite the contrary. When solving a traditional algebraic task, such as finding the solution for an equation (Kieran, 2007) with paper and pencil, the mathematical discourse of the students includes the use of fundamental properties of algebraic structures, such as the distributive rule. When solving the equation with a CAS, the mathematical discourse becomes very different, and most fundamental properties of algebraic structures are blackboxed (Carlsen, 2019).

In our study, we will consider the lever potential of CAS and explore the aspect, of the lever potential, of how CAS can be used to make the mathematical discourse explicit in the teaching of elementary algebra. Furthermore, we will study how the four categories of task, technique, (technical) discourse and theoretical discourse can be used to design appropriate activities in elementary algebra for the implementation of CAS.

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BACKGROUND

In this section, we will introduce the lever potential of CAS for the teaching of school algebra, and outline results from some of the studies depicting the lever potential of CAS. In addition, the role of CAS, in the selected papers, is to perform part of the mathematical tasks such as time-consuming algebraic manipulations, i.e., solve an equation or simplify an expression.

The lever potential of CAS was first introduced by Dreyfus (1994) and later further elaborated and named, as the lever potential, by Winsløw (2003). The idea is

"for students to operate at a high conceptual level; in other words, they can concentrate on the operations that are intended to be the focus of attention and leave the lower-level operations to the computer." Dreyfus (1994) p 205.

That is, instead of focusing on applying a series of techniques e.g., to solve an equation, students can instead progress to focus on more theoretical aspects such as technical discourse or theoretical discourse e.g., the number of solutions in relation to the type of equation.

One type of lever potential can be described as using CAS to swiftly consider multiple representation of a mathematical object. Then each different representation will accentuate different aspects of the mathematical object (Dreyfus, 1994; Gjone, 2009; Kieran & Yerushalmy, 2004; Gyöngyösi, Solovej & Winsløw, 2011). Thus, supporting further development of the students' concept of the mathematical object.

Another type of affordance of CAS is the ability to swiftly examine a series of comparable examples. This enables students to consider similarities and differences between the examples and explicitly develop and formulate their concept of the mathematical object in focus. Several papers have studied such possible activities. For example, the doctoral thesis by Drijvers (2003) is a testimony to how carefully crafted CAS-based activities can support students' explicit development and formulation of their "definition" of parameters. As an example, Heid et al. (2002) suggests working with the formula $A(time) = deposit\ (1 + rate)^{\text{time}}$ to have students develop and formulate their concept of parameters. Other suitable mathematical topics have also been explored such as algebraic syntax (Artigue, 2002b; Guin & Delgoulet, 1997) and algebraic equivalence (Gjone, 2009; Lagrange, 2005) has also been conducted.

A different type of lever potential of CAS can be described as, despite outsourcing algebraic manipulations to CAS such as expanding an expression. The pen-and-paper techniques can be used as arguments explicitly stated for validating the patterns produced and discovered with CAS. In a series of papers, students work through a course of carefully constructed problems exploring the factorisation of the polynomial x^n-1 using CAS and developing the telescoping technique (Kieran & Drijvers, 2006; Hitt & Kieran, 2009; Martínez, Kieran, & Guzmán, 2012).

The research mentioned, illustrating the different lever potentials, all have in common that the activities are very carefully designed with very specific purposes in mind. In the last articles mentioned, the activities are designed utilizing the dialectic between practice and reasoning (Artigue, 2002a). This dialectic is viewed as a structural model using the three notions of task (the task to be solved), technique (that is applied to solve the task), and theory (the reasoning that explains and validates the technique). The idea is taken from the Anthropological Theory of Didactic (ATD). However, ATD suggests to further consider the theory part of the dialectic as consisting of two parts. One part is technology, which is the discourse that explains and justifies the technique. The second part is theory, which is the discourse that justifies and validates the technology. This structural division can further benefit the design of CAS-based activities.

In our study, we want to show how the quartet-notion of praxeology can be used to design activities that realise the lever potential of CAS. The fist activity will focus on developing and having the students formulate their definition of equations, while the second activity will include traditional algebraic technical discourses, such as manipulation of equations, and link this to the theoretical discourse of having the solution stay the same.

FRAMEWORK

In general, the ATD suggests considering human activity, such as cooking a soup, as an amalgam of practice and reasoning (Bosch & Gascón, 2014; Chevallard, 1998). To cook soup, one could start with frying onions in a pot (the practice), with the reasoning being that onions give a nice flavour, and frying them sweetens and enhances their flavour. Alternatively, one might be frying onions because it is generally perceived as a good way to start cooking a soup. Our knowledge is based on practice, and our practice is shaped by our knowledge.

The ATD proposes denoting the dialectic of practice and knowledge by *praxeology*. Further to structure the practice as a twofold. One part *the type of task*, which is the type of task that is aimed at being solved. For example, cooking a soup or solving an equation of the type $s(x–t) = n·s–s·m·x$, where s,t,m and n are integers and $s \neq 0$. The other part is *the technique*, which is the actions utilized in order to solve the type of task, such as frying onions, or in the case of solving the equation above: rewriting the equation into $x–t = n–m·x$, and then another rewriting into the equation $x = \frac{n+t}{1+m}$. The knowledge related to the practice, which is called *logos* from Greek meaning theory, can be split into two components. *The technology*, which is the discourse that justifies and validates the technique, such as “we multiply with $\frac{1}{s}$ on both sides of the equation” and “we can multiply with the same number on both sides of the equation”. The discourse can also contain motivation for the actions included in the technique, the effectiveness of the technique, etc. *The theory* justifies and explains the technology, such as the solution set must remain the same. Again, the theory also comprises of discourse such as the effectiveness of the technology and motivation for the technology. However, the praxeology taught usually differs from the praxeology learned. Indeed, theory, or even technology, does not need to be “formal mathematics” but can, to individuals and even large institutions, also include mere convictions and even falsities, such as “we must maintain the balance between the right-hand side and the left-hand side of the equation” and “when we move a number from one side of the equation to the other side, then the sign changes” or “it is correct because it is the same method our teacher illustrated”. In general, technology is close to the actions of the technique and can also include motivation for the actions of the technique, effectiveness of the technique etc. (Chaachoua, Bessot, Romo, & Castela, 2019). While theory is the technology of the technology.

The following is an example of what a typical praxeology for a grade 8 student could look like when solving the equation $3(x–2) + 7 = 14 + 2x$. The type of task is to solve the equation $n(x–m) + (n·m+1) = t + (n–1)x$, where n,m and t are positive integers. The technique involves rewriting $n(x–m) \rightarrow nx–nm$ (applying the left distributive rule from left to right), rewriting such that all terms containing the variable are on the left side of the equation, etc. The prevailing technology would involve multiplying inside the parentheses, when multiplying a positive number with a negative number, the result is a negative number; whenever we encounter a parenthesis, we “dissolve” it, etc. The theory: an equation must be kept in balance meaning that each side of the equal sign must be equally heavy, the solution to the equation is a number such that each side of the equal sign in the equation has the same value, parenthesis induce a different order of operations, etc.

The notion of praxeology is useful because the four components -type of task, technique, technology, and theory- enable us to describe and analyse explicitly the elements in play in solving a type of task. This tool lets us consider structures and relations between praxeologies within and across educational institutions and mathematical themes, such as solving equations (of different types) or manipulating algebraic expressions. A praxeology is determined by its type of task (and the institution in which it lives). An action or an ingredient of a technique can be applied in several different praxeologies or several times within the same praxeology. For example, we can reduce an expression even when the task is not necessarily to solve an equation. One technology can unify several actions and even whole techniques. For instance, multiplying both sides of the equation by a number does not necessarily reduce the equation. The theory, such as we must maintain the balance between the right-hand side and the left-hand side of the equation, can join several technologies, since it is also the theory for adding on both sides of the equation. Therefore, we can classify a type of lever potential of CAS as to focus on (parts of) the technology or theory. By targeting the development of students’ technology or theory, this will strengthen related techniques and the scope of those techniques.

As hinted above, the praxeology taught and the praxeology learned differ, particularly in the logos block. This is because in a teaching and learning situation, there is not a direct transfer of knowledge and practice. However, the praxeology is transposed, and this transposition is conditioned and regulated within the institutional context. Therefore, we can refer to students’ personal praxeologies (Chaachoua, Bessot, Romo, & Castela, 2019). Studying the structure of students’ praxeologies can help us identify some of the difficulties they encounter in learning algebra. For instance, if we consider the two tasks of expanding the following expressions: $3(x–2)$ and $(x–2)3$, students often find the second expression much harder to expand than the first (the case of Danish upper secondary students (Poulsen, 2015)). For students at lower secondary school, the two praxeologies generated by the tasks are not related, not even by theory. This is because these students are not yet familiar with the formalised commutative property of the distributive rule, $(a + b)c = c(a + b)$, which would link the two praxeologies. This example emphasizes that if the technology and theory block is not made explicit in mathematics education, such as putting it into words or making it the object of study in a lesson, then many of the praxeologies are not linked, and thus, techniques will not be related to each other. This makes the more than 90 elementary algebraic techniques at lower secondary school (Poulsen, 2015) into a wilderness of rules.

If we consider the praxeologies of students at lower secondary school for solving equations, the use of CAS will dramatically change part of the praxeology (Carlsen, 2019). Without CAS, it would make sense to distinguish between solving the equations $(x–3)2 = 5\cdot 2–2x$, and $12–3x + 2 = –2$, because they require different techniques and technology to solve, such as multiplying inside the parenthesis. However, when solving with a CAS, the two equations do not require different techniques to solve, so the praxeologies would be the same. Furthermore, since the technique has changed into entering the equation, entering the command **solve**, and interpret the output, the technology has changed accordingly. However, a part of the theory block remains unchanged, namely, that the solution to the equation is a number such that when substituted with x, both sides of the equation yield the same number. By assigning technology and theory the same category and notion, as in previous studies (Hitt & Kieran, 2009; Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) we might overlook possible information or design opportunities for CAS-based activities.

If we consider the lever potential of CAS through the lens of praxeology and ATD, it can be viewed as making explicit and formulating elements of the algebraic logos, including both technology and theory, through carefully designed tasks that can only be reasonably approached with CAS-based techniques. As this will strengthen the theory and/or the technology, it will benefit a range of algebraic praxeologies.

Research questions

- How can CAS be used to engage students to work with elements of the theory block for praxeologies in school algebra?
- How can CAS be used to strengthen students' technology related to standard techniques (such as rewriting equations) in school algebra?
- How can one design tasks that realise the use of CAS described in the previous research questions?

CONTEXT AND METHODOLOGY

To explore how the notion of praxeology can guide the design of tasks for teaching of school algebra realising the potential of CAS a qualitative approach has been chosen. We have conditioned the design of CAS-based activities to fit into two regular math lessons. It was decided that one lesson would focus on a theoretical aspect of solving equations, while the other lesson should focus on strengthening tradition pen and paper techniques for solving equations. A series of tasks was designed, and two lesson plans were created to describe the different didactical situations throughout the lessons. The two lessons were conducted in three Danish grade 8 (14 - 15-year-old) classes by their regular mathematics teachers. Prior to carrying out the lesson plans, a meeting was held with each teacher to familiarize them with GeoGebra, a program for recording of screen activities, and the learning objectives of the lessons. The first two participating classes were already familiar with using GeoGebra, though they had never used the CAS application, while the third class had no prior experience with GeoGebra, CAS tools, or other dynamic geometric environments.

To collect data for the study, seven recording devices were placed in the classroom to record the students' and the teacher's voices. Additionally, the students' work in GeoGebra was recorded with Screencast-O-Matic, software that records screen activities. Everything written on the blackboard was photographed, and field notes were written by the researcher during each lesson.

For each of the three lessons, one recording was chosen for a full transcript. For the remaining recordings, only episodes including not yet transcribed work or discussions were transcribed. All the transcriptions were analysed with the notion of praxeology, i.e., categorized by task, technique, technology, and theory. The students' written materials were also analysed using the notion of praxeology. The screencasts and written materials were used to support the transcription and analysis of students praxeologies.

THE LESSON PLANS AND THE RATIONAL OF THE DESIGNS

In this section, we will give a description of the lesson design based on the lesson plans written for the teachers, and part of the a priori analysis of the lesson. The description will be supplemented by the rationale behind the design. It is assumed that the students already have some familiarity with working with simple equations, such as manipulating them and describing simple relations using equations.

Lesson A: Describe what is an equation

The first lesson, we can, as in the Background section on the lever potential of CAS, describe the learning objective as to further develop the students' concept of a mathematical object, in this case: equations. The focus of the lesson is to make explicit and formulate what makes an equation, such as the possible solutions, and whether the equal sign or a variable can be omitted. It is not the purpose of the lesson that the students should develop a definition in par with scholars of mathematics, such as a statement that expresses an equality between two expressions, but rather develop and formulate a definition that is appropriate for the institution and tradition of lower secondary school.

The first task given to the students in the lesson is to write down an equation of their own and then try to describe what an equation is. An important aspect of this activity is that the teacher does not provide the students with an example of an equation earlier in the lesson or interfere with the students' work. The use of CAS is not intended for this activity. The expectation is for the students to give a somewhat vague definition, mentioning an unknown, and that "the unknown must be found".

Next, the students are given a series of equations to solve in the CAS window of GeoGebra, while recording their screen work with Screencast-O-Matic. For the series of equations containing different types of equations, see Figure 1. For each equation that is solved with CAS, the students are asked to reconsider their description of an equation.

Figure 1. A minimized version of the students' worksheet for solving equations with CAS including the a priori analysis of added description.

	Equation	Addition to the description of what an equation is
1)	$14\,a + 2 = 72$	An equation can have a fraction as a solution
2)	$14\,y + 4 = 7 + (y–11)34$	Unknowns can be named other than x, and have a fraction as a solution
3)	$x–2 = 0$	
4)	$(x–1)(x–2) = 0$	An equation can have two solutions
5)	$(x–1)(x–2)(x–3) = 0$	An equation can have three solutions
6)	$2x + 4 = (x–2)2 + 8$	An equation can have infinitely many solutions
7)	$r–23 = (3r–4)2–5r$	An equation can have no solution
8)	$4(x–436)–326 = t–6434$	An equation can have an unknown as part of a solution
9)	$2(3–4) + 7 = 5–3(4–2) + 18$	Equations must contain an unknown
10)	$3(23–11) = \frac{1}{5} + \frac{22(3–5)}{2}$	Equations must contain an unknown
11)	$3t + 12 = 3(t+4) = 4(t–2) + 20–t$	An equation must only contain one equal sign
12)	$12(3–7) + 4 + y$	An equation must contain an equal sign

The first equation is intended to help the students become familiar with entering and solving an equation using CAS in GeoGebra. The second equation yields the output $\left\{y=\frac{371}{20}\right\}$. This equation was included because we have observed in other schools that many students implicitly expect an integer solution, particularly if students are primarily working in a traditional paper-and-pencil environment. Therefore, this type of equation is intended to serve as a gentle transition into questioning and further developing the logos block for solving equations. The solutions for the fourth and the fifth equations are 1 and 2, and 1, 2, and 3 respectively, the output from CAS is $\{x=1,\ x=2\}$ and $\{x=1,\ x=2,\ x=3\}$ respectively. The equations are constructed in a way that makes it easier for the students to test or find the solutions by trial and error by hand, in order to relate the output of CAS to the solutions for the equations. It is expected that the students will add that it is possible for an equation to have multiple solutions to their description of an equation. Equation six is true for all values of x, which CAS writes $\{x=x\}$, while equation seven does not have a solution, which CAS writes { }. It is expected that the students will spend a considerable amount of time interpreting the output of CAS but will ultimately conclude that equations can have an infinite number of solutions or no solution at all. Equation eight has the solution $\frac{1}{4}t–1091$, which is to expose the students to a type of equation where the solution contains a parameter. However, we do not expect the students to know the difference between a parameter and a variable since it is not an explicit part of the curriculum. Despite this, the difference between parameters and variables could be a topic of study for a future CAS-based lesson. We await that the students will conclude that a solution for an equation can contain another variable. It is expected that the students will add that a solution to an equation can include an expression that contains an unknown. Equations nine and ten are equations without an unknown, and CAS gives the output { } and $\{x=x\}$ respectively. We expect that all students, in their first description of an equation, will mention something about an unknown, and thus equation nine and ten are included to prompt a discussion of whether an unknown is a requirement for an equation. It is also expected that the students will not regard the expressions as equations. These examples are given as formulating a definition requires considering the limitations of the object. Expression eleven and twelve contain two or no equal sign, respectively, and CAS gives the outputs $\{t=t\}$ and $\{y=44\}$, respectively. The expressions have been included in the hope that the students will discuss the role of the equal sign in an equation. It is expected that, due to CAS being able to give a solution for both expressions, the students will question the role of the equal sign.

For the next section of the lesson, the teacher conducts the students' sharing of their discoveries of additional descriptions and clarifications, followed by the teacher's reformulation and clarification of the descriptions. In the last section of the lesson, the teacher further challenges the students' description, i.e., asking if an equation can have four and a half solution. This is done in case any of the students have written that equations can have any number of solutions, and that "number" must be specified as positive integers. Another follow-up reflection questions is if it is fair that you can manipulate an equation (with an unknown) into something that is not an equation (equation without an unknown). This question is asked to challenge the perceived concept of equations in lower secondary school as having an unknown.

Lesson B: Make an equation where $x = 2$

For the second lesson, the learning objective is for the students to strengthen their praxeology for solving equations, in particular, the relation between the techniques for manipulating an equation, such as adding a number to both sides of the equation, and the theory that the solution must remain the same.

The lesson starts with the teacher telling the students that today they will have a competition about making "ugly" equations. He then proceeds to write his two contributions on the board $\frac{2(x+4)}{3} - \frac{7x+2}{4} = \frac{3x+2}{2} - \frac{33x-18}{12}$ and $3x–4 = \frac{4x–2}{3}$. He further shows how to solve the two equations in CAS, and that both equations have the solution 2.

Then, the students are given five minutes to create their own "ugly" equation, but the solution must be 2. The students are asked to use CAS to experiment and to verify the solution. After the five minutes, one student for each group presents their equation to the class, and the class votes to determine the "ugliest" equation.

The next section of the lesson is introduced by the teacher. The students will continue working on making the "ugliest" equations, but the focus is now, given an equation with solution two, to find, develop, and describe techniques that complicates the equation further, but the solution is still two. The students are given a series of equations that all have solution two, but they are free to start with any equation that has solution two, see example in Figure 2. The students are asked to use CAS to experiment and verify that the equations have solution two.

Figure 2. An extract of the worksheet for developing methods for complicating equations.

Old equation with solution x=2	New equation with solution x=2
$3x–4 = \frac{(4x–2)}{3}$	
Method:	

The students get 15 minutes to develop their methods. It is expected that the students will develop techniques such as adding a number to both sides of the equation, multiplying both sides of the equation with a non-zero number, adding an expression equal to zero on one side of the equation, multiplying one side of the equation with an expression equal to one, etc. The teacher then orchestrates the students' sharing of different methods while taking notes on the board.

The last section of the lesson involves a new competition, in which the students have five minutes to make the "ugliest" equation with solution 2. They are encouraged to try out the collection of shared methods. The students are asked to use CAS to experiment and verify that the equations have solution two. After the allotted time, a student from each group presents their "ugliest" equation, and the class choses the "ugliest" equation them all.

The teacher ends the lesson by repeating the learning objective, particularly relating the traditional algebraic technology for manipulating an equation, with the theory that the solution remains the same.

EMPIRICAL FINDINGS

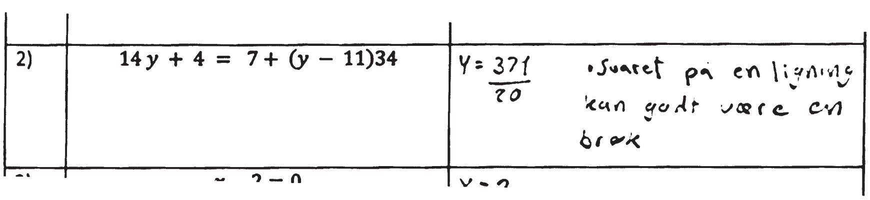
In this section, we will present the students' praxeologies based on the data collected. In lesson A, our focus is on how and what theory is developed and clarified through the lesson. The discussions cited and the replay of the lesson are based on voice recording transcripts, while the students' work presented are cuttings from their worksheets. In lesson B, we are interested in the techniques and the technology developed related to making equations that has solution two. We mainly use the transcripts to give a replay of the lesson, and the students' production noted on their worksheets to illustrate the students' technology related to the praxeology of retaining the solution for the equations.

Lesson A: Describe what is an equation

Though the lesson was carried out in three different classes, the students' praxeologies related to the learning objective of the lesson were similar. The most noticeable difference between the classes was the difference in time spent on introducing the use of CAS and Screencast-O-Matic. In our analysis, we will focus on the theory that the students developed related to the praxeologies of solving equations.

Of interest for our study of the lesson A are the episodes around solving new types of equations in CAS and revising the description of an equation. We will go through those parts of the lesson chronologically and relay and present episodes from our data.

In this section of the lesson, the students work in groups. We will now consider the students' work and discussions. Equation two, $14y + 4 = 7 + (y–11)34$, which has the solution $\frac{371}{20}$. For the first two classes who regularly solve equations with a non-integer solution and use calculators to perform the calculations, the output $\left\{ x = \frac{371}{20} \right\}$ did not generate any discussion regarding its viability as a solution. Those groups who do reflect on the output come to the conclusion that the fraction is a division that has not yet been performed, and therefore, should be written as a decimal number instead. Many of these groups went on to perform the calculation and write the first three to four decimal places of the number. In the third class, the solution $\frac{371}{20}$ prompted the first discussion in the groups. Initially, the students were not accepting a fraction as a valid solution for the equation. However, after either consulting the teacher or solving the equation by hand, the groups eventually accept a fraction as a possible output for CAS. During the discussions, some groups expressed that while CAS had successfully isolated x on the left-hand side of the equation, it had yet to complete the calculations on the right-hand side of the equation that would convert the fraction into a decimal number. One group disappointedly remarked that CAS cannot do everything for them. Subsequently each group added a statement like "The solution to an equation can be a fraction", see Figure 3.



2)	$14y + 4 = 7 + (y - 11)34$	$y = \frac{371}{20}$ •Svaret på en ligning kan godt være en brøk

Figure 3. Addition of a fraction being a possible solution to an equation to the description of an equation, cut out from worksheet.

The fourth equation, $(x–1)(x–2) = 0$, with two solutions, sparked the most discussion among the groups. After double-checking the input in CAS, it was generally accepted that CAS can solve the equation, and the (one) solution to the equation can be read from the output $\{x = 1, x = 2\}$. The disconnect between the students' view of equations having exactly one solution and the solution offered by CAS sparks varied discussions. One group proposes the hypothesis that you can choose between one or two, while another group that one and two must be added together, resulting in the solution of three. Yet another group has the following discussion about substitution and the role of the unknown in an equation:

Aza: But, it is x equal to one, and x equal to two!

Bab: Well, it [x] is the same [referring to the equation].

Aza: What do you mean? It is two different x's [referring to the output]. [Small pause]

Cala: Are there no multiplication sign in between [the expression (x-1) and (x-2)]?

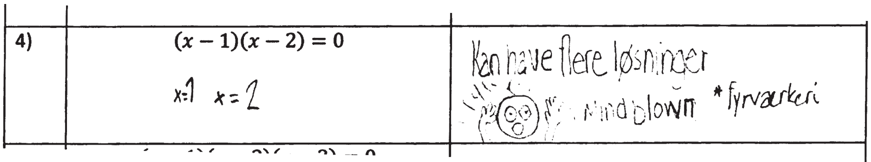
Aza: There is at least a multiplication sign there.

Cala: I think it makes sense. x minus one is one thing[equation], and x minus two is another thing[equation], and then they both must be solved.

Bab: But, both x's [in the equation] must be the same.

Cala: Well, yes, it is like that.

Most of the groups eventually conclude that an equation can have two solutions by substituting (by hand) x with a value, and then discovering that either the term $(x–1)$ or the term $(x–2)$ must be zero, realising thus that both 1 and 2 are solutions to the equation. As a result, the groups all wrote additions to their description of an equation, see Figure 4 for example.



4)	$(x-1)(x-2)=0$ x=1 x=2	Kan have flere løsninger Mind blown *fyrværkeri

Figure 4. Addition of «can have several solutions» to the description of an equation, cut out from worksheet.

We have cited the above part of a discussion as it demonstrates how the students are beginning to develop and articulate elements of theory for solving equations. Specifically, they are learning what the syntax is for writing equations, and what the role of the unknown is if it appears more than one time in the equation. They are also learning how to approach equations with multiple solutions how to use substitution in these cases.

In some groups, unforeseen inquiries into the theory continues. In one group, a student asks how one knows if an equation has two solutions or only one solution. In another group, they discuss whether the number of solutions for an equation can be two and a half.

The sixth equation, $2x+4 = (x–2)2+8$, which has an infinite number of solutions, is not an example that all groups reach. The groups that reach this equation will get the output $\{x = x\}$. In general, the students are not confused about the CAS output, but instead, they are confused about what the meaning of the expression $x = x$.

Xia: x is equal to x. That makes sense [ironic].

Yao: Then the solution is x -that is just super [ironic]. [small pause] I do not understand x equal to x, how can that [x=x] be?

Xia: I don't think we entered everything correctly.

Zhou: We entered correctly.

Yao: What does it mean: x is equal to x [directed towards the teacher]?

Teacher: That is a very good question.

Yao: We entered correctly.

Teacher: Yes, you did. But there is something very interesting here.

…

Xia: It [CAS] wrote that the x, that the unknown is equal to the unknown.

Teacher: So, what numbers can you substitute with x in order to solve the equation?

Xia: All.

Teacher: That is an interesting observation.

Yao: So, no matter if we substitute four or seven, then it would be correct.

Teacher: So, how many solutions does the equation have?

Zhou: Infinite.

Teacher: Okay Zhou, that is an interesting observation, which you might want to note on your paper.

Yao: And x is equal to x [writes aloud while she writes]

Zhou: Does it also apply for negative numbers?

Xia: I don't think so. We can ask our teacher.

Zhou: Teacher, does it also apply for the negative numbers?

Teacher: Try. What happens if you substitute with a negative number?

…

We have chosen this extract from one of the groups' transcript, as it illustrates not only the formulation of a new type of equation (containing an unknown) with an infinite many solutions, but also prompts students to reflect on and formulate further elements of the theory block.

Equation seven, $r–23 = (3r–4)2–5r$, is an equation with no solution, which GeoGebra prints as the output { }. The groups of students who have reached the equation conclude that a solution does not exist since the curly parenthesis that beforehand have encompassed the solution(s) are now empty. In one group, before reaching the conclusion that the equation has no solution, the students discuss, whether the unknown, r in this case, can be substituted by a curly parenthesis:

Sekai: What does it say?

Tabia: It just says the wrong [type of] parenthesis.

Sekai: Okay, we interpret it as if there should be a parenthesis on both.

Tabia: On both?

Sekai: Yes, because we must keep the balance [of the equation].

Tabia: No, that must be wrong, it [the solution] cannot be parentheses.

Sekai: If this is an equation, then there was a parenthesis here, and it is really heavy, then you also need a parenthesis here [on the other side of the equation], so it becomes equally heavy. Do you get it?

Ronja: No!

It is finally dismissed with the argument that a parenthesis is not a solution for an equation.

This bit of discussion between the students shows that questioning a part of the theory leads to further exposure of the theory, such as the view that an equation is a seesaw where the equilibrium must be maintained and a clarification of what is possible and not possible to substitute with the unknown. Furthermore, the students implicitly raise the question of what do you substitute the unknown with if there is no solution to the equation?

The eighth equation, $4(x–436) – 326 = t–6434$, is an equation with two different variables, and the output of CAS is $\left\{x=\frac{1}{4}t–1091\right\}$. The groups that reach this equation do not seem perplexed with now having a solution for an equation that contains an unknown and relates the solution to previous obtained solution type with infinitely many solutions. In one group, a student remarks, “x is equal to one fourth t [stops mid-sentence]. It is that thing where x can have one or several unknown [as a solution]. Next!” Later, she reflects, “I had not considered that one could put an equation in relation to, err, an unknown in relation to another unknown”.

The ninth equation, $2(3–4) + 7 = 5 – 3(4 – 2) + 18$, is an equation with no unknowns, and it is true. If each side of the equation is simplified by calculations, it will yield 5=5. The output of CAS is $\{x = x\}$. The groups of students who reach the equation immediately dismiss the expression as an equation since it does not contain an unknown: “There is no x -that is not an equation!”.

The next episode of interest is the last episode of the lesson, where the students share their findings, and the teacher reformulates the students’ findings and writes key phrases on the board. Following this, the teacher outsets further discussions and clarifications. In one of the lessons, the output $\{x = x\}$ is further discussed. During the students’ presentation of the implications of this type of output, i.e., an equation can have an infinite number of solutions, it is implicitly understood that the solution type is considered a number. The teacher relates this type of equation to equation eight, where the solution contains an expression with a second unknown, and asks the students if, perhaps when $x = x$, the solution can also be an expression containing an unknown.

The curly parentheses are also one of the objects that the teachers consider. After the students have had their try at guessing the meaning of the curly parentheses, the teacher explains that the curly parentheses are usually used to denote a set. The students have not only encountered equations with more than one solution but are also taught how to write such solutions.

Equation nine and ten, $2(3–4) + 7 = 5–3(4–2) + 18$ and $3(23–11) = \frac{1}{5} + \frac{22(3–5)}{2}$, are also brought to the attention of the students since they do not include an unknown. The students all agree that they do not qualify as equations, even though CAS can solve them with an output of $\{x = x\}$ or $\{\ \}$. In one of the lessons, the teacher suggests that they can be called statements, and these statements can either be true or false, just as the so-called equations can either be true or false.

In the last episode of the lesson, the students formulate elements of their theory. The teacher then further questions the theory, and together, the students and the teacher develop and formulate theory regarding equations, as well as other related subjects.

Overall, the students developed and clarified a wide range of theory related to solving equations, including rational or decimal numbers as solution, equations having several or infinitely many solutions, and the limitation of parenthesis as a solution. They also established syntax for equations i.e., whether there can exist 2-in-1 equations and syntax relating to parenthesis. In addition, the students determined the rules for substitution in equations with multiple solutions. At the end of the class, the teacher institutionalised theory related to what can a solution for an equation be/ not be.

Lesson B: Make an equation where $x = 2$

We will now study the empirical findings for lesson B. In our analysis, we will focus on the techniques and the technology that the students developed related to the tasks of making and further developing equations with a solution of two. As the problems are open, there are many possible ways of attacking the task. Although the lesson was carried out three times, the students' praxeologies developed in the lesson were similar. We will present our findings in chronological order, i.e., first the task of making an equation with a solution of two, and then the task of further developing an equation with a solution of two while maintaining the same solution.

The first section of interest is the students' first try to create an equation with a solution equal to two. One group of students suggest three main techniques. The first main technique:

Irene: Can't we just make a random equation and then compensate?

Leo: What! Oh, you mean like just add minus [a number] at the end.

…

Leo: We just copy the one up there [from the board] but we make it longer.

…

Irene: I have got the greatest idea ever! Can we just do like last lesson, where x is equal to x. Infinitely many answers.

Joules: How do you get x equal to x?

Leo: Then you have to make both sides [of the equation] such that when reduced they are the same.

The group of students has formulated the three main techniques for this section of the lesson. We will refer to them as π_1, which is the main technique for making a "random" equation, checking the solution with CAS, and then compensating until the solution is two. π_2, which starts with an equation with a solution of two, complicating the equation further, and checking the solution with CAS. π_3, which is making an equation that is always true by making the right-hand side equivalent to the left-hand side.

The group cited above ends up trying out π_1. After entering a "random" equation, they get the output from CAS $\{x = 4\}$. This prompts the discussion on how to compensate such that the solution is two.

Leo: What if we divide it [the equation] with two?

Irene: Then it would be one-half x equal to two, that would still give the result x equal to four. If we want two less on the right side perhaps, we should either add or subtract two up here.

The work and discussion of the group show the variety of techniques the students hold, their formulation of their technique, and the choice of technique. Even more specific technique of how to manipulate a simple equation with solution of four such that the solution becomes two.

All the groups work with similar or variations of the above mentioned three main techniques.
Another group that works with π_3 uses the technique of making two expressions that are equal to five, and then place the expressions on opposite sides of the equal sign so that the equation has infinitely many solutions. The result is $\dfrac{4x + x - \frac{1}{2}x}{x + \frac{1}{2}x} + x = 3+x$.

Another group employing π_3 uses the technique of adding and subtracting the same on both sides of the equation.

Gerð: We use a lot of difficult numbers, and then we subtract them in the end.

Fróði: Minus one thousand-seven-hundred-and-sixty-one minus something plus x minus seven-hundred-and-eleven. Then x is equal to x!

Gerð: I think we should use pi.

…

The discussion among the students and the results of their work show that the students have the theory that equations with infinitely many solutions exist and how such equations can be made with the technology of making an equation where the right-hand side of the equation is equivalent with the left-hand side.

Although some groups have discussed π_2, further developing an equation with solution two, none of these groups choose to work with this strategy.

In the next section of the lesson of interest, the classes are tasked with π_2. They are given a list of equations, all of which have a solution of two, to make more complex. Our main interest is not the ugly equations produced, but rather the techniques and technology that the students develop and employ. We will first recount the main techniques and technologies developed and formulated by the students, and then recount some of the theory that was discussed and clarified.

We will present two of the techniques developed and formulated by the students. The first technique, which we will denote by τ_1, is to divide a part of the equation by an expression that is equal to one when x is equal to two, i.e., $3–x$. In the following transcript, the technique is developed, and the expression $3x–4$ is rewritten into $\frac{3x–4}{(x^3–2x)–3}$.

Margrethe: Perhaps we can make a fraction here, and then make some weird expression [as the denominator] that is just equal to one … Then we say four minus two, all we need is for it to equal one. x and we have two, we need [pause]. Here we have two x, thus we need four, minus x squared. Here we have got. No. x to the power of three, then we have minus four, then we need to subtract with three on the other side somehow [$(x^3–2x)–3$]…That is a very overcomplicated way of writing one.

As a method the group writes “overcomplicating by dividing with one”.

The other technique, which we will denote as τ_2, was developed, and clarified by the students. It is to add the same number on both sides of the equation. A group has the following discussion:

Emmy: We must make this one ugly.

Elliot: You can change this to three and then minus thee here [referring to the other side of the equation sign in the equation].

Emmy: Plus six, plus two.

Elliot: [interrupts] You are doing it wrong. You cannot put plus on each side [of the equal sign] because you have to put a minus. [Directed to the teacher] Isn’t it true that if you put a plus on one side, then on the other side is has to be minus?

Teacher: Hmm, you have to keep the balance of the equation. If I have an equation and I make one of the sides heavier, then in order of keeping the balance, the other sides must also get heavier. I.e. if I add something on one side then I need to add the same thing on the other side.

Elliot: Does this make sense? [shows the teacher her equation]

Teacher: Test it [with CAS]

…

Elliot: Okay, we add plus three and twenty on both sides. And minus nineteen on both sides. And plus four on both sides.

...

The extracts show the two techniques, τ_1 and τ_2, developed, formulated, and employed by students. τ_1, dividing a term of the equation with an expression that is equivalent to one, and τ_2, adding the same number on both sides of the equation. The two techniques developed and formulated play a fundamental role in school algebra, as they are often the key to simplifying algebraic expressions or equations. In addition, they are pivotal for the application and handling of school algebra, such as simple formulas, in other subjects and further education.

Although the focus of the activity was on developing techniques and technology, discussions and clarification of elements of the theory also emerged. The main theory present was that the solution must stay the same, as it was the condition for rewriting the equations. During the lesson, the students often referred to another main theory, which is that an equation can be seen as a seesaw, and that the seesaw must be kept in balance, i.e., if you add a number to one side of the equation, you must add a number or an expression that is of equal weight on the other side of the equation.

We will now present some more of the technology and theory that emerged in this section of the lesson. In one group of students that were employing τ_1, a student wanted to divide by zero. In another group, also employing τ_1, a student asked what happens if you divided by x. As time was of the essence, the respective groups did not engage in theoretical discussion but refocused on the task at hand and developing techniques for which the technology and theory were already part of the students' inventory.

The activity of handling equations all with solution two seemed to generate, in several groups, a questioning about whether the equations were all the same. After seeing the teachers' two examples of equations with solution equal to two, a student asks

Uliuk: How can they all give the same result? Such a long equation can give the same result as that one.

Vaaltimaat: That is the question!

Teacher: I think that is a very good question!

Uliuk: Then you can just write that [the first equation] instead of all that [the second equation]. That [the second] one takes a lot longer to reduce.

In the second section of the lesson, other students commented that it did not feel like they had made new equations, as it was a rewriting of the first equation. This happened in particularly if the students had done simpler manipulations, such as rewriting $4x$ into $3x + x$ or $2x$ into $x + x$.

The extracts show that elements of theory are discussed and clarified, but they also inspire further questioning of the theory, which could lead to entire new types of praxeologies such as when are two equations the same? The students are becoming aware of equations as an object of study.

In the next part of the lesson, the students share their methods while the teacher reformulates and writes them on the board. The techniques shared are: multiplying a term of the equation with an expression that is equal to one when x is equal to two; dividing a part of the equation with an expression that is equal to one when x is equal to two; adding an expression that is equal to zero when x is equal to two, to one side of the equation; adding the same number or equivalent expressions on both sides of the equation; and multiplying with the same number or expression on both sides of the equation. In two out of the three classes involved in the project, the teacher ends the sharing of methods by emphasises the connection to the theory that the solution of the equation stays the same. In the third class, the teacher ends the sharing of methods by reflecting that many of the methods developed for uglifying an equation, are the same methods one could use to simplify an equation. Many of the discussions on techniques, technology, and questioning of theory is not shared, and neither is a relation to Lesson A made in this lesson.

The students' sharing of their developed methods shows the variety of the different technologies present in the lesson and the similarity to the technology for solving equations. Thus, embedding the theory element that the solution is maintained into the praxeology for solving equations.

In the last attempt at making an ugly equation with solution two led to the emergence of another piece of theory concerning the equal sign. In one group the necessity of the equal sign in an equation is questioned. However, after wondering how to solve such an equation, or how to check if the solution is two, they agree that the equal sign is a requirement for an expression to be an equation.

The recount of the group's discussion shows that the students are once again considering equations as an object of study, and though the question of whether or not the equal sign is a necessity for an equation was raised in the first lesson, the discussion in this account includes technology.

CONCLUSION

In this section, we will try an answer our research questions regarding the potential of CAS and the notion of praxeology as a tool for designing tasks.

The use of CAS in the activities allowed for the students to approach new tasks (more exotic types of equations) that would otherwise have required time-consuming algebraic work such as solving equations. The new tasks involved themes within school algebra that strengthened, developed, and clarified both technology and theory. In lesson A, the students worked with describing what constitutes an equation. The students developed, formulated, and clarified a wide range of technology and theory involving equations and related topics such as substitution, thus strengthening an entire series of related praxeologies. In lesson B, the students worked on establishing a stronger connection between the theory of maintaining the solution of an equation with the technology and techniques used for manipulating equations. Thus, sustaining a grander logos for school algebra. However, lesson B did not solve all problems related to the praxeologies of solving equations. During the lesson, both students and teachers mentioned the metaphor of an equation as a seesaw, where the balance must be kept, as a valid argument for justifying techniques. There were instances where this magic trick could have been questioned, such as when a student wanted to divide by zero or when one student wanted to take the square root of both sides of the equation (without following using the absolute value), which could possibly change the solution space. But it did not happen.

By outsourcing time-consuming algebraic work, CAS can be used to introduce new types of equation i.e., equations with more than one solution, making it possible to study equations as an object and not just using equations as a tool. This in turn developed both elements of the students' theory and technology such as in lesson A. In lesson B, the use of CAS provided students with a technique for verifying the solution of their equations, allowing students to experiment with creating and developing the equations. This led the students to develop and formulate not only an abundance of technology, but also formulate and clarify elements of theory.

The students' work with the logos further prompted them to question, formulate and discuss elements of theory, such as rules and the role of substitution (from lesson A) or what happens if you divide by the unknown, thus further enhancing the activity.

As both lesson A and lesson B contained many questions inquiring further into elements of technology and theory, we conjecture that the activities could be further enriched by not being constrained by a lesson plan. This would allow the students to direct the path of inquiry and development of technology and theory. In addition, the object of lesson B, which is to further develop and explicitly state the relation between the techniques and technology (of manipulating equations) with the theory (of the solution staying the same), the institutionalisation must be further emphasised.

The use of the notion praxeology, particularly the four notions of task, technique, technology, and theory, allowed us to analyse the prevailing paper-and-pencil praxeology for solving an equation, and thus select the themes "the students' definition of an equation", and "the relation between algebraic manipulations of an equation, and maintaining the same solution". Both themes are within school algebra, thus the designed activities would strengthen the students' logos by further developing, formulating, and clarifying it.

The notion of praxeology lets us further study the lever potential of CAS, enabling the design of future activities more easily. In lesson A, the lever potential can be described as the development and formulation of the students' concept of equation, which is an element of the theory block. This characterisation can also be used to describe the potential of CAS studied in the thesis by Drijvers (2003). In lesson B, the lever potential is the strengthening of the relation between the theory of maintaining the same solution, with the techniques and technology for traditional algebraic manipulations. In the article by Hitt and Kieran (2009), the lever potential can be described as the telescoping technique (using paper and pencil) being used not only as a technique but also as technology for developing, justifying, and validating conjectures. The use of the notion praxeology enables us to describe the students' learning explicit and justifies the lever potential of CAS.

As a final note, we would like to reflect on how these more theoretical CAS-based activities can upset the traditional epistemological model for school algebra and thus necessitate new design tools for activities. Traditionally, second degree equations with two, one or no solutions are introduced by their graphical representation at the end of lower secondary school. However, a more inquiry-based teaching approach combined with CAS will naturally lead to question what an equation is, and following an examination of a series of examples. These examples would feasibly include second- and higher-degree equations as the strength of CAS is to handle more exotic examples than is possible in a pen-and-paper-only environment. A change in the traditional curriculum when implementing CAS is not unheard of (Kendal, Stacey, & Pierce, 2005) and perhaps even necessary. This would, however, only stress the need for design tools to support teachers in analysing and crafting CAS-based activities.

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