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AIMS AND SCOPE

Hiroshima Journal of Mathematics Education (HJME) is the official English-language journal of the Japan Academic Society of Mathematics Education (JASME), and is devoted to research that seeks to improve the mathematics education. It is an annual peer-reviewed and open-access international journal that publishes original papers written in English. HJME was founded by the Department of Mathematics Education at Hiroshima University. Articles have been published by JASME since Volume 12 was issued in 2019.

The journal is dedicated to the dissemination of research findings on mathematics education at all levels (e.g., from preschool to university, professional development, lifelong education) on a variety of issues related to the teaching and learning of mathematics (e.g., students' understanding, classroom teaching, curricula, policy, teacher education). It is open to any type of research (e.g., theoretical, empirical, methodological, qualitative/quantitative). Papers dealing with Japanese mathematics education issues or issues of special interest to the Japanese mathematics education community are particularly welcome.

Paper submissions are welcome from contributors from different sectors (e.g., researchers, educators, and practitioners) who are interested in contributing to the development of mathematics education and its research. HJME is expected to promote reflection on mathematics education and communication among international researchers, educators, and practitioners in mathematics education. Papers are published in either of two HJME sections: original paper and lecture notes. The original paper reflects the main topics of the journal. All papers submitted to this section are peer-reviewed, with an emphasis on papers of an excellent level. Lecture note includes records or materials of scientific events (e.g., invited talks, symposia) organized by JASME

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Editorial for Volume 16

Tatsuya Mizoguchi

Editor-in-Chief of the *Hiroshima Journal of Mathematics Education*,

Japan Academic Society of Mathematics Education

Domestic and international conferences have gradually reverted to face-to-face mode. Again, we enjoy the pleasure of meeting the faces that we used to encounter on a screen. Discussions in each community naturally produce positive research outcomes. In many fields, the number of papers submitted has increased. We look forward to your contribution to the Hiroshima Journal of Mathematics Education (HJME).

As each journal has its own peer review system, HJME operates on the system shown in Figure 1 (the actual process is more complex, but the necessary procedures for authors are simplified).

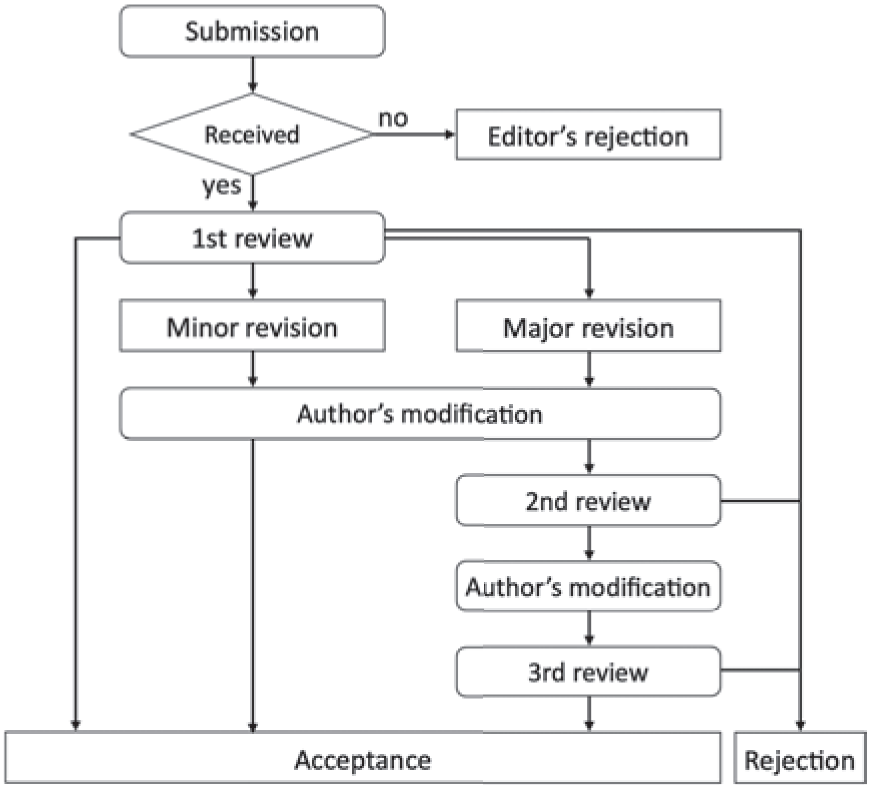

Figure 1. HJME peer review system

Along with peer review systems, the type of peer review is often discussed. This journal currently employs a double-blind peer review process. Each type of review has its own advantages and disadvantages. In general, double-blind peer review prevents any type of bias or mutual pressure. Not knowing the author's background emphasizes the content of the paper rather than the assessment of the individual; whereas, excluding knowledge of the context of the research topic makes it difficult to obtain the necessary information to peer review.

The Special Issue of this volume has been peer-reviewed independently of the journal. An open-review design was adopted. Open-review is considered to ensure transparency in the review process and make the tension for reviewers, while simultaneously maintaining authors' satisfaction and restraining negative criticism. In recent peer reviews of applications for presentations at international conferences, the open-type has often been adopted. This creates a community around the research topic and enriches discussions during conferences. Peer review systems and review types are important issues at the heart of journal operations. At every opportunity in the future, we, the editorial team, would like to have the courage to be proactive in making improvements.

In addition, because HJME is an online journal, the editorial team discussed copyrights. Consequently, we decided on a Creative Commons [CC BY] licence, starting with Volume 16 to maintain alignment with other international mathematics education journals. We thank the authors whose contributions appear in Volume 16 for their agreement in this regard.

Volume 16 features one contributing paper and five Special Issue papers. The contribution by Louise Meier Carlsen analyses activities related to pupils' study of equations using CAS with praxeology from the anthropological theory of the didactic. It draws the interesting conclusion that students learn to use equations not only as tools but also as objects, and that this has a positive effect on technology and theory, that is, the logos part of praxeology.

The introduction of Special Issue papers will be in the Editorial for Special Issue by the guest editors. The HJME editorial team would like to thank Angelika Bikner-Ahsbahs, Ivy Kidron, Yusuke Shinno and Takeshi Miyakawa for their great contributions as guest editors.

Finally, as I repeat the same thing every time, please note that HJME is published annually. The accepted papers are published online before the publication of the next volume. Submissions are accepted at any time. We look forward to receiving your contributions.

HIROSHIMA JOURNAL OF MATHEMATICS EDUCATION

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DESIGNING ACTIVITIES FOR CAS-BASED STUDENT WORK REALISING THE LEVER POTENTIAL

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Abstract

This paper explores two types of lever potentials of CAS and a tool for designing such activities. In the first activity, CAS is used for students to study the concept of equations. In the second activity, CAS is used to strengthen the relation between traditional algebraic paper-and-pencil manipulations of equations and the theory that the solution of the equation must remain the same. As the design tool for CAS-based activities that capitalise on the potential of CAS, the notion of praxeology from the Anthropological Theory of Didactical is employed.

Keywords: Computer algebra system; school algebra; task design; lever potential; the notion of praxeology.

INTRODUCTION

In the early days of research on implementing CAS in mathematics education, the entrance of this digital tool was met with applause by the community of researchers, and the focus was on the potential that CAS could offer mathematics education (Dreyfus, 1994). The idea is, that, at the level of upper secondary school, 'low level' work, such as solving an equation, can be outsourced (Bang, Grønbæk, & Larsen, 2017) to the CAS. This would leave more time to focus on more complex issues such as the mathematical discourse. This potential is denoted the lever potential (Winsløw, 2003).

The lever potential has been documented in several articles since the 1990s (which we will further elaborate on in the background section). However, using a CAS on any problem in the subject of algebra does not automatically lift the students' learning to a higher level, quite the contrary. When solving a traditional algebraic task, such as finding the solution for an equation (Kieran, 2007) with paper and pencil, the mathematical discourse of the students includes the use of fundamental properties of algebraic structures, such as the distributive rule. When solving the equation with a CAS, the mathematical discourse becomes very different, and most fundamental properties of algebraic structures are blackboxed (Carlsen, 2019).

In our study, we will consider the lever potential of CAS and explore the aspect, of the lever potential, of how CAS can be used to make the mathematical discourse explicit in the teaching of elementary algebra. Furthermore, we will study how the four categories of task, technique, (technical) discourse and theoretical discourse can be used to design appropriate activities in elementary algebra for the implementation of CAS.

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BACKGROUND

In this section, we will introduce the lever potential of CAS for the teaching of school algebra, and outline results from some of the studies depicting the lever potential of CAS. In addition, the role of CAS, in the selected papers, is to perform part of the mathematical tasks such as time-consuming algebraic manipulations, i.e., solve an equation or simplify an expression.

The lever potential of CAS was first introduced by Dreyfus (1994) and later further elaborated and named, as the lever potential, by Winsløw (2003). The idea is

"for students to operate at a high conceptual level; in other words, they can concentrate on the operations that are intended to be the focus of attention and leave the lower-level operations to the computer." Dreyfus (1994) p 205.

That is, instead of focusing on applying a series of techniques e.g., to solve an equation, students can instead progress to focus on more theoretical aspects such as technical discourse or theoretical discourse e.g., the number of solutions in relation to the type of equation.

One type of lever potential can be described as using CAS to swiftly consider multiple representation of a mathematical object. Then each different representation will accentuate different aspects of the mathematical object (Dreyfus, 1994; Gjone, 2009; Kieran & Yerushalmy, 2004; Gyöngyösi, Solovej & Winsløw, 2011). Thus, supporting further development of the students' concept of the mathematical object.

Another type of affordance of CAS is the ability to swiftly examine a series of comparable examples. This enables students to consider similarities and differences between the examples and explicitly develop and formulate their concept of the mathematical object in focus. Several papers have studied such possible activities. For example, the doctoral thesis by Drijvers (2003) is a testimony to how carefully crafted CAS-based activities can support students' explicit development and formulation of their "definition" of parameters. As an example, Heid et al. (2002) suggests working with the formula $A(time) = deposit\ (1 + rate)^{\text{time}}$ to have students develop and formulate their concept of parameters. Other suitable mathematical topics have also been explored such as algebraic syntax (Artigue, 2002b; Guin & Delgoulet, 1997) and algebraic equivalence (Gjone, 2009; Lagrange, 2005) has also been conducted.

A different type of lever potential of CAS can be described as, despite outsourcing algebraic manipulations to CAS such as expanding an expression. The pen-and-paper techniques can be used as arguments explicitly stated for validating the patterns produced and discovered with CAS. In a series of papers, students work through a course of carefully constructed problems exploring the factorisation of the polynomial x^n-1 using CAS and developing the telescoping technique (Kieran & Drijvers, 2006; Hitt & Kieran, 2009; Martínez, Kieran, & Guzmán, 2012).

The research mentioned, illustrating the different lever potentials, all have in common that the activities are very carefully designed with very specific purposes in mind. In the last articles mentioned, the activities are designed utilizing the dialectic between practice and reasoning (Artigue, 2002a). This dialectic is viewed as a structural model using the three notions of task (the task to be solved), technique (that is applied to solve the task), and theory (the reasoning that explains and validates the technique). The idea is taken from the Anthropological Theory of Didactic (ATD). However, ATD suggests to further consider the theory part of the dialectic as consisting of two parts. One part is technology, which is the discourse that explains and justifies the technique. The second part is theory, which is the discourse that justifies and validates the technology. This structural division can further benefit the design of CAS-based activities.

In our study, we want to show how the quartet-notion of praxeology can be used to design activities that realise the lever potential of CAS. The fist activity will focus on developing and having the students formulate their definition of equations, while the second activity will include traditional algebraic technical discourses, such as manipulation of equations, and link this to the theoretical discourse of having the solution stay the same.

FRAMEWORK

In general, the ATD suggests considering human activity, such as cooking a soup, as an amalgam of practice and reasoning (Bosch & Gascón, 2014; Chevallard, 1998). To cook soup, one could start with frying onions in a pot (the practice), with the reasoning being that onions give a nice flavour, and frying them sweetens and enhances their flavour. Alternatively, one might be frying onions because it is generally perceived as a good way to start cooking a soup. Our knowledge is based on practice, and our practice is shaped by our knowledge.

The ATD proposes denoting the dialectic of practice and knowledge by *praxeology*. Further to structure the practice as a twofold. One part *the type of task*, which is the type of task that is aimed at being solved. For example, cooking a soup or solving an equation of the type $s(x–t) = n{\cdot}s–s{\cdot}m{\cdot}x$, where s,t,m and n are integers and $s \neq 0$. The other part is *the technique*, which is the actions utilized in order to solve the type of task, such as frying onions, or in the case of solving the equation above: rewriting the equation into $x–t = n–m{\cdot}x$, and then another rewriting into the equation $x = \frac{n+t}{1+m}$. The knowledge related to the practice, which is called *logos* from Greek meaning theory, can be split into two components. *The technology*, which is the discourse that justifies and validates the technique, such as “we multiply with $\frac{1}{s}$ on both sides of the equation” and “we can multiply with the same number on both sides of the equation”. The discourse can also contain motivation for the actions included in the technique, the effectiveness of the technique, etc. *The theory* justifies and explains the technology, such as the solution set must remain the same. Again, the theory also comprises of discourse such as the effectiveness of the technology and motivation for the technology. However, the praxeology taught usually differs from the praxeology learned. Indeed, theory, or even technology, does not need to be “formal mathematics” but can, to individuals and even large institutions, also include mere convictions and even falsities, such as “we must maintain the balance between the right-hand side and the left-hand side of the equation” and “when we move a number from one side of the equation to the other side, then the sign changes” or “it is correct because it is the same method our teacher illustrated”. In general, technology is close to the actions of the technique and can also include motivation for the actions of the technique, effectiveness of the technique etc. (Chaachoua, Bessot, Romo, & Castela, 2019). While theory is the technology of the technology.

The following is an example of what a typical praxeology for a grade 8 student could look like when solving the equation $3(x–2) + 7 = 14 + 2x$. The type of task is to solve the equation $n(x–m) + (n{\cdot}m+1) = t + (n–1)x$, where n,m and t are positive integers. The technique involves rewriting $n(x–m) \rightarrow nx–nm$ (applying the left distributive rule from left to right), rewriting such that all terms containing the variable are on the left side of the equation, etc. The prevailing technology would involve multiplying inside the parentheses, when multiplying a positive number with a negative number, the result is a negative number; whenever we encounter a parenthesis, we “dissolve” it, etc. The theory: an equation must be kept in balance meaning that each side of the equal sign must be equally heavy, the solution to the equation is a number such that each side of the equal sign in the equation has the same value, parenthesis induce a different order of operations, etc.

The notion of praxeology is useful because the four components -type of task, technique, technology, and theory- enable us to describe and analyse explicitly the elements in play in solving a type of task. This tool lets us consider structures and relations between praxeologies within and across educational institutions and mathematical themes, such as solving equations (of different types) or manipulating algebraic expressions. A praxeology is determined by its type of task (and the institution in which it lives). An action or an ingredient of a technique can be applied in several different praxeologies or several times within the same praxeology. For example, we can reduce an expression even when the task is not necessarily to solve an equation. One technology can unify several actions and even whole techniques. For instance, multiplying both sides of the equation by a number does not necessarily reduce the equation. The theory, such as we must maintain the balance between the right-hand side and the left-hand side of the equation, can join several technologies, since it is also the theory for adding on both sides of the equation. Therefore, we can classify a type of lever potential of CAS as to focus on (parts of) the technology or theory. By targeting the development of students’ technology or theory, this will strengthen related techniques and the scope of those techniques.

As hinted above, the praxeology taught and the praxeology learned differ, particularly in the logos block. This is because in a teaching and learning situation, there is not a direct transfer of knowledge and practice. However, the praxeology is transposed, and this transposition is conditioned and regulated within the institutional context. Therefore, we can refer to students’ personal praxeologies (Chaachoua, Bessot, Romo, & Castela, 2019). Studying the structure of students’ praxeologies can help us identify some of the difficulties they encounter in learning algebra. For instance, if we consider the two tasks of expanding the following expressions: $3(x–2)$ and $(x–2)3$, students often find the second expression much harder to expand than the first (the case of Danish upper secondary students (Poulsen, 2015)). For students at lower secondary school, the two praxeologies generated by the tasks are not related, not even by theory. This is because these students are not yet familiar with the formalised commutative property of the distributive rule, $(a + b)c = c(a + b)$, which would link the two praxeologies. This example emphasizes that if the technology and theory block is not made explicit in mathematics education, such as putting it into words or making it the object of study in a lesson, then many of the praxeologies are not linked, and thus, techniques will not be related to each other. This makes the more than 90 elementary algebraic techniques at lower secondary school (Poulsen, 2015) into a wilderness of rules.

If we consider the praxeologies of students at lower secondary school for solving equations, the use of CAS will dramatically change part of the praxeology (Carlsen, 2019). Without CAS, it would make sense to distinguish between solving the equations $(x–3)2 = 5·2–2x$, and $12–3x + 2 = –2$, because they require different techniques and technology to solve, such as multiplying inside the parenthesis. However, when solving with a CAS, the two equations do not require different techniques to solve, so the praxeologies would be the same. Furthermore, since the technique has changed into entering the equation, entering the command **solve**, and interpret the output, the technology has changed accordingly. However, a part of the theory block remains unchanged, namely, that the solution to the equation is a number such that when substituted with x, both sides of the equation yield the same number. By assigning technology and theory the same category and notion, as in previous studies (Hitt & Kieran, 2009; Kieran & Drijvers, 2006; Kieran & Saldanha, 2008) we might overlook possible information or design opportunities for CAS-based activities.

If we consider the lever potential of CAS through the lens of praxeology and ATD, it can be viewed as making explicit and formulating elements of the algebraic logos, including both technology and theory, through carefully designed tasks that can only be reasonably approached with CAS-based techniques. As this will strengthen the theory and/or the technology, it will benefit a range of algebraic praxeologies.

Research questions

- How can CAS be used to engage students to work with elements of the theory block for praxeologies in school algebra?
- How can CAS be used to strengthen students' technology related to standard techniques (such as rewriting equations) in school algebra?
- How can one design tasks that realise the use of CAS described in the previous research questions?

CONTEXT AND METHODOLOGY

To explore how the notion of praxeology can guide the design of tasks for teaching of school algebra realising the potential of CAS a qualitative approach has been chosen. We have conditioned the design of CAS-based activities to fit into two regular math lessons. It was decided that one lesson would focus on a theoretical aspect of solving equations, while the other lesson should focus on strengthening tradition pen and paper techniques for solving equations. A series of tasks was designed, and two lesson plans were created to describe the different didactical situations throughout the lessons. The two lessons were conducted in three Danish grade 8 (14 - 15-year-old) classes by their regular mathematics teachers. Prior to carrying out the lesson plans, a meeting was held with each teacher to familiarize them with GeoGebra, a program for recording of screen activities, and the learning objectives of the lessons. The first two participating classes were already familiar with using GeoGebra, though they had never used the CAS application, while the third class had no prior experience with GeoGebra, CAS tools, or other dynamic geometric environments.

To collect data for the study, seven recording devices were placed in the classroom to record the students' and the teacher's voices. Additionally, the students' work in GeoGebra was recorded with Screencast-O-Matic, software that records screen activities. Everything written on the blackboard was photographed, and field notes were written by the researcher during each lesson.

For each of the three lessons, one recording was chosen for a full transcript. For the remaining recordings, only episodes including not yet transcribed work or discussions were transcribed. All the transcriptions were analysed with the notion of praxeology, i.e., categorized by task, technique, technology, and theory. The students' written materials were also analysed using the notion of praxeology. The screencasts and written materials were used to support the transcription and analysis of students praxeologies.

THE LESSON PLANS AND THE RATIONAL OF THE DESIGNS

In this section, we will give a description of the lesson design based on the lesson plans written for the teachers, and part of the a priori analysis of the lesson. The description will be supplemented by the rationale behind the design. It is assumed that the students already have some familiarity with working with simple equations, such as manipulating them and describing simple relations using equations.

Lesson A: Describe what is an equation

The first lesson, we can, as in the Background section on the lever potential of CAS, describe the learning objective as to further develop the students' concept of a mathematical object, in this case: equations. The focus of the lesson is to make explicit and formulate what makes an equation, such as the possible solutions, and whether the equal sign or a variable can be omitted. It is not the purpose of the lesson that the students should develop a definition in par with scholars of mathematics, such as a statement that expresses an equality between two expressions, but rather develop and formulate a definition that is appropriate for the institution and tradition of lower secondary school.

The first task given to the students in the lesson is to write down an equation of their own and then try to describe what an equation is. An important aspect of this activity is that the teacher does not provide the students with an example of an equation earlier in the lesson or interfere with the students' work. The use of CAS is not intended for this activity. The expectation is for the students to give a somewhat vague definition, mentioning an unknown, and that "the unknown must be found".

Next, the students are given a series of equations to solve in the CAS window of GeoGebra, while recording their screen work with Screencast-O-Matic. For the series of equations containing different types of equations, see Figure 1. For each equation that is solved with CAS, the students are asked to reconsider their description of an equation.

Figure 1. A minimized version of the students' worksheet for solving equations with CAS including the a priori analysis of added description.

	Equation	Addition to the description of what an equation is
1)	$14\ a + 2 = 72$	An equation can have a fraction as a solution
2)	$14\ y + 4 = 7 + (y–11)34$	Unknowns can be named other than x, and have a fraction as a solution
3)	$x–2 = 0$	
4)	$(x–1)(x–2) = 0$	An equation can have two solutions
5)	$(x–1)(x–2)(x–3) = 0$	An equation can have three solutions
6)	$2x + 4 = (x–2)2 + 8$	An equation can have infinitely many solutions
7)	$r–23 = (3r–4)2–5r$	An equation can have no solution
8)	$4(x–436)–326 = t–6434$	An equation can have an unknown as part of a solution
9)	$2(3–4) + 7 = 5–3(4–2) + 18$	Equations must contain an unknown
10)	$3(23–11) = \frac{1}{5} + \frac{22(3–5)}{2}$	Equations must contain an unknown
11)	$3t + 12 = 3(t+4) = 4(t–2) + 20–t$	An equation must only contain one equal sign
12)	$12(3–7) + 4 + y$	An equation must contain an equal sign

The first equation is intended to help the students become familiar with entering and solving an equation using CAS in GeoGebra. The second equation yields the output $\left\{y=\dfrac{371}{20}\right\}$. This equation was included because we have observed in other schools that many students implicitly expect an integer solution, particularly if students are primarily working in a traditional paper-and-pencil environment. Therefore, this type of equation is intended to serve as a gentle transition into questioning and further developing the logos block for solving equations. The solutions for the fourth and the fifth equations are 1 and 2, and 1, 2, and 3 respectively, the output from CAS is $\{x=1,\ x=2\}$ and $\{x=1,\ x=2,\ x=3\}$ respectively. The equations are constructed in a way that makes it easier for the students to test or find the solutions by trial and error by hand, in order to relate the output of CAS to the solutions for the equations. It is expected that the students will add that it is possible for an equation to have multiple solutions to their description of an equation. Equation six is true for all values of x, which CAS writes $\{x=x\}$, while equation seven does not have a solution, which CAS writes { }. It is expected that the students will spend a considerable amount of time interpreting the output of CAS but will ultimately conclude that equations can have an infinite number of solutions or no solution at all. Equation eight has the solution $\frac{1}{4}\,t–1091$, which is to expose the students to a type of equation where the solution contains a parameter. However, we do not expect the students to know the difference between a parameter and a variable since it is not an explicit part of the curriculum. Despite this, the difference between parameters and variables could be a topic of study for a future CAS-based lesson. We await that the students will conclude that a solution for an equation can contain another variable. It is expected that the students will add that a solution to an equation can include an expression that contains an unknown. Equations nine and ten are equations without an unknown, and CAS gives the output { } and $\{x=x\}$ respectively. We expect that all students, in their first description of an equation, will mention something about an unknown, and thus equation nine and ten are included to prompt a discussion of whether an unknown is a requirement for an equation. It is also expected that the students will not regard the expressions as equations. These examples are given as formulating a definition requires considering the limitations of the object. Expression eleven and twelve contain two or no equal sign, respectively, and CAS gives the outputs $\{t = t\}$ and $\{y = 44\}$, respectively. The expressions have been included in the hope that the students will discuss the role of the equal sign in an equation. It is expected that, due to CAS being able to give a solution for both expressions, the students will question the role of the equal sign.

For the next section of the lesson, the teacher conducts the students' sharing of their discoveries of additional descriptions and clarifications, followed by the teacher's reformulation and clarification of the descriptions. In the last section of the lesson, the teacher further challenges the students' description, i.e., asking if an equation can have four and a half solution. This is done in case any of the students have written that equations can have any number of solutions, and that "number" must be specified as positive integers. Another follow-up reflection questions is if it is fair that you can manipulate an equation (with an unknown) into something that is not an equation (equation without an unknown). This question is asked to challenge the perceived concept of equations in lower secondary school as having an unknown.

Lesson B: Make an equation where $x = 2$

For the second lesson, the learning objective is for the students to strengthen their praxeology for solving equations, in particular, the relation between the techniques for manipulating an equation, such as adding a number to both sides of the equation, and the theory that the solution must remain the same.

The lesson starts with the teacher telling the students that today they will have a competition about making “ugly” equations. He then proceeds to write his two contributions on the board $\frac{2(x+4)}{3} - \frac{7x+2}{4} = \frac{3x+2}{2} - \frac{33x-18}{12}$ and $3x–4 = \frac{4x–2}{3}$. He further shows how to solve the two equations in CAS, and that both equations have the solution 2.

Then, the students are given five minutes to create their own “ugly” equation, but the solution must be 2. The students are asked to use CAS to experiment and to verify the solution. After the five minutes, one student for each group presents their equation to the class, and the class votes to determine the “ugliest” equation.

The next section of the lesson is introduced by the teacher. The students will continue working on making the “ugliest” equations, but the focus is now, given an equation with solution two, to find, develop, and describe techniques that complicates the equation further, but the solution is still two. The students are given a series of equations that all have solution two, but they are free to start with any equation that has solution two, see example in Figure 2. The students are asked to use CAS to experiment and verify that the equations have solution two.

Figure 2. An extract of the worksheet for developing methods for complicating equations.

Old equation with solution x=2	New equation with solution x=2
$3x–4 = \frac{(4x–2)}{3}$	
Method:	

The students get 15 minutes to develop their methods. It is expected that the students will develop techniques such as adding a number to both sides of the equation, multiplying both sides of the equation with a non-zero number, adding an expression equal to zero on one side of the equation, multiplying one side of the equation with an expression equal to one, etc. The teacher then orchestrates the students’ sharing of different methods while taking notes on the board.

The last section of the lesson involves a new competition, in which the students have five minutes to make the “ugliest” equation with solution 2. They are encouraged to try out the collection of shared methods. The students are asked to use CAS to experiment and verify that the equations have solution two. After the allotted time, a student from each group presents their “ugliest” equation, and the class choses the “ugliest” equation them all.

The teacher ends the lesson by repeating the learning objective, particularly relating the traditional algebraic technology for manipulating an equation, with the theory that the solution remains the same.

EMPIRICAL FINDINGS

In this section, we will present the students' praxeologies based on the data collected. In lesson A, our focus is on how and what theory is developed and clarified through the lesson. The discussions cited and the replay of the lesson are based on voice recording transcripts, while the students' work presented are cuttings from their worksheets. In lesson B, we are interested in the techniques and the technology developed related to making equations that has solution two. We mainly use the transcripts to give a replay of the lesson, and the students' production noted on their worksheets to illustrate the students' technology related to the praxeology of retaining the solution for the equations.

Lesson A: Describe what is an equation

Though the lesson was carried out in three different classes, the students' praxeologies related to the learning objective of the lesson were similar. The most noticeable difference between the classes was the difference in time spent on introducing the use of CAS and Screencast-O-Matic. In our analysis, we will focus on the theory that the students developed related to the praxeologies of solving equations.

Of interest for our study of the lesson A are the episodes around solving new types of equations in CAS and revising the description of an equation. We will go through those parts of the lesson chronologically and relay and present episodes from our data.

In this section of the lesson, the students work in groups. We will now consider the students' work and discussions. Equation two, $14y + 4 = 7 + (y–11)34$, which has the solution $\frac{371}{20}$. For the first two classes who regularly solve equations with a non-integer solution and use calculators to perform the calculations, the output $\left\{ x = \frac{371}{20} \right\}$ did not generate any discussion regarding its viability as a solution. Those groups who do reflect on the output come to the conclusion that the fraction is a division that has not yet been performed, and therefore, should be written as a decimal number instead. Many of these groups went on to perform the calculation and write the first three to four decimal places of the number. In the third class, the solution $\frac{371}{20}$ prompted the first discussion in the groups. Initially, the students were not accepting a fraction as a valid solution for the equation. However, after either consulting the teacher or solving the equation by hand, the groups eventually accept a fraction as a possible output for CAS. During the discussions, some groups expressed that while CAS had successfully isolated x on the left-hand side of the equation, it had yet to complete the calculations on the right-hand side of the equation that would convert the fraction into a decimal number. One group disappointedly remarked that CAS cannot do everything for them. Subsequently each group added a statement like "The solution to an equation can be a fraction", see Figure 3.

2)	$14y + 4 = 7 + (y - 11)34$	$y = \frac{371}{20}$	• Svaret på en ligning kan godt være en brøk

Figure 3. Addition of a fraction being a possible solution to an equation to the description of an equation, cut out from worksheet.

The fourth equation, $(x–1)(x–2) = 0$, with two solutions, sparked the most discussion among the groups. After double-checking the input in CAS, it was generally accepted that CAS can solve the equation, and the (one) solution to the equation can be read from the output $\{x = 1, x = 2\}$. The disconnect between the students' view of equations having exactly one solution and the solution offered by CAS sparks varied discussions. One group proposes the hypothesis that you can choose between one or two, while another group that one and two must be added together, resulting in the solution of three. Yet another group has the following discussion about substitution and the role of the unknown in an equation:

Aza: But, it is x equal to one, and x equal to two!

Bab: Well, it [x] is the same [referring to the equation].

Aza: What do you mean? It is two different x's [referring to the output]. [Small pause]

Cala: Are there no multiplication sign in between [the expression (x-1) and (x-2)]?

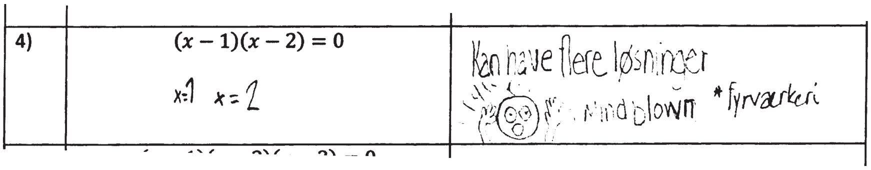
Aza: There is at least a multiplication sign there.

Cala: I think it makes sense. x minus one is one thing[equation], and x minus two is another thing[equation], and then they both must be solved.

Bab: But, both x's [in the equation] must be the same.

Cala: Well, yes, it is like that.

Most of the groups eventually conclude that an equation can have two solutions by substituting (by hand) x with a value, and then discovering that either the term $(x–1)$ or the term $(x–2)$ must be zero, realising thus that both 1 and 2 are solutions to the equation. As a result, the groups all wrote additions to their description of an equation, see Figure 4 for example.



4)	$(x-1)(x-2)=0$ x=1 x=2	Kan have flere løsninger Mind blown *fyrværkeri

Figure 4. Addition of «can have several solutions» to the description of an equation, cut out from worksheet.

We have cited the above part of a discussion as it demonstrates how the students are beginning to develop and articulate elements of theory for solving equations. Specifically, they are learning what the syntax is for writing equations, and what the role of the unknown is if it appears more than one time in the equation. They are also learning how to approach equations with multiple solutions how to use substitution in these cases.

In some groups, unforeseen inquiries into the theory continues. In one group, a student asks how one knows if an equation has two solutions or only one solution. In another group, they discuss whether the number of solutions for an equation can be two and a half.

The sixth equation, $2x+4 = (x–2)2+8$, which has an infinite number of solutions, is not an example that all groups reach. The groups that reach this equation will get the output $\{x = x\}$. In general, the students are not confused about the CAS output, but instead, they are confused about what the meaning of the expression $x = x$.

Xia: x is equal to x. That makes sense [ironic].

Yao: Then the solution is x -that is just super [ironic]. [small pause] I do not understand x equal to x, how can that [x=x] be?

Xia: I don't think we entered everything correctly.

Zhou: We entered correctly.

Yao: What does it mean: x is equal to x [directed towards the teacher]?

Teacher: That is a very good question.

Yao: We entered correctly.

Teacher: Yes, you did. But there is something very interesting here.

…

Xia: It [CAS] wrote that the x, that the unknown is equal to the unknown.

Teacher: So, what numbers can you substitute with x in order to solve the equation?

Xia: All.

Teacher: That is an interesting observation.

Yao: So, no matter if we substitute four or seven, then it would be correct.

Teacher: So, how many solutions does the equation have?

Zhou: Infinite.

Teacher: Okay Zhou, that is an interesting observation, which you might want to note on your paper.

Yao: And x is equal to x [writes aloud while she writes]

Zhou: Does it also apply for negative numbers?

Xia: I don't think so. We can ask our teacher.

Zhou: Teacher, does it also apply for the negative numbers?

Teacher: Try. What happens if you substitute with a negative number?

…

We have chosen this extract from one of the groups' transcript, as it illustrates not only the formulation of a new type of equation (containing an unknown) with an infinite many solutions, but also prompts students to reflect on and formulate further elements of the theory block.

Equation seven, $r–23 = (3r–4)2–5r$, is an equation with no solution, which GeoGebra prints as the output { }. The groups of students who have reached the equation conclude that a solution does not exist since the curly parenthesis that beforehand have encompassed the solution(s) are now empty. In one group, before reaching the conclusion that the equation has no solution, the students discuss, whether the unknown, r in this case, can be substituted by a curly parenthesis:

Sekai: What does it say?

Tabia: It just says the wrong [type of] parenthesis.

Sekai: Okay, we interpret it as if there should be a parenthesis on both.

Tabia: On both?

Sekai: Yes, because we must keep the balance [of the equation].

Tabia: No, that must be wrong, it [the solution] cannot be parentheses.

Sekai: If this is an equation, then there was a parenthesis here, and it is really heavy, then you also need a parenthesis here [on the other side of the equation], so it becomes equally heavy. Do you get it?

Ronja: No!

It is finally dismissed with the argument that a parenthesis is not a solution for an equation.

This bit of discussion between the students shows that questioning a part of the theory leads to further exposure of the theory, such as the view that an equation is a seesaw where the equilibrium must be maintained and a clarification of what is possible and not possible to substitute with the unknown. Furthermore, the students implicitly raise the question of what do you substitute the unknown with if there is no solution to the equation?

The eighth equation, $4(x–436) – 326 = t–6434$, is an equation with two different variables, and the output of CAS is $\left\{x=\frac{1}{4}t–1091\right\}$. The groups that reach this equation do not seem perplexed with now having a solution for an equation that contains an unknown and relates the solution to previous obtained solution type with infinitely many solutions. In one group, a student remarks, "x is equal to one fourth t [stops mid-sentence]. It is that thing where x can have one or several unknown [as a solution]. Next!" Later, she reflects, "I had not considered that one could put an equation in relation to, err, an unknown in relation to another unknown".

The ninth equation, $2(3–4) + 7 = 5 – 3(4 – 2) + 18$, is an equation with no unknowns, and it is true. If each side of the equation is simplified by calculations, it will yield 5=5. The output of CAS is $\{x = x\}$. The groups of students who reach the equation immediately dismiss the expression as an equation since it does not contain an unknown: "There is no x -that is not an equation!".

The next episode of interest is the last episode of the lesson, where the students share their findings, and the teacher reformulates the students' findings and writes key phrases on the board. Following this, the teacher outsets further discussions and clarifications. In one of the lessons, the output $\{x = x\}$ is further discussed. During the students' presentation of the implications of this type of output, i.e., an equation can have an infinite number of solutions, it is implicitly understood that the solution type is considered a number. The teacher relates this type of equation to equation eight, where the solution contains an expression with a second unknown, and asks the students if, perhaps when $x = x$, the solution can also be an expression containing an unknown.

The curly parentheses are also one of the objects that the teachers consider. After the students have had their try at guessing the meaning of the curly parentheses, the teacher explains that the curly parentheses are usually used to denote a set. The students have not only encountered equations with more than one solution but are also taught how to write such solutions.

Equation nine and ten, $2(3–4) + 7 = 5–3(4–2) + 18$ and $3(23–11) = \frac{1}{5} + \frac{22(3–5)}{2}$, are also brought to the attention of the students since they do not include an unknown. The students all agree that they do not qualify as equations, even though CAS can solve them with an output of $\{x = x\}$ or $\{\ \}$. In one of the lessons, the teacher suggests that they can be called statements, and these statements can either be true or false, just as the so-called equations can either be true or false.

In the last episode of the lesson, the students formulate elements of their theory. The teacher then further questions the theory, and together, the students and the teacher develop and formulate theory regarding equations, as well as other related subjects.

Overall, the students developed and clarified a wide range of theory related to solving equations, including rational or decimal numbers as solution, equations having several or infinitely many solutions, and the limitation of parenthesis as a solution. They also established syntax for equations i.e., whether there can exist 2-in-1 equations and syntax relating to parenthesis. In addition, the students determined the rules for substitution in equations with multiple solutions. At the end of the class, the teacher institutionalised theory related to what can a solution for an equation be/ not be.

Lesson B: Make an equation where $x = 2$

We will now study the empirical findings for lesson B. In our analysis, we will focus on the techniques and the technology that the students developed related to the tasks of making and further developing equations with a solution of two. As the problems are open, there are many possible ways of attacking the task. Although the lesson was carried out three times, the students' praxeologies developed in the lesson were similar. We will present our findings in chronological order, i.e., first the task of making an equation with a solution of two, and then the task of further developing an equation with a solution of two while maintaining the same solution.

The first section of interest is the students' first try to create an equation with a solution equal to two. One group of students suggest three main techniques. The first main technique:

Irene: Can't we just make a random equation and then compensate?

Leo: What! Oh, you mean like just add minus [a number] at the end.

…

Leo: We just copy the one up there [from the board] but we make it longer.

…

Irene: I have got the greatest idea ever! Can we just do like last lesson, where x is equal to x. Infinitely many answers.

Joules: How do you get x equal to x?

Leo: Then you have to make both sides [of the equation] such that when reduced they are the same.

The group of students has formulated the three main techniques for this section of the lesson. We will refer to them as π_1, which is the main technique for making a "random" equation, checking the solution with CAS, and then compensating until the solution is two. π_2, which starts with an equation with a solution of two, complicating the equation further, and checking the solution with CAS. π_3, which is making an equation that is always true by making the right-hand side equivalent to the left-hand side.

The group cited above ends up trying out π_1. After entering a "random" equation, they get the output from CAS $\{x = 4\}$. This prompts the discussion on how to compensate such that the solution is two.

Leo: What if we divide it [the equation] with two?

Irene: Then it would be one-half x equal to two, that would still give the result x equal to four. If we want two less on the right side perhaps, we should either add or subtract two up here.

The work and discussion of the group show the variety of techniques the students hold, their formulation of their technique, and the choice of technique. Even more specific technique of how to manipulate a simple equation with solution of four such that the solution becomes two.

All the groups work with similar or variations of the above mentioned three main techniques. Another group that works with π_3 uses the technique of making two expressions that are equal to five, and then place the expressions on opposite sides of the equal sign so that the equation has infinitely many solutions. The result is $\dfrac{4x + x - \frac{1}{2}x}{x + \frac{1}{2}x} + x = 3+x$.

Another group employing π_3 uses the technique of adding and subtracting the same on both sides of the equation.

Gerð: We use a lot of difficult numbers, and then we subtract them in the end.

Fróði: Minus one thousand-seven-hundred-and-sixty-one minus something plus x minus seven-hundred-and-eleven. Then x is equal to x!

Gerð: I think we should use pi.

…

The discussion among the students and the results of their work show that the students have the theory that equations with infinitely many solutions exist and how such equations can be made with the technology of making an equation where the right-hand side of the equation is equivalent with the left-hand side.

Although some groups have discussed π_2, further developing an equation with solution two, none of these groups choose to work with this strategy.

In the next section of the lesson of interest, the classes are tasked with π_2. They are given a list of equations, all of which have a solution of two, to make more complex. Our main interest is not the ugly equations produced, but rather the techniques and technology that the students develop and employ. We will first recount the main techniques and technologies developed and formulated by the students, and then recount some of the theory that was discussed and clarified.

We will present two of the techniques developed and formulated by the students. The first technique, which we will denote by τ_1, is to divide a part of the equation by an expression that is equal to one when x is equal to two, i.e., $3–x$. In the following transcript, the technique is developed, and the expression $3x–4$ is rewritten into $\frac{3x-4}{(x^3-2x)-3}$.

Margrethe: Perhaps we can make a fraction here, and then make some weird expression [as the denominator] that is just equal to one … Then we say four minus two, all we need is for it to equal one. x and we have two, we need [pause]. Here we have two x, thus we need four, minus x squared. Here we have got. No. x to the power of three, then we have minus four, then we need to subtract with three on the other side somehow [$(x^3–2x) –3$]…That is a very overcomplicated way of writing one.

As a method the group writes “overcomplicating by dividing with one”.

The other technique, which we will denote as τ_2, was developed, and clarified by the students. It is to add the same number on both sides of the equation. A group has the following discussion:

Emmy: We must make this one ugly.

Elliot: You can change this to three and then minus thee here [referring to the other side of the equation sign in the equation].

Emmy: Plus six, plus two.

Elliot: [interrupts] You are doing it wrong. You cannot put plus on each side [of the equal sign] because you have to put a minus. [Directed to the teacher] Isn’t it true that if you put a plus on one side, then on the other side is has to be minus?

Teacher: Hmm, you have to keep the balance of the equation. If I have an equation and I make one of the sides heavier, then in order of keeping the balance, the other sides must also get heavier. I.e. if I add something on one side then I need to add the same thing on the other side.

Elliot: Does this make sense? [shows the teacher her equation]

Teacher: Test it [with CAS]

…

Elliot: Okay, we add plus three and twenty on both sides. And minus nineteen on both sides. And plus four on both sides.

...

The extracts show the two techniques, τ_1 and τ_2, developed, formulated, and employed by students. τ_1, dividing a term of the equation with an expression that is equivalent to one, and τ_2, adding the same number on both sides of the equation. The two techniques developed and formulated play a fundamental role in school algebra, as they are often the key to simplifying algebraic expressions or equations. In addition, they are pivotal for the application and handling of school algebra, such as simple formulas, in other subjects and further education.

Although the focus of the activity was on developing techniques and technology, discussions and clarification of elements of the theory also emerged. The main theory present was that the solution must stay the same, as it was the condition for rewriting the equations. During the lesson, the students often referred to another main theory, which is that an equation can be seen as a seesaw, and that the seesaw must be kept in balance, i.e., if you add a number to one side of the equation, you must add a number or an expression that is of equal weight on the other side of the equation.

We will now present some more of the technology and theory that emerged in this section of the lesson. In one group of students that were employing τ_1, a student wanted to divide by zero. In another group, also employing τ_1, a student asked what happens if you divided by x. As time was of the essence, the respective groups did not engage in theoretical discussion but refocused on the task at hand and developing techniques for which the technology and theory were already part of the students' inventory.

The activity of handling equations all with solution two seemed to generate, in several groups, a questioning about whether the equations were all the same. After seeing the teachers' two examples of equations with solution equal to two, a student asks

Uliuk: How can they all give the same result? Such a long equation can give the same result as that one.

Vaaltimaat: That is the question!

Teacher: I think that is a very good question!

Uliuk: Then you can just write that [the first equation] instead of all that [the second equation]. That [the second] one takes a lot longer to reduce.

In the second section of the lesson, other students commented that it did not feel like they had made new equations, as it was a rewriting of the first equation. This happened in particularly if the students had done simpler manipulations, such as rewriting $4x$ into $3x + x$ or $2x$ into $x + x$.

The extracts show that elements of theory are discussed and clarified, but they also inspire further questioning of the theory, which could lead to entire new types of praxeologies such as when are two equations the same? The students are becoming aware of equations as an object of study.

In the next part of the lesson, the students share their methods while the teacher reformulates and writes them on the board. The techniques shared are: multiplying a term of the equation with an expression that is equal to one when x is equal to two; dividing a part of the equation with an expression that is equal to one when x is equal to two; adding an expression that is equal to zero when x is equal to two, to one side of the equation; adding the same number or equivalent expressions on both sides of the equation; and multiplying with the same number or expression on both sides of the equation. In two out of the three classes involved in the project, the teacher ends the sharing of methods by emphasises the connection to the theory that the solution of the equation stays the same. In the third class, the teacher ends the sharing of methods by reflecting that many of the methods developed for uglifying an equation, are the same methods one could use to simplify an equation. Many of the discussions on techniques, technology, and questioning of theory is not shared, and neither is a relation to Lesson A made in this lesson.

The students' sharing of their developed methods shows the variety of the different technologies present in the lesson and the similarity to the technology for solving equations. Thus, embedding the theory element that the solution is maintained into the praxeology for solving equations.

In the last attempt at making an ugly equation with solution two led to the emergence of another piece of theory concerning the equal sign. In one group the necessity of the equal sign in an equation is questioned. However, after wondering how to solve such an equation, or how to check if the solution is two, they agree that the equal sign is a requirement for an expression to be an equation.

The recount of the group's discussion shows that the students are once again considering equations as an object of study, and though the question of whether or not the equal sign is a necessity for an equation was raised in the first lesson, the discussion in this account includes technology.

CONCLUSION

In this section, we will try an answer our research questions regarding the potential of CAS and the notion of praxeology as a tool for designing tasks.

The use of CAS in the activities allowed for the students to approach new tasks (more exotic types of equations) that would otherwise have required time-consuming algebraic work such as solving equations. The new tasks involved themes within school algebra that strengthened, developed, and clarified both technology and theory. In lesson A, the students worked with describing what constitutes an equation. The students developed, formulated, and clarified a wide range of technology and theory involving equations and related topics such as substitution, thus strengthening an entire series of related praxeologies. In lesson B, the students worked on establishing a stronger connection between the theory of maintaining the solution of an equation with the technology and techniques used for manipulating equations. Thus, sustaining a grander logos for school algebra. However, lesson B did not solve all problems related to the praxeologies of solving equations. During the lesson, both students and teachers mentioned the metaphor of an equation as a seesaw, where the balance must be kept, as a valid argument for justifying techniques. There were instances where this magic trick could have been questioned, such as when a student wanted to divide by zero or when one student wanted to take the square root of both sides of the equation (without following using the absolute value), which could possibly change the solution space. But it did not happen.

By outsourcing time-consuming algebraic work, CAS can be used to introduce new types of equation i.e., equations with more than one solution, making it possible to study equations as an object and not just using equations as a tool. This in turn developed both elements of the students' theory and technology such as in lesson A. In lesson B, the use of CAS provided students with a technique for verifying the solution of their equations, allowing students to experiment with creating and developing the equations. This led the students to develop and formulate not only an abundance of technology, but also formulate and clarify elements of theory.

The students' work with the logos further prompted them to question, formulate and discuss elements of theory, such as rules and the role of substitution (from lesson A) or what happens if you divide by the unknown, thus further enhancing the activity.

As both lesson A and lesson B contained many questions inquiring further into elements of technology and theory, we conjecture that the activities could be further enriched by not being constrained by a lesson plan. This would allow the students to direct the path of inquiry and development of technology and theory. In addition, the object of lesson B, which is to further develop and explicitly state the relation between the techniques and technology (of manipulating equations) with the theory (of the solution staying the same), the institutionalisation must be further emphasised.

The use of the notion praxeology, particularly the four notions of task, technique, technology, and theory, allowed us to analyse the prevailing paper-and-pencil praxeology for solving an equation, and thus select the themes "the students' definition of an equation", and "the relation between algebraic manipulations of an equation, and maintaining the same solution". Both themes are within school algebra, thus the designed activities would strengthen the students' logos by further developing, formulating, and clarifying it.

The notion of praxeology lets us further study the lever potential of CAS, enabling the design of future activities more easily. In lesson A, the lever potential can be described as the development and formulation of the students' concept of equation, which is an element of the theory block. This characterisation can also be used to describe the potential of CAS studied in the thesis by Drijvers (2003). In lesson B, the lever potential is the strengthening of the relation between the theory of maintaining the same solution, with the techniques and technology for traditional algebraic manipulations. In the article by Hitt and Kieran (2009), the lever potential can be described as the telescoping technique (using paper and pencil) being used not only as a technique but also as technology for developing, justifying, and validating conjectures. The use of the notion praxeology enables us to describe the students' learning explicit and justifies the lever potential of CAS.

As a final note, we would like to reflect on how these more theoretical CAS-based activities can upset the traditional epistemological model for school algebra and thus necessitate new design tools for activities. Traditionally, second degree equations with two, one or no solutions are introduced by their graphical representation at the end of lower secondary school. However, a more inquiry-based teaching approach combined with CAS will naturally lead to question what an equation is, and following an examination of a series of examples. These examples would feasibly include second- and higher-degree equations as the strength of CAS is to handle more exotic examples than is possible in a pen-and-paper-only environment. A change in the traditional curriculum when implementing CAS is not unheard of (Kendal, Stacey, & Pierce, 2005) and perhaps even necessary. This would, however, only stress the need for design tools to support teachers in analysing and crafting CAS-based activities.

REFERENCES

Artigue, M. (2002a). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245-274. https://doi.org/10.1023/A:1022103903080

Artigue, M. (2002b). Quelques leçons des ingénieries didactiques. In D. Guin & L. Trouche (Eds.), *L'instrumentation de calculatrices symboliques: un probl'eme didactique.* Grenoble: La Pensée Sauvage.

Bang, H. P., Grønbæk, N., & Larsen, C. R. (2017). Out-and in-sourcing, an analysis model for the use of instrumented techniques. Paper presented at the Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education, Dublin: DCU Institute of Education and ERME. Henrik P. Bang, Niels Grønbaek, Claus R. Larsen. Out- and in-sourcing, an analysis model for the use of instrumented techniques. *CERME 10*, Feb 2017, Dublin, Ireland. ffhal-01942147

Bosch, M., Gascón, J. (2014). Introduction to the Anthropological Theory of the Didactic (ATD). In: Bikner-Ahsbahs, A., Prediger, S. (eds) *Networking of Theories as a Research Practice in Mathematics Education* (pp. 67-83). Advances in Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-319-05389-9_5

Carlsen, L. M. (2019). What algebraic knowledge may not be learned with CAS-a praxeological analysis of Faroese exam exercises. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 21.*

Chaachoua, H., Bessot, A., Romo, A., & Castela, C. (2019). Developments and functionalities in the praxeological model. In M. Bosch, Y. Chevallard, F. García, & J. Monaghan (Eds.) *Working with the Anthropological Theory of the Didactic in Mathematics Education* (pp. 41-60). Routledge.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques: l'approche anthropologique. *Actes de l'UE de la Rochelle,* 91-118.

Dreyfus, T. (1994). The role of cognitive tools in mathematics education. In R. Biehler, R. W. Scholz, R. Sträszer & B. Winkelmann (Eds.) *Didactics of mathematics as a scientific discipline,* (1 st. ed., pp. 201-211). Kluwer Academic Publishers.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment: Design research on the understanding of the concept of parameter* [Doctoral Thesis, Utrecht University]. Dissertation Utrecht University.

Gjone, G. (2009). The use of CAS in school mathematics: Possibilities and limitations. In *Proceedingsof the 6th CAME Symposium* (p. 19).

Guin, D., & Delgoulet, J. (1997). Etude des modes d'appropriation de calculatrices graphiques et symboliques dans une classe de seconde. *Rapport de recherche, IREM de Montpellier.*

Gyöngyösi, E., Solovej, J. P., & Winsløw, C. (2011). Using CAS based work to ease the transition from calculus to real analysis. In *Proceedings of the 7th conference of European researchers in mathematics education* (pp. 2002-2011).

Heid, M. K., Blume, G. W., Hollebrands, K., & Piez, C. (2002). Computer algebra systems in mathematics instruction: Implications from research. *The Mathematics Teacher, 95*(8), 586-591.

Hitt, F., & Kieran, C. (2009). Constructing Knowledge Via a Peer Interaction in a CAS Environment with Tasks Designed from a Task–Technique–Theory Perspective. *International Journal of Computers for Mathematical Learning, 14*(2), 121-152. doi:10.1007/s10758-009-9151-0

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels In F. Lester (ed.). *Second Handbook of Research on Mathematics Teaching and Learning: A project of the National Council of Teachers of Mathematics,* (vol. 2, pp. 669-705). National Council of Teachers of Mathematics.

Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning, 11*(2), 205-263. https://doi.org/10.1007/s10758-006-0006-7

Kieran, C., & Saldanha, L. (2008). Designing tasks for the co-development of conceptual and technical knowledge in CAS activity: An example from factoring. In G. W. Blume & M. K. Heid (Eds.) *Research on technology and the teaching and learning of mathematics,* (vol. 2, pp. 393-414). Information Age Publishing.

Kieran, C., & Yerushalmy, M. (2004). Research on the role of technological environments in algebra learning and teaching. In K. Stacey, H. Chick, & M. Kendal, (Eds.), *The Future of the Teaching and Learning of Algebra The 12th ICMI Study* (vol. 8, pp. 97-152). Springer, Dordrecht. https://doi.org/10.1007/1-4020-8131-6_6

Lagrange, J.-B. (2005). Using symbolic calculators to study mathematics -the case of task and technique. In D. Guin, K. Ruthven, & L. Trouche (Eds.) *The didactical challenge of symbolic calculators Turning a computational device into a mathematical instrument* (vol. 36, pp. 113-135). Springer, Boston, MA. https://doi.org/10.1007/0-387-23435-7_6

Martínez, C., Kieran, C., & Guzmán, J. (2012). The Use of CAS in the Simplifi cation of Rational Expressions and Emerging Paper-and-Pencil Techniques. In L. R. Van Zoest, J.-J. Lo & J. L. Kratky (Eds.), *Proceedings of the 34'th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* 1089-1096.

Poulsen, C. S. (2015). Basic algebra in the transition from lower secondary school to high school. (Master thesis). University of Copenhagen.

Winsløw, C. (2003). Semiotic and discursive variables in CAS-based didactical engineering. *Educational Studies in Mathematics, 52*(3), 271-288. https://doi.org/10.1023/A:1024201714126

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Special Issue:

Rethinking the Diversity of Theories in Mathematics Education. Contributions Related to the Topic Study Group 57 of ICME 14

Guest Editorial

RETHINKING THE DIVERSITY OF THEORIES IN MATHEMATICS EDUCATION. CONTRIBUTIONS RELATED TO THE TOPIC STUDY GROUP 57 OF ICME 14

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This Special Issue emerged from Topic Study Group 57 (TSG57) on the “Diversity of Theories in Mathematics Education” conducted at ICME 14 in July 2021 (Bikner-Ahsbahs et al., in print). Based on the presentations and discussions in the TSG57 at the conference, five papers have been extended and elaborated for the special issue. To introduce this special issue, we would like to look back briefly at what we have discussed in the TSG. Main topic was the Networking of Theories in mathematics education research and this also resonates with this special issue.

In recent years, the theory networking enterprise has advanced (Kidron et al. 2018). Networking of Theories means to relate different theories, to make use of them in research and scientifically reflect this process. Key assumption of this approach consists of respecting theoretical identities that are often based on cultural backgrounds and at the same time requiring from researchers to keep a critical stance on theoretical compatibility in the theoretical networking practice. To keep this balance, Prediger et al. (2008) have developed pairs of networking strategies that express increasing degrees of integration; these are *understanding other and making understandable* own theories, *comparing and contrasting* theories, *combining and coordinating* theories, and *local integration and synthesizing* of theories. Many authors in this issue recount how they used these strategies in their research.

The TSG57 addressed three subthemes in three sessions: Reasons for the diversity of theories; methodological approaches in design research related to the diversity of theories; and reconsidering specific commitments in the diversity of theories. In the first session, the TSG articulated a double aim, to respect theory development coming from various educational cultures and research areas in the field and at the same time to overcome theoretical fragmentation in the field. This requires a balance of keeping the diversity needed but simultaneously bridging theories to make communication across the theory cultures possible. Two themes were discussed, first, networking of theories can be a source of (as well as it may unpack) ‘tensions’ between theoretical discourses, depending on basic assumptions and, second, networking of theories can offer ‘flexibility’ when relating basic assumptions and thus creating a new discourse. From the second session we learned that each theory is built on the three commitments, ontology, axiology, and epistemology, in specific ways that underlie how teaching and learning are understood, in research as well as in practice. Main postulation raised in the discussion was to increase scientific sensitivity for these commitments, specifically in terms of theoretical choices in transformative research. From the third session we have learned to make these commitments visible in research and to look for methodological ways to practice this sensitivity, as for instance by a vertical analysis grounded in historical or philosophical considerations. A crucial part of the discussion resulted in the insight that each theory on teaching and learning entails a certain understanding of ethics grounded in this theory.

Finally, two questions were raised: To what extent is it necessary or even mandatory to take ethical issues of the teaching/learning into consideration for theorizing and what are the relevant concepts of ethics for that? More generally, what are the pitfalls when such commitments go unnoticed?

This Special Issue does not address all the aspects and questions mentioned in the TSG57, but some relevant aspects are captured by the publications. All the five papers focus on and 'rethink' the diversity of theories from different perspectives. In the first contribution "Facing the Challenge of Theoretical Diversity: The Digital Case" Michèle Artigue explores the advancement in the field of technology in mathematics education in terms of research on two concepts conducted by networking of theory approaches. In the second article "The Role of a Priori Analysis in Theories", Ivy Kidron explores the notions of a priori analyses related to two theories showing how the epistemological grounding of theories determines them. In the third paper "Ethics in the Mathematics Classroom", Luis Radford elaborates the concept of communitarian ethics for the Theory of Objectification making explicit how theory and ethics are entwined. The fourth contribution "Networking Praxeologies and Theoretical Grain Sizes in Mathematics Education: Cultural Issues Illustrated by Three Examples from the Japanese Research Context" by Yusuke Shinno and Tatsuya Mizoguchi is a comparison of three Japanese networking examples that shows clearly the relevance of culture when networking theories. The final contribution by Angelika Bikner-Ahsbahs et al. is a networking case on design research leading to local integration of two theoretical approaches that allows to explain how learners transformed their behavior when acting with a digital artifact.

In the following, we are going to summarize each article and thereby extract their main contributions to the field.

SUMMARIES OF THE ARTICLES IN THE SPECIAL ISSUE

The theoretical diversity offered by different theoretical cultures is a real challenge in mathematics education research. The analysis in the paper by Artigue (2023) provides an important overview of the theoretical landscape and its main tendencies in theorizing the use of technology in mathematics education. Artigue aims to contribute to the reflection on the question: "To what extent, are we now better equipped to meet the challenge of theoretical diversity in relation to technology-based mathematics teaching and learning?" To this end, two conceptual tools are introduced in the paper. Both tools have proven their effectiveness in addressing issues of theoretical diversity: The scale of networking strategies and the concept of research praxeology (introduced in Artigue et al., 2011). Two case studies are described: The Instrumental Approach (IA) and the Documentational Approach to Didactics (DAD). The emergence and development of these approaches illustrate well the global dynamic of the field towards increasing theoretical diversity, the questions raised by this dynamic and the insightful efforts made to deal with it. The evolution of teachers' documentational work was induced by technological advances. One important contribution of Artigue's paper is her analysis that demonstrates how knowledge has advanced both, in terms of understanding the diversity challenge and in terms of developing strategies to address it.

Kidron (2023) investigates the role of a priori analysis in different theories in the context of networking theories. A priori analysis here means an analysis, which is often carried out prior to the experiment or data collection. While the a priori analysis was initially proposed as a part of the methodology of *didactic engineering* developed within *the Theory of Didactical Situations* (TDS) (Artigue, 2020), it is today shared in different theories. As the methodology is often considered a part of the theory (Radford, 2008), the comparison of the methodologies reveal the nature of each theory as well as the relationship between them. In the previous work of networking theories, several case studies have been carried out by analyzing the common empirical data from different theoretical perspectives to shed light on their differences, commonalities, and complementarities (see Prediger & Bikner-Ahsbahs, 2014), that is to say, these studies consist of the comparative analyses of the processes and results of a posteriori analyses. Kidron's paper is unique in that it focuses on a priori analysis instead of a posteriori analysis. It shows by comparing the a priori analyses of TDS and the theory of *Abstraction in Context* (AiC), each theory has a different focus, while both theories place importance on the epistemological perspective: on the one hand, AiC focuses on the learner's construction of knowledge with a question "What is the epistemic process of the student?"; on the other hand, TDS focuses on the didactic system with a question "How this [epistemic] process is possible?" One of the important contributions of Kidron's paper is that it shows the relevance of analyzing a priori analyses in the context of networking theories. As the focus of the theory affects the data collection, the study of a priori analyses allows us to better understand the choice of data.

Radford (2023) begins his elaboration of educational ethics by an example, sensitizing readers for seeing teaching and learning as a deeply ethical form of activity. Although always present, ethics seem to have lost sight in mathematics education. Radford sees this being grounded in the conception of mathematics education that has been reduced "to a matter of acquiring knowledge, making the question of being and becoming peripheral aspects of teaching and learning" (p. 60). He traces two fundamentally distinguishable forms of ethical action back to Hobbes (1841) and Kant (2006). With reference to Hobbes, the notion of ethics consists of contractual obligations that regulate acting in society. Under the maxim of universal reason, Kant anchors moral action, hence, ethical acting, in the individual. According to Radford, Hobbes' conception of ethics informs the transmissive idea of teaching and learning, while ethics according to Kant rather inform constructivist approaches. Thus, ethics "appear framed by the way in which we understand teaching and learning" (Radford, 2023, p. 60), hence, by theories of teaching and learning. Radford elaborates the concept of communitarian ethics for the Theory of Objectification as an example. His contribution is unique as he makes explicit how the Theory of Objectification and communitarian ethics are interrelated by elaborating the theory's relational conception of the social and how this hands over to ethics. He defines communitarian ethics through three dimensions: responsibility, commitment, and care. Making use of Lévinas (1982)' understanding of responsibility he regards responsibility as "living and acting with and for others" (p. 69). The key function of communitarian commitment is "to participate in the creation of the classroom common work" (p. 69) of joint labor. A core ingredient of educational ethics is care, meaning here that "the importance of caring for the Other is to go beyond ourselves and to be dragged powerfully into the world, to position ourselves there, with-the-Other" (p. 70). Communitarian ethics are thus relational and, consistent with the Theory of Objectification, permanently materializing themselves within the classroom.

Networking of theories is a way to bring different theoretical perspectives into a dialogue often used to unpack implicit theoretical assumptions. However, there is a lack of meta-theoretical knowledge for conducting a networking theories study. With their article, Yusuke Shinno and Tatsuya Mizoguchi (2023) want to contribute to filling this gap by answering the question *How can we characterise researchers' theoretical work on networking endeavours in terms of praxeology?* Leaning on the work of Artigue and Bosch (2014), they use the concept of praxeology introduced by the Anthropological Theory of the Didactic (Chevallard, 2019) and elaborate the concept of *grain size of theories* to characterize the researchers' theoretical work in three Japanese research and development examples that follow networking theories approaches. Their networking technique, one of the elements of praxeology, consists of networking strategies. Shinno and Mizoguchi use the strategy of comparing and contrasting theories to explore a design approach (lesson study or didactical engineering), with the strategy of combining and coordinating theories they investigate an empirical classroom study, and they use local integration for the case of a curriculum development study where the reference epistemological model (Bosch & Gascón, 2006) plays a key role. Shinno and Mizoguchi identify and compare the three networking praxeologies of the examples to describe the theoretical discourses involved. Their contribution is unique as it shows that these discourses are intrinsically related to culture, i.e., classroom culture, research culture, and curriculum culture. As reflected by the authors, these forms of culture are specific in the Japanese culture of mathematics education in that *theory for research and theory for practice* (p. 7) are distinguished.

Bikner-Ahsbahs et al. (2023) address theoretical networking between *Activity Theory* (Leontyev, 2009) and *Instrumental Approach* (Rabardel, 2002) to investigate the role of digital feedback in supporting the teaching and learning of negative numbers with a multimodal algebra learning (MAL) system. Within Activity Theory, tools such as the MAL system play an essential role as cultural objects. From the perspective of Instrumental Approach, the process of how the MAL system becomes an individual instrument can be conceived as instrumental genesis, shaped by two interrelated processes: instrumentation and instrumentalization. Bikner-Ahsbahs et al. (2023) examine the teaching and learning of negative numbers with the MAL system using Activity Theory and a model about feedback loops. They characterize the role of digital feedback as mediating between different types of action expressed by a four-layered model. One of the important contributions of Bikner-Ahsbahs et al.'s paper is their theorizing of an empirical phenomenon at the boundary of the two theories leading to 'local integration', a relatively challenging strategy of theory networking that allows to go beyond the mere understanding of a special empirical phenomenon (Prediger et al., 2008). In their networking work, after coordinating the two theories with the help of the layered model, they clearly demonstrate how local integration in terms of Radford's (2008) theory concept is achieved.

Four of the five articles are about the networking of theories, and two papers address philosophical commitments underlying educational theories. While Artigue elaborates the state of the art of research and its advancement in technology in mathematics education realized by theory networking approaches, the other three networking papers contribute directly to advance its meta-theoretical sphere. Shinno and

Mizoguchi elaborate that cultural backgrounds affect notions of theory and thus also notions of theory networking. Bikner-Ahsbahs et al. explore the role of digital feedback at the boundary of two theories as a new phenomenon and show the advancement made by the take-up of the networking strategy of local integration. Kidron, through theory networking, contributes to our understanding of epistemological commitments of theories and their methodological and conceptual consequences. Grounded in philosophical considerations, Radford improves our knowledge on ethical commitments, underlying theories on teaching/learning in general. All the five papers advance the field by meta-theoretical considerations included into research resulting in deepening our understanding of what can make a difference in the diversity of theories in mathematics education.

Acknowledgement

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References

Artigue, M. (2023). Facing the challenge of theoretical diversity: The digital case. *Hiroshima Journal of Mathematics Education, 16*. 27–43.

Artigue, M. (2020). Didactical engineering. In S. Lerman (Ed.), *Encyclopedia of mathematics education. Second Edition* (pp. 202–206). New York: Springer. https://doi.org/10.1007/978-3-030-15789-0_44

Artigue, M., & Bosch, M. (2014). Reflection on networking through the praxeological lens. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 249–265). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_15

Artigue, M., Bosch, M., & Gascón, J. (2011). La TAD face au problème de l'interaction entre cadres théoriques en didactique des mathématiques. In M. Bosch et al. (Eds.), *Un panorama de la TAD. Actes du troisième congrès de la TAD* (pp. 33–56). Centre de Recerca Matemàtica, University of Barcelona.

Bikner-Ahsbahs, A., Kidron, I., Bullock, E., Shinno, Y., & Zhang, Q. (in print). Topic Study Group 57: Diversity of theories in mathematics education. In Jianpan Wang (Ed.) *Proceedings of the 14th International Congress on Mathematical Education*. World Scientific Publishing House.

Bikner-Ahsbahs, A., Vallejo-Vargas, E., Rohde, S., Janßen, T., Reid, D., Alexandrovsky, D., Reinschluessel, A., Döring, T., & Malaka, R. (2023). The role of feedback when learning with a digital artifact: A theory networking case on multimodal algebra learning. *Hiroshima Journal of Mathematics Education*, *16*. 95–116.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition [ICMI bulletin], (Vol. 58) (pp. 51–65).

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://doi.org/10.24529/hjme.1205

Hobbes, T. (1841). *The English works of Thomas Hobbes of Malmesbury: Vol. 2* (W. Molesworth, Ed.). John Bohn.

Kant, I. (2006). *Groundwork of the metaphysic of morals* (Mary Gregor, Trans.). Cambridge University Press.

Kidron, I. (2023). The role of a priori analysis in theories. *Hiroshima Journal of Mathematics Education, 16*. 45–56.

Kidron, I, Bosch, M., Monaghan, J., & Palmer, H. (2018). Theoretical perspectives and approaches in mathematics education research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 254–268). Routledge. https://doi.org/10.4324/9781315113562

Leontyev, A. N. (2009). Activity and consciousness. *Marxists Internet Archive* https://www.marxists.org/archive/leontev/works/1978/index.htm

Lévinas, E. (1982). *Éthique et infini*. [Ethics and infinity]. Fayard.

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs & S. Prediger (Eds.) *Networking of theories as a research practice in mathematics education.* (pp. 117–125). Springer. https://doi.org/10.1007/978-3-319-05389-9_8

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. Université paris 8. https://hal.archives-ouvertes.fr/hal-01020705

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM - The International Journal on Mathematics Education*, *40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Radford, L. (2023). Ethics in the mathematics classroom. *Hiroshima Journal of Mathematics Education, 16*. 57–75.

Shinno, Y. & Mizoguchi, T. (2023). Networking praxeologies and theoretical grain sizes in mathematics education: cultural issues illustrated by three examples from the Japanese research context. *Hiroshima Journal of Mathematics Education, 16*. 77–94.

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FACING THE CHALLENGE OF THEORETICAL DIVERSITY: THE DIGITAL CASE

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Abstract

The presentation of TSG57 at ICME-14 outlined that mathematics education is a scientific field in which many theoretical cultures coexist, and that "this diversity can be regarded as richness but it also challenges research as well as communication and cooperation in the field". It was added that "how the scientific community can cope with this diversity with scientific integrity remains an open question". In this paper, we propose to contribute to the reflection on this challenging issue by considering research on technology-based teaching and learning. First, we present a brief overview of the theoretical landscape in this area of research, highlighting its diversity. We then introduce two conceptual tools that have proven their effectiveness in addressing issues of theoretical diversity: the scale of networking strategies and the concept of research praxeology, before focusing on two case studies. These regard the instrumental approach and the documentational approach to didactics, the emergence and development of which illustrate well the global dynamic of the field towards increasing theoretical diversity, the questions raised by this dynamic and the insightful efforts made to deal with it.

Keywords: mathematics education, digital technology, research praxeology, scale of networking strategies, instrumental approach, documentational approach to didactics

1. INTRODUCTION

Increasing theoretical diversity is a general phenomenon in mathematics education, and research focused on technological issues is no exception, as we already showed it in our plenary lecture at CERME5 fifteen years ago (Artigue, 2007). In fact, in that lecture we pointed out that, for decades, this area of research has reflected the general trends and major developments in the field but has also been a source of inspiration for it, as shown for instance, by the contribution of research in this area to constructivist and situated perspectives as well as to semiotic and embodied approaches. We also argued that such a situation made this area of research an interesting window through which to look at theoretical diversity and at its practical implications. This is all the more so as the development and the use of digital technologies, since their emergence, have been given a transformative aim in mathematics education.

A similar vision is proposed in the volume resulting from ICMI Study 17, the second ICMI Study dedicated to the teaching and learning of mathematics with digital technologies (Hoyles & Lagrange, 2010). In fact, in this volume, Chapter 7 (Drijvers et al., 2010) is especially devoted to theoretical perspectives. The

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authors point to the influence on this area of research of the global evolution of the field from constructivist to socio-cultural perspectives, and they illustrate this with three examples: the webbing and situated abstraction frame in reference to (Noss & Hoyles, 1996), the theory of didactical situations and the related notion of "antagonist milieu" (Brousseau, 1998) and the perceptuo-motor activity frame developed by Nemirovski for use in research on modelling environments involving physical apparatus, such as the well-known water wheel (Rasmussen & Nemirovski, 2003). They also highlight the theoretical creativity of this area of research. Two "current developments" are then presented with more details: the instrumental approach on the one hand, and mediation and semiotic mediation on the other hand. In the first case, the authors rightly point out the fact that the theoretical combination at stake between cognitive ergonomics and the anthropological theory of the didactic has led to different variations reflecting the particular weight given to these two perspectives, the first one cognitive, the second one institutional (see Section 3 for more details), although all variations share fundamental distinctions and concepts, such as the distinction between artefact and instrument, and the concept of instrumental genesis. For the second, the researchers involved share the claim that the epistemological nature of mathematical objects makes them accessible only through the mediation of representations, and that technological devices substantially alter these mediation possibilities. However, once again, as the authors point out, these premises are dealt with through different theoretical constructions, for instance the concept of human-with-media introduced by Borba and Villareal (2005), which emphasizes the unity between humans and tools, or the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), which emphasizes the essential role of the teacher in the transition from personal meanings rooted in the context of the artefact to mathematical meanings. Despite the fact that the chapter begins with the assertion that, due to technological evolution, "communication has become a more integrated part of technology use"(p. 98), the theoretical influence of this evolution does not seem very strong at the time, rather a line for future research. Looking to the future, the authors write that the key word should be "connectivity", because connectivity is a key word for approaching a technological evolution that is fundamentally changing our modes of communication and learning opportunities in and out-of-school. In fact, this issue is mainly addressed in Chapter 11 (Beatty & Geiger, 2010), where it is emphasized that technological evolution calls for theoretical frameworks that allow researchers to approach mathematical learning as a collaborative endeavor in technologically enhanced communities of practice. Various examples are presented where the influence of theoretical constructs associated with communities of practice (Wenger, 1998) is clear. However, there is no doubt that at the time of the ICMI Study 17 research on these newer technological affordances was not so much developed.

The chapter on technology in the book published for the 20th anniversary of ERME (Dreyfus et al., 2018) confirms the theoretical diversity observed in the ICMI Study volume (Hoyles & Lagrange, 2010), and organizes it in relation to a didactic tetrahedron with knowledge, technology, students and teacher as edges (Trgalová et al., 2018). However, it does not show an obvious movement of research towards the latest technologies. In fact, the evolution observed in the last decade, at least until recently, seems to be mainly due to the increase of research on teachers, their knowledge and their practices in digital environments, their preparation and professional development, their documentational work. For example, the documentational approach to didactics began to develop about fifteen years ago, motivated both by this attention to teachers and by the evolution of their documentational work induced by technological advances, as well explained in

(Gueudet & Trouche, 2010). This increased attention is especially addressed in the book *The Mathematics Teacher in the Digital Era* (Clark-Wilson et al., 2014) that makes clear the variety of theoretical constructs developed to address these issues. These include the adaptation of Ball's mathematical knowledge for teaching model ((MKT) to the technological pedagogical and content knowledge model (TPACK) by Mishra and Koehler (2006), Ruthven's structures features of classroom practice framework (Ruthven, 2007), and the extension of the instrumental approach to the teacher, leading to specific constructs such as instrumental orchestration (Trouche, 2005), double instrumental genesis (Haspekian, 2011) and genesis of use (Abboud & Vandebrouck, 2013). They also include, of course, the specific constructs associated with the documentational approach, such as the concepts of documentational genesis, resource system and documentational trajectory (for an updated vision, see Trouche, Gueudet & Pepin, 2019). Of particular interest for a reflection about networking is the chapter written by Ruthven, which explores the similarities, complementarities and contrasts between instrumental orchestration, TPACK and his own framework (Ruthven, 2014). Another point worth highlighting is the place given to the instrumental approach and its extensions in these various syntheses. For instance, in (Clark-Wilson et al., 2014) an entire section with six chapters is dedicated to related research.

In recent years, however, the context has been transformed by major technological changes: the massification of mobile multi-touch technologies, the development of virtual or augmented reality devices, and the growing influence of social networks on practices. The context has also been transformed by the disruptions caused by the pandemic situation, with the abrupt shift to online or hybrid forms of teaching at all levels of education, and the efforts made to limit the growth of educational inequalities by investing massively in the most widely accessible technologies, such as mobile phones. Actions and research are multiplying in response to the new challenges encountered, which will undoubtedly add to the existing theoretical diversity.

In the aforementioned CERME lecture (Artigue, 2007), we argued for the collective development of networking activities in order to limit the growing risk of fragmentation of the field. This need is also stressed in the ICMI Study 17 volume where it is written that the key word for the future should be "connectivity", not only for reasons of technological evolution, but also because efforts should be made to better connect the existing diversity of theoretical perspectives. This is even more necessary today. Since then, however, the situation has changed with the undeniable advances of the networking enterprise (Bikner-Ahsbahs & Prediger, 2014; Kidron et al., 2018). These are both conceptual, methodological and practical. To what extent, then, are we now better equipped to meet the challenge of theoretical diversity in relation to technology-based mathematics teaching and learning? Our aim in this paper is to contribute to the reflection on this question. To this end, we introduce two conceptual tools in the next section before exploiting them to two case studies. Over the last decade, we have indeed experienced the strong potential offered by the combined use of these two conceptual tools to address the challenges posed by theoretical and linguistic diversity (Artigue, 2019, 2021; Artigue & Bosch, 2014; Mesiti et al., 2021).

2. INTRODUCING TWO CONCEPTUAL TOOLS

2.1. The scale of networking strategies

As explained in (Bikner-Ahsbahs & Prediger, 2010), this construct aims to show the variety of strategies that can be used to create connections between theories. The scale distinguishes eight strategies, paired and ordered between two extreme positions: "ignoring other theories" and "unifying globally", both of which considered not desirable. However, as the authors themselves point out, this idea of a linear order must be taken flexibly, as the degree of integration depends on the precise work done, not just on the strategy.

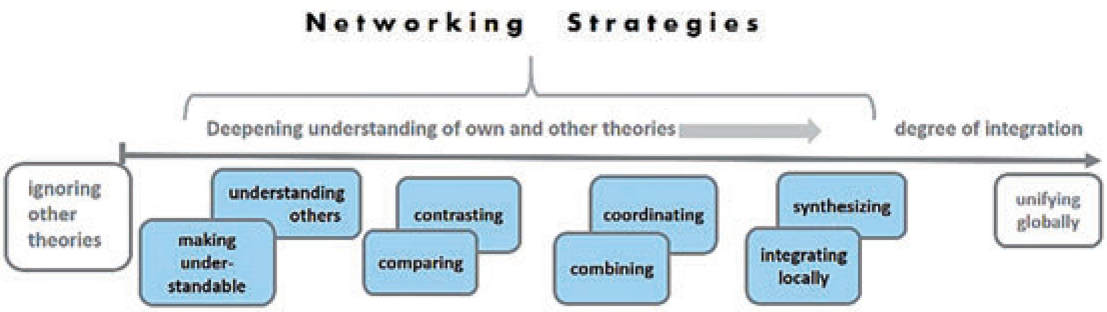

Figure 1: Scale of networking strategies (Bikner-Ahsbahs, 2016, p. 34, adapted from (Bikner-Ahsbahs & Prediger, 2010, p. 492) (CC BY 4.0))

The authors provide precise definitions for these different strategies. For example, on page 495, they explain that coordinating means that "a conceptual framework is built by well-fitting elements from different theories" and that this strategy therefore presupposes the complementarity of the theoretical approaches involved, whereas combining means that "the theoretical approaches are only juxtaposed according to a specific aspect". The combining strategy can thus involve theories with some conflicting basic assumptions. On page 496, we are told that integrating locally and synthesizing label a pair of strategies that focus "on the development of theories by putting together a small number of theories or theoretical approaches into a new framework", and the authors differentiate between the two strategies by considering the dissymmetry/symmetry of the theories involved in terms of scope and degree of development. As explained in (Artigue 2019), we find it more appropriate to differentiate between these two strategies according to the dissymmetrical/symmetrical contribution of the theories involved in the resulting construction, and to consider that any time there is a significant dissymmetry in the connection, even for theories having a similar state of development, there is local integration. This is the case, for example, when a broad theory is enriched by theoretical constructs from another well-established theory in order to build an integrated theoretical framework that takes into account research findings in a particular area produced in a different theoretical culture.

2.2. The concept of research praxeology

The notion of research praxeology introduced in (Artigue et al., 2011) extends to research practices the concept of praxeology at the heart of the anthropological theory of the didactic (hereafter ATD) (Chevallard, 2019). Indeed, a basic principle of ATD is that all human practices can be modelled in terms of praxeologies. These consist of a praxis block and a theoretical block in dialectical interaction. A priori, thus, this is also the case for research practices. By definition, the praxis block of research praxeologies includes the different types of tasks that the research activity requires, and the associated techniques of study. Their theoretical block consists of the discourse used to describe, justify and interpret research techniques (technological discourse in ATD language), and a theory consisting of "statements of a more general and abstract character, with a generally strong justifying and generating power." (Bosch & Chevallard, 2020). At the simplest level of point praxeologies, dealing with a single type of task, a typical research praxeology in mathematics education is associated with a research question and a technique of study for this question. These form its praxis block. This block is dialectically linked to a theoretical block consisting of technological and theoretical discourses. The theoretical block includes at least some methodological discourse that explains and justifies the technical choices; some research background, namely existing knowledge regarding the question at hand or related questions, and associated constructs that are considered of interest, plus at least one theory that underlies the whole construction.

Of course, this is a very simple case. Most research questions cannot be solved by using a single technique, and the theoretical block of research praxeologies, even point praxeologies, often, does not mobilize a single theory but some theoretical combination. Most often, research praxeologies emerge from questions about teaching and learning processes, or more globally about the functioning of didactic systems or institutions, that is, through their praxis block. But the questions and their formulation are influenced by the "theoretical" already present in the researcher's or the research team's environment. The study techniques are guided by those that are implemented in close research praxeologies, or that are familiar to the researcher. In this way, the praxis and the theoretical blocks interact from the outset. Moreover, the study of a research question generally requires more than a single point praxeology. It is necessary to develop a coherent set of research praxeologies that share, at least in part, the same theoretical block. Local and regional praxeologies thus emerge. New questions also arise from research results; new elements enter the technological discourse in terms of distinctions, categorizations, didactic phenomena. If they become reasonably shared, they will enrich the theory itself, thus contributing to the dynamic of research praxeologies.

We claim that approaching the relationships between theories through this praxeological lens can help researchers to take better account of the functional role of theories in the research activity. The interest in adopting such a functional stance towards theories became clear to us while working in the European project TELMA whose aim was to capitalize European knowledge about technology enhanced learning in mathematics (Artigue, 2009). At that time, however, the concept of research praxeology had not yet been introduced. In the context of research on the educational use of digital technologies, its first use took place in a retrospective analysis of the theoretical activities carried out in the ReMath European project that had emerged from TELMA (Artigue & Mariotti, 2014).

In the following two sections, we use these tools to present and discuss two case studies. These regard the instrumental approach and the documentational approach to didactics, respectively. Both approaches have obvious connections and both have generated a multiplicity of theoretical interactions, given the diversity of researchers who have contributed to their development or simply used them. They provide a limited, but sufficiently rich, ground to support the reflection aimed at in this article.

3. A FIRST CASE STUDY: THE INSTRUMENTAL APPROACH

3.1. Overview of the instrumental approach

The instrumental approach (IA in the following) emerged in France in the mid-1990s in a specific technological context, that of CAS technology (Artigue, 2002), and with two theoretical pillars: the ergonomic perspective on contemporary tools developed by Rabardel and Vérillon (Rabardel, 1995; Vérillon & Rabardel, 1995) and ATD[1]. In fact, this approach quickly spread beyond the small community in which it was born. The rapid publication of books, both in French and in English, and of many articles certainly favored this dissemination. All over the world, researchers began to incorporate IA constructs into their theoretical frameworks. This phenomenon was already evident at the time of the ICMI Study 17 mentioned above, and in the last decade the theoretical diversity has even increased. All constructions share the reference to the Rabardel's and Vérillon's ergonomic perspective, in particular:

- the distinction between artefact and instrument;
- the concept of instrumental genesis with its two movements dialectically connected: instrumentalization from the user to the artefact and instrumentation from the artefact to the user;
- and the conceptualization of instrumental genesis in terms of the elaboration or appropriation of schemes.

However, these constructs are combined with or integrated into different theories, which leads to significantly different research praxeologies. This phenomenon was already visible in the first two doctoral theses based on IA, those of Defouad (2000) and Trouche (1997), the latter having the theory of conceptual fields (Vergnaud, 1991, 2009) as its main theoretical component. The impact on the different components of their respective research praxeologies is clear, with obvious consequences on the results obtained. In Defouad's thesis, in line with the ATD conceptualization of human activities in terms of praxeologies, more emphasis is placed on the instrumented techniques developed, the material signs (ostensives in ATD language) used in them, and the discourse explaining and justifying the techniques (the technological discourse in the praxeological model), rather than on the schemes underlying them. Furthermore, special attention is paid to the relationships between the students' instrumental geneses and the classroom instrumental genesis, and their management by the teacher. The results are mainly expressed in terms of regularities identified in the students' instrumental geneses and of didactic phenomena that show links between the characteristics of the students' instrumental geneses and the institutional conditions and constraints that shape the ecology of instrumented techniques in the classroom. In Trouche's thesis, the results are mainly expressed in terms of schemes of instrumented action and of their evolution in the transition from graphic to symbolic calculators.

A few years later, in the ReMath project (Kynigos & Lagrange, 2014), thanks to the development of specific methodological tools, we were able to analyze the effect on research praxeologies of the integration of instrumental perspectives in theories born outside the French didactic tradition such as the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) or constructionism (Papert, 1980). New local integrations have also resulted from the extension of IA to the teacher, such as its integration in the dual

1 More detailed information about the instrumental approach, its emergence and development, can be found in the Michèle Artigue's Unit of the ICMI AMOR project: https://www.mathunion.org/icmi/awards/amor/michele-artigue-unit

ergonomic and didactic approach of teachers' practices (Robert and Rogalski, 2002), well illustrated by the GUPTEN project in France (Lagrange, 2013).

3.2. Networking challenges: the scheme/technique case

There is no doubt that all these connections have contributed to the wealth of knowledge and results produced under the IA umbrella, but they have also raised serious challenges. We will illustrate these challenges with an example, the scheme/technique case, which has been the source of intense debate for almost a decade (see the synthesis offered in (Monaghan, 2007)). IA emerged as a synthesis between Rabardel's and Vérillon's instrumental perspective and ATD. In such a process, ATD categories and discourses were intertwined with those proposed by cognitive ergonomics. Instrumental geneses were thought in terms of the development of praxeologies, and thus approached in terms of instrumented techniques and associated technological discourses, rather than only in terms of schemes, as was the case in ergonomic publications. This duality was discussed extensively in the first decade of IA. The collective work on this issue helped to clarify the different points of view and, in particular, to reject simplistic assimilations such as those that reduced techniques to gestures or to the observable part of schemes. In fact, thinking in terms of praxeologies means that techniques cannot be isolated from the technological discourse that describes, explains and justifies them. In a sense, reducing techniques to gestures is akin to reducing schemes to their observable characteristics without considering the essential component of schemes that the operational invariants underlying the observed regularities are. Indeed, the many contributions to the scheme/technique debate have made it clear that schemes and techniques correspond to two different and complementary ways of approaching instrumental issues, both insightful but irreducible to each other.

However, a new element deserves to be taken into account in today‘s reflection on this topic. In fact, while maintaining its institutional anchorage, ATD has progressively incorporated the concept of personal (mathematics) praxeology, following research by Crozet and Chaachoua (2016)[2]. Like institutional praxeologies, personal praxeologies are modelled in terms of quadruplets that dialectically articulate a praxis block consisting of types of tasks and techniques and a theoretical discursive block. However, this personal dimension, the very nature of the students‘ verbalizations used to identify the theoretical block of personal praxeologies, should allow for more productive connections between the language of schemes and the praxeological language.

3.3.The ReMath contribution

The European ReMath project also had important spin-offs in terms of networking involving IA, as it soon became clear that, despite the diversity of their theoretical cultures, five of the six teams involved were sensitive to Rabardel's and Vérillon's conceptualizations and had incorporated them in some way into their theoretical background. Thanks to a carefully designed methodology, we were able to document and analyze the effects of these incorporations both in the design and development of digital dynamic artefacts (DDAs) – six DDAs were actually created or improved as part of this project – and in the use of these DDAs by two different teams, the team in charge of the DDA design and another team from another country with a

2 As explained in Module 9 of the Artigue's unit of the AMOR project, the idea of personal praxeology had been already introduced in Defouad's thesis, but it did not disseminate at the time and was soon forgotten.

different theoretical background. In this paper, we focus on the case of constructionism, and a comparison involving the Greek team in the Educational Technology Lab (ETL), for which constructionism was the main theoretical support, and the French team in DIDIREM, now the Laboratory of Didactics André Revuz (LDAR), whose theoretical background combined IA with the theory of didactic situations (Brousseau, 1998) and Duval's semiotic theory (Duval, 1995), both very influential in French didactic research on digital technologies (Artigue & Trouche, 2021). As explained in the special issue (Kynigos & Lagrange, 2014), in ReMath ETL was in charge of two DDAs, Cruislet and MaLT, and DIDIREM was in charge of the DDA Casyopée. Their constructionist culture made the Greek researchers particularly sensitive to the expressive power of digital artefacts, which they had theorized in terms of "half-baked microworld", that is to say "pieces of software explicitly designed so that their users would want to build on them, change them or de-compose parts of them in order to construct an artifact for themselves or one designed for instrumentation by others" (Kynigos, 2007, p. 336). This obviously influenced their (local) integration of Rabardel's and Vérillon's ergonomic perspective into constructionism, leading them to emphasize the instrumentalization dimension of the process of instrumental genesis. Indeed, instrumentalization captures the vision that artefacts can be transformed by their users in ways anticipated by the designers, but also in ways that are not anticipated (catachresis phenomena). Instrumental genesis results from the dialectic interaction between instrumentalization and instrumentation processes, and a constructionist perspective leads to emphasize the role of instrumentalization in this dialectic interaction. In the design of Cruislet and MALT, and in the tasks they proposed to students, the ETL researchers were particularly sensitive to the need to support this instrumentalization dimension of instrumental geneses and the students' creativity associated with it. In line with the constructionist culture, programmable simulations using the Logo language, open and challenging tasks with strong potential for the students' expressiveness were essential tools for this. The contrast with the DDA Casyopée was clear. This DDA was very innovative in terms of dynamic connections between representations – for example, it offered advanced tools for the functional modelling of geometric situations involving covariation of lengths and areas –, but the focus was on institutional graphic and symbolic representations, and the support offered by the DDA for their treatment within a given semiotic register or conversion from one register to another one, in line with Duval's semiotics. In terms of design priorities, therefore, more emphasis was placed on supporting instrumentation processes than on supporting instrumentalization processes, and the learning potential offered by the emergence and progressive transformation of alternative representations to the institutional ones was not considered. Such comparisons sharpened our vision of instrumentalization, of the dialectical games between instrumentalization and instrumentation, and made clear the influence on our vision of the theories with which we had networked ergonomic perspectives.

The cross-case study regarding the DDA Cruislet, involving ETL and DIDIREM, was also particularly insightful. It highlighted the profound differences between scenarios based on the theory of didactic situations and constructionism, although both theories share socio-constructivist principles. The concept of fundamental situation, the aim of optimizing the adidactic potential of situations through the selection and management of didactic variables, and the attention paid to the organization of the dual processes of devolution and institutionalization, are at the very heart of the theory of didactic situations and of didactic engineering designs based on this theory. Such constructs are alien to constructionist scenarios. In these scenarios, the aim of ensuring that the tasks designed and their technological and social environment offer strong potential for supporting the students' expressiveness is more important than the aim of achieving a precise learning goal and optimizing the possible trajectories. The Cruislet software, which offers various possibilities for simulating airplane flights over a map of Greece, was perfectly adapted to a constructionist approach to learning. For the French teachers and researchers, experimenting with it was a real challenge, and ultimately only possible within a specific project device, less constrained than the usual lessons. However, this experience was also a source of questioning and enrichment for us, as shown in (Le Feuvre, Meyrier & Lagrange 2010). Today, the evolution of the concept of didactic engineering carried out by ATD through the concept of study and research path and the related methodological tools such as question-answer graphs (Barquero & Bosch, 2015), undoubtedly provides a more appropriate framework for exploiting the potential of such constructionist DDAs in the French didactic culture.

Beyond this particular case, the ReMath cross-experiments showed that the productive networking of theories does not obey the same constraints when theories are networked, a posteriori, to analyze and interpret collected data, and when they are networked to build a conceptual framework to support didactic design, be it the design of digital artefacts or the design of tasks and learning situations involving them. The experimentation of Cruislet mentioned above showed that the logics underlying constructionist design and TDS design were so far apart that it was very difficult, if not impossible, to coordinate them in a conceptual framework. Significant distance was also observed in the conceptual frameworks supporting the design of situations involving the DDA Casyopée by the French DIDIREM team and the Italian UNISI team from the University of Siena. The UNISI design was guided by the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008), while the DIDIREM design, once again, was guided by TDS and Duval's semiotics. In line with their theoretical background, the Italian researchers conceived Casyopée as an instrument of semiotic mediation, and, in their design, they carefully organized the transition from semiotic signs attached to the artefact to mathematical signs detached from the artefact. Much time was devoted to this transition, in collective discussions led by the teacher without access to Casyopée. For the French researchers, in line with the TDS, Casyopée was a central element of the a-didactic milieu. Their design, while paying attention to the connection to be made between paper-pencil and instrumented techniques, did not conceive this connection in terms of transition. This was clear in the institutionalization phase where both instrumented and non-instrumented forms of knowledge were combined. Conversely, the various cross-case studies carried out in ReMath have shown the potential offered by the combination of constructionism, TDS and the theory of semiotic mediation in the a posteriori analysis of the experiments (see, for example, Maracci et al., 2013). In summary, the a posteriori combination of complementary and even conflicting theoretical perspectives in the analysis and interpretation of data can be very productive, while the combination of theoretical perspectives that are too far apart can make a coherent design impossible. More compatibility is needed. This is consistent with the distinction made between the combining and coordinating strategies in the scale of networking strategies (see Section 2) and confirms the pertinence of this distinction. Design based on theoretical networking requires coordination, and not just combination, of theories or theoretical constructs.

4. A SECOND CASE STUDY: THE DOCUMENTATIONAL APPROACH TO DIDACTICS

The documentational approach to didactics (hereafter DAD) was born in a course given at the French Summer school of didactics of mathematics in 2007 (Gueudet & Trouche, 2009). It was motivated by the significant changes in the documentational work of teachers brought about by technological evolution, the increasing number of digital and online resources available to and used by teachers, and the growing role of professional and social networks in their professional activity. Digital technologies are therefore at the heart of DAD. As Gueudet (2009) explains in her analysis of the development of DAD, its source was her and Trouche's earlier work, which was mainly concerned with digital technologies. For Trouche, it was the contribution to the development of IA and to the SFODEM project, "a project of in-service teacher training, mainly at distance, aiming to support the integration of ICT in the teachers' practices by a rich offer of resources on a platform" (Gueudet, 2019, p. 19). For Gueudet, it was her research on learning processes with online exercises, first at university level and then extended at all levels of schooling, in particular in connection with the association of teachers Sésamath, an association that has produced very much used online resources, since its creation more than 20 years ago (https://www.sesamath.net).

The development of DAD in just fifteen years is impressive, as is impressive the number of researchers from different cultures and backgrounds who have contributed to it, as shown in (Trouche, Gueudet & Pepin, 2019). Another interesting element for our reflection is that, from the beginning, DAD has combined different sources of inspiration: IA and it can be seen as an extension of IA to the documentational work of teachers, but also many others. For instance, Adler's conception that "resources for school mathematics extend beyond basic material and human resources to include a range of other human and material resources, as well as mathematical, cultural and social-resources" (Adler, 2000, p. 210) is explicitly used in the extended definition of resources proposed in the DAD. Remillard's participative approach to the use of curriculum resources (Remillard, 2005), or Wenger's community of practice approach (Wenger, 1998), to mention just but a few, have also been very influential.

Over the last decade, we can observe a progressive evolution of the connections, both on the side of the founders of DAD and on the side of its contributors and users. Analyzing the first book published on this approach (Gueudet & Trouche, 2010), we have shown in (Artigue, 2019) that Gueudet and Trouche mention many authors in the two chapters they co-author, but that few theoretical connections are really elaborated beyond the foundational one with IA. For example, we observe only one example of local integration beyond the already mentioned one with Adler's conception of resources. It occurs when the authors link teachers' systems of documents and systems of activities by introducing a categorization of teachers' activities explicitly inspired by the study moments of ATD (Chevallard 2002). The foundational connection with IA leads to the distinction between resource and document, analogous to the distinction between artefact and instrument. Indeed, a document is defined as a hybrid entity consisting of updated/recombined resources and of a scheme of utilization of these. This connection also leads to the concept of documentational genesis with the dual processes of instrumentation and instrumentalization, and the development of schemes.

An interesting point is the awareness of researchers, since this emerging state of DAD, of the need for specific methodologies that allow them to approach the teachers' documentational activity in-class and out-of-class, with a long-term follow-up, and a close involvement of the teacher in the collection of data leading to a reflective stance. Very soon, this methodology was given a name: "reflective investigation" and a specific tool was associated with it: the SRRS (schematic representation of resource system), and both became emblematic of the research praxeologies based on DAD. In the chapters of the same book written by other researchers, one can observe many cross-references but, once again, rather few advanced forms of networking, which is not surprising in this nascent state of DAD. In most chapters, the main purpose seems to be to make another approach understandable, or to contrast it with DAD. Only one chapter really goes further and opens the way to local integration (Trgalová, 2010). As explained in (Artigue, 2019, p. 100), "the author connects DAD with the model of teacher activity developed by Margolinas (2002) and Balacheff's cK¢ model of conceptions (Balacheff 1995), two constructions already connected by these two authors to analyze teachers' didactic decisions in (Balacheff and Margolinas, 2005)". It is interesting to note that, in this particular case, the data used comes from a doctoral thesis whose methodology has stimulated some form of documentational work, which certainly makes the networking more practicable.

As shown in (Artigue, 2019), the theoretical core of DAD research praxeologies stabilized quickly, which contributed to the coherent structuring of these praxeologies, together with the privileged role given to the methodology of reflective investigation and the SRRS tool. This did not prevent the diversification of research *problématiques*, to which the increasing cultural diversity of the researchers contributing to DAD certainly contributed. When preparing our plenary lecture at the Re(s)sources 2018 conference, we asked Luc Trouche for a selection of references to get an updated vision of the theoretical connections in DAD, and we received a list of theoretical crossings with one or two references for each. The list mentioned thirteen theories and theoretical approaches. Some of them were already present in the first book such as information and communication sciences, Remillard's approach or communities of practice, but also were new such as cultural historical activity theory, the theory of social creativity, constructionism or the meta-didactical transposition approach. Even if it remains partial, the analysis of the connections carried out in (Artigue, 2019) shows a clear progression in their quality. The collaborative work that has been developed, evidenced by the fact that most publications are co-authored, has obviously made this progress possible. There is no doubt that DAD researchers have been able to create communication devices and forms of collaborative work that help to meet the challenges raised by the increasing diversity of theoretical connections involved in the development of the theory, and also by the linguistic diversity of their community as shown by the DAD-MULTILINGUAL project (https://hal.science/DAD-MULTILINGUAL/).

In terms of research praxeologies, DAD today unifies, at a regional level, a diversity of research praxeologies aimed at better understanding of the transformations of teachers' activity induced by the use of digital technologies, and at developing innovative uses of the increasing diversity of these technologies. As already mentioned, DAD research has created a methodological tool, the methodology of reflective investigation. This methodology, which provides a study technique well adapted to the research questions addressed by DAD, has become emblematic of its research praxeologies, and has been progressively enriched as shown in (Trouche, Gueudet & Pepin, 2019). The theoretical block of DAD research praxeologies has also been progressively enriched in dialectic interaction with the enrichment of their praxis block. For example, the study of documentational genesis over longer periods of time than was initially the case has led to the concept of documentational trajectory, which in turn has led to new methodological and representational tools (see, for example, Rocha, 2018).

In this second case study, the combined use of the networking and praxeological lenses helps us to understand a dynamic particularly rich in theoretical connections that has developed over the last fifteen years, to understand why we observe such a theoretical diversity in it, and how this diversity has gradually been better managed through more advanced forms of networking, and also thanks to the unifying characteristics of the DAD research praxeologies.

5. CONCLUSION

In this paper, we have shared some elements of the knowledge we have gained through our experience of networking between theories, focusing on research on teaching and learning in digital environments and the ways in which these influence teaching and learning processes in mathematics. If we return to the question that has guided the reflection developed in this paper: "To what extent are we now better equipped to face the challenge of theoretical diversity in what regards technology-based mathematics teaching and learning?", our reflection makes clear that conceptual and methodological tools have come a long way since the issues raised by theoretical diversity were seriously put on the agenda of the international mathematics education community at the beginning of this century. As a result, as we have tried to show, knowledge has advanced both in terms of understanding the diversity challenge and in terms of developing strategies to address it. All the forms of networking identified in the scale of networking strategies have been used productively, involving a number of theories with different state of development, aims and scope. The connections established and the knowledge built now form a solid background for supporting research in this area. Networking research has also shown the interest of developing specific research praxeologies. These allow us to examine our respective research practices, their theoretical frameworks and their results without denaturing the theories and theoretical constructs involved, in order to build significant connections, and make clear their potential and limitations, as has been done, for example, in ReMath and in (Bikner-Ahsbahs & Prediger, 2014). More globally, our reflection confirms the interest of adopting a praxeological lens to approach the diversity challenge.

The theoretical landscape of research on technology-based teaching and learning is a dynamic one. On the one hand, the development of multi-touch mobile technologies, of virtual and augmented reality devices, is the source of new interesting theoretical connections, such as the connection between IA and radical embodied cognitive science approaches presented in (Shvarts et al., 2021). On the other hand, the massive move towards online and hybrid mathematics teaching provoked by the pandemic situation has abruptly placed technology-based teaching and learning at the center of the educational agenda. This situation, and the resulting increase in educational inequalities, is also a source of new theoretical needs and connections, as illustrated by Borba (2021), for example. In this contribution to the special call on mathematics education research in pandemic times launched by the *journal Educational Studies in Mathematics*, Borba indeed reworks the theoretical construct of humans-with-media that he introduced years ago to emphasize that the production of knowledge results from the collective agency of humans and media (Borba & Villareal, 2005) to take into account the impact and agency of non-living things such as COVID-19. He also suggests combining this theoretical perspective with that offered by critical mathematics education (Skovsmose,

1994) to address the crucial issue of educational inequalities exacerbated by the pandemic. These are only two examples but there is no doubt that networking needs will continue to grow in this area of research as well as in mathematics education more generally.

REFERENCES

Abboud, M., & Vandebrouck, F. (2013). Geneses of technology uses: A theoretical model to study the development of teachers' practices in technology environments. In B. Ubuz, C. Haser & M.A. Mariotti (Eds.), *Proceedings of CERME 8* (pp. 2504–2513). Middle East Technical University and ERME. http://erme.site/wp-content/uploads/2021/06/CERME8_2013_Proceedings.pdf

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224. https://doi.org/10.1023/A:1009903206236

Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematics Learning*, *7*(3), 245–274. https://www.learntechlib.org/p/96049/.

Artigue, M. (2007). Digital technologies: A window on theoretical issues in mathematics education. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of CERME 5* (pp. 68–82). Cyprus University and ERME. http://erme.site/cerme-conferences/cerme-5/

Artigue, M. (Ed.) (2009). Connecting approaches to technology enhanced learning in mathematics: The TELMA experience. *International Journal of Computers for Mathematical Learning*, *14*(3). https://www.learntechlib.org/j/ISSN-1382-3892/v/14/n/3/

Artigue, M. (2019). Reflecting on a theoretical approach from a networking perspective: The case of the documentational approach to didactics. In L. Trouche, G. Gueudet & B. Pepin (Eds.), *The 'Resource' Approach to Mathematics Education* (pp. 89–120). Springer. https://doi.org/10.1007/978-3-030-20393-1_5

Artigue, M. (2022). From the Networking of Theories to the Discussion of the Educational Implications of Research. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás & N. Ruiz-Munzón (Eds.), *Advances in the anthropological theory of the didactic* (pp. 25-36). Birkhäuser. https://doi.org/10.1007/978-3-030-76791-4_

Artigue, M., Bosch, M., & Gascón, J. (2011). La TAD face au problème de l'interaction entre cadres théoriques en didactique des mathématiques. In M. Bosch et al. (Eds.), *Un panorama de la TAD. Actes du troisième congrès de la TAD* (pp. 33–56). Centre de Recerca Matemàtica, University of Barcelona.

Artigue, M., & Bosch, M. (2014). Reflection on networking through the praxeological lens. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 249–266). Springer. https://doi.org/10.1007/978-3-319-05389-9_15

Artigue, M., & Mariotti, M.A. (2014). Networking theoretical frames: the ReMath enterprise. *Educational Studies in Mathematics, 85*(3), 329-356. https://doi.org/10.1007/s10649-013-9522-2

Artigue, M., & Trouche, L. (2021). Revisiting the French didactic tradition through technological lenses. *MPDI Highlights of Research in the Didactics of Mathematics: The Insider's Perspective, Mathematics*, *9*(6), 629. https://doi.org/10.3390/math9060629

Balacheff, N. (1995). Conception, connaissance et concept. *Didactique et technologies cognitives en mathématiques. Séminaires 1994–95* (pp. 219–244). Université Joseph Fourier. https://hal.science/hal-01072247

Balacheff, N., & Margolinas, C. (2005). cK¢ Modèle de connaissances pour le calcul de situations didactiques. In A. Mercier & C. Margolinas (Eds.), *Balises en Didactique des Mathématiques* (pp. 75–106). La Pensée Sauvage éditions. https://revue-rdm.com/ouvrage/balises-pour-la-didactique-des-mathematiques/

Barquero, B., & Bosch, M. (2015). Didactic engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *c*(pp. 249–272). Springer. https://doi.org/10.1007/978-3-319-09629-2_13

Bartolini Bussi, M., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom. Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini-Bussi, G. Jones, R. Lesh & D. Tirosh (Eds.), *Handbook of international research in mathematics education, second revised edition* (pp. 746–805). Lawrence Erlbaum. https://doi.org/10.4324/9780203930236.ch28

Beatty, R., & Geiger, V. (2010). Technology, communication, and collaboration: Re-thinking communities of inquiry, learning and practice. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology – Rethinking the terrain.* (pp. 251–284). Springer. https://doi.org/10.1007/978-1-4419-0146-0_11

Bikner-Ahsbahs, A. (2016). Networking of theories in the tradition of TME. In Bikner-Ahsbahs, A. et al. (Eds.), *Theories in and of mathematics education. Theory strands in German-speaking countries* (pp. 33–42). Springer. https://doi.org/10.1007/978-3-319-42589-4_5

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories in mathematics education* (pp. 483–506). Springer. https://doi.org/10.1007/978-3-642-00742-2_46

Bikner-Ahsbahs, A., & Prediger, S. (Eds.) (2014). *Networking of theories as a research practice in mathematics education*. Springer. https://doi.org/10.1007/978-3-319-05389-9

Borba, M. (2021). The future of mathematics education since COVID-19: humans-with-media or humans-with-non-living-things. *Educational Studies in Mathematics, 108,* 385–400. https://doi.org/10.1007/s10649-021-10043-2

Borba, M., & Villareal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking*. Springer. https://doi.org/10.1007/b105001

Bosch, M., & Chevallard, Y. (2020). The Anthropological theory of the didactic. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education. Second Edition* (pp. 53–61). New York: Springer. https://doi.org/10.1007/978-3-030-15789-0_100034

Brousseau, G. (1998). *Theory of didactical situations in mathematics*. Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47211-2

Chevallard, Y. (2002). Organiser l'étude. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la Xème École d'été de didactique des mathématiques* (pp. 3–22, 41–56). La Pensée Sauvage éditions.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114. https://doi.org/10.24529/hjme.1205

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.) (2014). *The mathematics teacher in the digital era*. Springer. https://doi.org/10.1007/978-94-007-4638-1

Crozet, M.-C., & Chaachoua, H. (2016). Une réponse à la prise en compte de l'apprenant dans la TAD : la praxéologie personnelle. *Recherches en Didactique des Mathématiques, 36*(2), 161–196. https://revue-rdm.com/2016/une-reponse-a-la-prise-en-compte/

Defouad B. (2000). *Etude de genèses instrumentales liées à l'utilisation d'une calculatrice symbolique en classe de première S.* Thèse de doctorat. Université Paris 7. https://tel.archives-ouvertes.fr/tel-01253860

Drijvers, P., Kieran, C., & Mariotti, M.A. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology – Rethinking the terrain.* (pp. 89–132). Springer. https://doi.org/10.1007/978-1-4419-0146-0_7

Duval, R. (1995). *Sémiosis & pensée humaine*. Peter Lang.

Gueudet, G. (2019). Studying teachers' documentation work: Emergence of a theoretical approach. In L. Trouche, G. Gueudet & B. Pepin (Eds.) *The 'Resource' approach to mathematics education* (pp. 17-42).. Springer. https://

doi.org/10.1007/978-3-030-20393-1_2

Gueudet, G., & Trouche, L. (2009). Vers de nouveaux systèmes documentaires des professeurs de mathématiques. In I. Bloch & F. Conne (Eds.), *Nouvelles perspectives en didactique des mathématiques* (pp. 109–133). La Pensée Sauvage éditions. https://revue-rdm.com/ouvrage/nouvelles-perspectives-en-didactique-des-mathematiques/

Gueudet, G., & Trouche, L. (Eds.) (2010). *Ressources vives. Le travail documentaire des professeurs de mathématiques*. Presses Universitaires de Rennes and INRP. https://www.pur-editions.fr/product/3915/ressources-vives

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of CERME 7* (pp. 2298–2307). University of Rzeszów and ERME. http://erme.site/wp-content/uploads/2021/06/CERME7.pdf

Hoyles, C., & Lagrange, J.-B. (Eds.) (2010). *Mathematics education and technology – Rethinking the terrain*. Springer. https://doi.org/10.1007/978-1-4419-0146-0

Kidron, I, Bosch, M., Monaghan, J., & Palmer, H. (2018). Theoretical perspectives and approaches in mathematics education research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 254–268). Routledge. https://doi.org/10.4324/9781315113562

Kynigos, C. (2007). Half-baked Logo microworlds as boundary objects in integrated design. *Informatics in Education, 6*(2), 335–358. https://doi.org/10.15388/infedu.2007.22

Kynigos, C., & Lagrange, J.-B. (Eds.) (2014). Special Issue: Representing mathematics with digital media: Working across theoretical and contextual boundaries. *Educational Studies in Mathematics*, *85*(3). https://link.springer.com/journal/10649/volumes-and-issues/85-3

Lagrange, J.-B. (Ed.) (2013). *Les technologies numériques pour l'enseignement : usages, dispositifs et genèses.* Octarès. https://www.octares.com/boutique/109-technologies-numeriques-enseignement.html

Le Feuvre, B., Meyrier, X, & Lagrange, j.-B. (2010). Apprendre des notions mathématiques, géographiques et algorithmiques à l'aide d'un environnement de navigation 3D au-dessus de la Grèce. *MathémaTICE*, 18. http://revue.sesamath.net/spip.php?article264

Maracci, M., Cazes, C., Vandebrouck, F., & Mariotti, M. A. (2013). Synergies between theoretical approaches to mathematics education with technology: A case study through a cross-analysis methodology. *Educational Studies in Mathematics Education, 84*(3), 461–485. https://doi.org/10.1007/s10649-013-9495-1

Margolinas, C. (2002). Situations, milieu, connaissances : analyse de l'activité du professeur. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot & R. Floris (Eds.), *Actes de la XIe École d'été de didactique des mathématiques* (pp. 141–156). La Pensée Sauvage éditions. https://revue-rdm.com/ouvrage/actes-de-la-xieme-ecole-d-ete-de-didactique-des-mathematiques/

Mesiti, C., Artigue, M., Grau, V., & Novotná, J. (2022). Towards an international lexicon. *ZDM – Mathematics Education, 54*(2), 239-255. https://doi.org/10.1007/s11858-022-01349-3

Mishra, P., & Koehler, M.J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017–1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Monaghan, J. (2007). Computer algebra, instrumentation and the anthropological approach. *The International Journal for Technology in Mathematics Education. 14*(2), 63–71. https://www.learntechlib.org/j/ISSN-1744-2710/v/14/n/2/

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas*. Harvester Press.

Rasmussen, C., & Nemirovski, R. (2003). Becoming friends with acceleration. The role of tools and bodily activity in mathematics learning. In N.A. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education*, Vol.1 (pp. 127–135). PME. https://www.igpme.org/publications/current-proceedings/

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings. Learning cultures and computers*. Kluwer

Academic Publishers. https://doi.org/10.1007/978-94-009-1696-8

Rabardel, P. (1995). *Les hommes et les technologies. Approche cognitive des instruments contemporains.* Armand Colin. . https://hal.science/hal-01017462

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246. https://doi.org/10.3102/00346543075002211

Robert, A., & Rogalski, J. (2002). Le système complexe et cohérent des pratiques des enseignants de mathématiques : une double approche. *Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies, 2*(4), 505–528. https://doi.org/10.1080/14926150209556538

Rocha, K.M. (2018). Uses of online resources and documentational trajectories: The case of Sésamath. In L. Fan, L. Trouche, S., Rezat, C. Qi, & J. Visnovska (Eds.), *Research on mathematics textbooks and teachers' resources: Advances and issues* (pp. 235–258). Springer. https://doi: 10.1007/978-3-319-73253-4_11

Ruthven, K. (2009). Towards a naturalistic conceptualization of technology integration in classroom practice: The example of school mathematics. *Education & Didactique, 3*(1), 131–149. https://www.pur-editions.fr/product/1286/education-et-didactique-3-1

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 373–394). Springer. https://doi.org/10.1007/978-94-007-4638-1_16

Shvarts, A., Alberto, R., Bakker, A., Doorman, M., & Drijvers, P. (2021). Embodied instrumentation in learning mathematics as the genesis of a body-artifact functional system. *Educational Studies in Mathematics, 107(3),* 447–469. https://doi.org/10.1007/s10649-021-10053-0

Skovsmose, O. (1994). Towards a critical mathematics education. *Educational Studies in Mathematics, 27,* 37–57. https://doi.org/10.1007/BF01284527

Trgalová, J. (2010). Documentation et décisions didactiques des professeurs. In G. Gueudet & L. Trouche (Eds.), *Ressources vives. Le travail documentaire des professeurs de mathématiques* (pp. 271–302). Presses Universitaires de Rennes and INRP. https://www.pur-editions.fr/product/3915/ressources-vives

Trgalová, J., Clark-Wilson, A., & Weigand, H.-G. (2018). Technology and resources in mathematics education. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 142–161). Routledge. https://doi.org/10.4324/9781315113562

Trouche L. (1997). *A propos de l'apprentissage de fonctions dans un environnement de calculatrices, étude des rapports entre processus de conceptualisation et processus d'instrumentation*. Thèse de doctorat. Université de Montpellier. https://tel.archives-ouvertes.fr/tel-02374810

Trouche, L. (2005). Construction et conduite des instruments dans les apprentissages mathématiques : nécessité des orchestrations. *Recherches en Didactique des Mathématiques, 25*(1), 91–138. https://revue-rdm.com/2005/construction-et-conduite-des-instruments-dans-les-apprentissages-mathematiques-necessite-des-orchestrations/

Trouche, L., Gueudet, G., & Pepin, B. (Eds.) (2019). *The 'Resource' approach to mathematics education*. Springer. https://doi.org/10.1007/978-3-030-20393-1

Vergnaud, G. (1991). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques, 10*(2/3), 133–170. https://revue-rdm.com/2005/la-theorie-des-champs-conceptuels/

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52,* 83-94. http://doi.org/10.1159/000202727

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: a contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101. https://doi.org/10.1007/bf03172796

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press. https://

doi.org/10.1017/CBO9780511803932

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THE ROLE OF A PRIORI ANALYSIS IN THEORIES

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Abstract

The paper offers a theoretical reflection on the role of a priori analysis in theory. For each theory the a priori analysis has a role of reference while comparing with the a posteriori analysis. Different theories offer different a priori analyses. The different a priori analyses result from the different priorities of the theories with regard to the focus of analysis. In the a priori analyses these priorities and their differences are made explicit. This study demonstrates the important role of the a priori analysis in the different theories and the benefit of comparing a priori analyses for networking the theories.

Keywords: A priori analysis, a posteriori analysis, networking theories, epistemological perspective.

INTRODUCTION

This paper offers a theoretical reflection on the role of a priori analysis in different theories. In the words “the role of a priori analysis in different theories” I mean both, the role of a priori analysis as a methodology within a given perspective and also the role of a priori analysis in the research of networking theories as a methodological tool to better understand the specificities of different theories. In the following, I explain why I consider both roles of a priori analyses, in a given theory and in networking theories.

In Kidron et al. (2018), the authors deal first with the diversity of theoretical perspectives and networking theories and only then questions concerning the role of theory are discussed with regard to the diversity of theoretical perspectives and with networking in the background. The authors explain how by means of networking theories researchers understand better what a theory is:

> In all the work related to networking theoretical approaches, deepening into the notion of theory appears as a crucial issue. (Kidron et al., p.260)

The present paper deals with the role of a priori analysis. Trying to give a general description of the term a priori analysis, we may describe an a priori analysis as an analysis, which is often carried out prior to the experiment or data collection as a part of methodology proposed within some theoretical perspective. Therefore, the answer to the question “What is an a priori analysis?” might be different for different theories.

In the present paper, I investigate the differences in the a priori analyses of some specific theories as well as the influence of the different a priori analyses on the networking of these theories.

Dealing with a priori analyses, the focus of the paper is on methodology. We will also observe how the

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notion of a priori analysis is closely related to the phase of design.

The main theories discussed in this paper are the Theory of Didactical Situations (TDS) and the theory of Abstraction in Context (AiC). There are three main reasons for the choice of these theories. The first one is that TDS as a well-established theory that belongs to the French school of didactics benefits from a strong a priori analysis. The second reason of this choice is that this paper is based on my networking experience with these theories (Kidron et al., 2008; Kidron et al., 2014). The third reason is that the a priori analyses take into account the mathematical epistemology of the given domain and for both theories the epistemological perspective is of importance.

Then, other examples of the differences in the a priori analyses of couple of theories (TDS and APC (Action, Production, Communication); CWS (Connected Working Spaces) and AiC) will be presented shortly as well as the influence of the different a priori analyses on the networking of these theories.

Only then, we will be in a better situation to understand the important role of the a priori analysis in the different theories and the benefit of comparing a priori analyses for networking theories.

A PRIORI ANALYSES IN DIFFERENT THEORIES: THEORY OF DIDACTICAL SITUATIONS (TDS) and THEORY OF ABSTRACTION in CONTEXT (AiC)

For both theories, one main focus of the a priori analysis is the epistemological perspective. We will observe how the two theories consider the epistemological dimension in different ways. TDS combines epistemological, cognitive, and didactical perspectives. TDS focuses on the epistemological potential of didactical situations. AiC analysis is essentially cognitive and focuses on the students' reasoning; mathematical meaning resides in the verticality of the knowledge constructing process and the added depth of the resulting constructs. An epistemological stance is underlying this idea of vertical reorganization (I will explain this term in more details in a next subsection). For both theories, the epistemological dimension has a significant role. In the literature, the important role of epistemology is discussed in detail in Artigue (1990, 1995). Kidron (2016) analyses the connection between epistemology and networking of theories.

In the next two subsections, I will present shortly each theory and deal with the question what is an a priori analysis for each theory. Then, I will deal with the differences of the a priori analyses and demonstrate that, in their effort of networking theories, researchers of both theories benefit of comparing their a priori analyses.

TDS a priori analysis

A short introduction to TDS is offered in Artigue et al. (2014). As mentioned earlier, this well-established theory belongs to the French school. It began to develop in the 1960s in France, initiated by Guy Brousseau (1997). In Artigue (2020, p. 203) we read "how the idea of didactical engineering which emerged in French didactics in the early 1980s contributed to firmly establish the place of design in mathematics education research". We also read that the Didactical Engineering (DE) is structured in four different phases. Design and a priori analysis is one of these four phases. Artigue (2020, p. 204) wrote: "The goal of the a priori analysis is… to build a reference with which classroom realizations will be contrasted in the a posteriori analysis". In Artigue et al. (2014, p. 48) we learn about the three characteristics of the way TDS considers the teaching and learning of mathematics. The first characteristic, the systemic perspective of TDS is expressed by the central object of the theory, the idea of situation, which is itself a system. In the present paper, a special attention is given to the second characteristic, the epistemology of mathematical knowledge, while discussing TDS a priori analysis. The third characteristic, the vision of learning as a combination of adaptation and acculturation, relates to the cognitive dimension. These characteristics determine the questions that TDS raises, as well as its methodology.

The theory is structured around the notions of a-didactical situation (in a-didactical situation there is no explicit didactical intentions: students are working as if there is no didactical intention and the teacher refrains from interfering) and didactical situations and includes concepts relevant for teaching and learning in mathematics classrooms. The social dimension also has a significant role in TDS. In essence, the central object of the theory, the situation, incorporates the idea of social interaction.

It is this systemic view that led to the concept of DE (Artigue, 1989, 2020) that we mentioned earlier in connection to TDS a priori analysis. Artigue et al. (2014, p.50) explains DE methodology:

> It is a methodology which is structured around a phase of preliminary analysis combining epistemological, cognitive, and didactical perspectives, and aiming at the understanding of the conditions and constraints to which the didactical system considered is submitted, a phase of design and a priori analysis of situations reflecting its optimization ambition; and, after the implementation, a phase of a posteriori analysis and validation.

The notion of "milieu" is an important construct in TDS. In Brousseau (1997, p.9) we read:

> Within a situation of action, everything that acts on the student or that she acts on is called the "*milieu*".

A-didactical situations are well explained in Brousseau (1997, pp, 54-72). The a-didactic milieu was initially defined by Brousseau as the system with which the student interacts in the a-didactic game. In Brousseau (1997, p.57) we read:

> The analysis of the didactical relationships implies the definition or the recognition of these "fundamental" and adidactical games, bringing together a *milieu* and a player, these games being such that knowledge – a given precise knowledge – will appear as the means of producing winning strategies".

In the design of learning situations, there is a special attention to the constituents of the milieu organized for the learner.

In her chapter "Perspectives on design research: the case of didactical engineering", Artigue (2015) presents the evolution of Didactical Engineering (DE) in the last three decades and explains its links with TDS. She also presents its characteristics as a research methodology. In this chapter we read that design has always played a fundamental role in the French school. We also read how design is connected to the a priori analysis.

In Artigue et al. (2014, pp. 54–60), we have a detailed example that explains the components of the TDS a priori analysis and the requests of the a priori analysis for developing the systemic analysis typical for TDS. For example, the need of information of the mathematical knowledge of the students, of the particular situation at stake, of the teacher's expectations regarding this situation. The methodology for analysis is described in the following sentence:

> We developed thus our analysis using the usual techniques of TDS, that is to say, preparing an a priori analysis focusing on the determination of the cognitive potential of an a-didactic interaction with the milieu, for a *generic* and *epistemic* student, that is, a student accepting the a-didactical game and able to invest in it the mathematical and instrumental knowledge supposed by the teacher. (Artigue et al., 2014, p. 63)

Therefore, in TDS a priori analysis:

– The researchers make assumptions about the supposed mathematical knowledge of the students which is required for a productive interaction with the "milieu".
– There is a need of information of the particular situation at stake, of the teacher's expectations regarding this situation.
– There is a focus on the determination of the cognitive potential of an a-didactic interaction with the milieu.
– The researchers make assumptions about the role of the teacher and how she extends the results of the a-didactical situation.

The a priori analysis must then play its role of reference as well as its role of revealing the didactic phenomena. Then the a posteriori analysis is compared to the a priori analysis and sometimes the hypotheses which were done in the a priori analysis are not in accord with the a posteriori analysis of the collected data. This comparison of the a priori analysis and the a posteriori analysis will allow the TDS researchers to deeply understand the functioning of the "situation".

AiC a priori analysis

Dreyfus & Kidron (2014) offers a short introduction to AiC. The theory is explained in more details in (Schwarz et al., 2009). AiC has been developed over the past 20 years with the purpose of providing a theoretical and methodological approach for researching, at the micro-level, learning processes in which learners construct deep structural mathematical knowledge. Methodologically (and this is the focus of the present study), the AiC researchers are offered tools that allow them to observe and analyze students' thinking processes. A detailed treatment of the methodology is offered in Dreyfus et al. (2015). AiC view of abstraction is grounded in the works of Davydov (1990) and Freudenthal (1991). I wrote earlier that AiC focuses on the students' reasoning and that mathematical meaning resides in the verticality of the knowledge constructing process and the added depth of the resulting constructs. Freudenthal ideas led his collaborators to the idea of "vertical mathematization". This idea is explained in Dreyfus et al. (2015, p. 186–187):

> Vertical mathematization points to a process that typically consists of the reorganization of previous mathematical constructs within mathematics and by mathematical means by which students construct a new abstract construct. As researchers in mathematics education, we preferred the expression "vertical reorganization" to the expression "vertical mathematization" to discern between what is intended by the teacher - the mathematization, and what often happens - a reorganization.... In vertical reorganization, previous constructs serve as building blocks in the process of constructing.

Thus, AiC defines abstraction as a process of vertically reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means in order to lead to a new construct (for the learner). The expression used in the previous sentence *within mathematics and by mathematical means* demonstrates the importance of the epistemological dimension for AiC. For the convenience of the readers, I will report some more details about AiC. The process of abstraction has three stages: the need for a new construct, the emergence of the new construct and the consolidation of this new construct. The second stage, the emergence of the new construct is analyzed by means of three observable epistemic actions: **R**ecognizing, **B**uilding-With and **C**onstructing. Recognizing takes place when the learner recognizes that a specific knowledge construct is relevant to the problem she or he is dealing with. Building-with is an action comprising the combination of recognized knowledge elements, in order to achieve a localized goal, such as the actualization of a strategy, or a justification, or the solution of a problem. Constructing consists of integrating previous constructs by vertical mathematization to produce a new construct.

In view of AiC essential cognitive perspective, the focus is on the students' processes of construction of knowledge. In the AiC approach, contextual aspects are considered to be integral factors of the learning process. Context is regarded in a wide sense, comprising historical, physical and social context. Historical context includes students' prior learning history, physical context includes artefacts such as computers and software, and social context refers to interaction with peers, teachers and others.

Design is important for AiC. This is in accord with the epistemological stance which is underlying the idea of vertical reorganization. The design is accompanied by its epistemological aspects. As a part of the AiC methodology, an effort is made to foresee students' expected processes of construction of knowledge and an a priori analysis of the activities is conducted.

The AiC a priori analysis consists first of assumptions about the previous mathematical knowledge of the students, in particular, previous constructs which have been constructed in the past and that may be helpful in the present task. Then, the AiC a priori analysis consists of intended constructs that are required in the given task. For each intended construct, the AiC researchers give in the a priori analysis an operational definition. The operational definition will help the researchers in their decision if the student did express the intended construct. It will offer a criterium for evidence if the intended construct has been constructed. Different researchers in the team perform separately their a priori analyses. Then the a priori analyses are discussed until there is agreement between the researchers.

Like for the TDS researchers, the a priori analysis serves as a system of reference for the AiC researchers. Comparing the a priori analysis and the a posteriori analysis, the AiC researchers note that sometimes the students achieve new constructs which were not expected in the AiC a priori analyses. This fact is an important and interesting stage in the research. Sometimes, students only achieve constructs that partially match a corresponding intended construct in the a priori analysis (Ron et al., 2010).

As explained in Dreyfus et al. (2015), the AiC a priori analysis is not only a list of intended constructs. It is more a structure of intended constructs with some interactions between the different constructs. For example, some constructs are contained in others. Some intended constructs might be a prerequisite for others. Sometimes, possible paths of thinking are taken into account. This is relevant, for example, for a priori analyses of justification tasks. Justification is a specific case of construction of knowledge. Each itinerary of thinking towards the justification might be in itself a kind of construction of knowledge and different itineraries of thinking, each with a structure of intended constructs might appear in the a priori analysis.

Comparing the a priori analyses of TDS and AiC and the benefit for networking

The two theories share the same aim: to understand the epistemological nature of the episode but different questions are asked:

AiC: What is the epistemic process of the student?

TDS: How this process is possible?

For both theories, TDS and AiC, the epistemological perspective is of importance but their a priori analyses have a different focus. In AiC the focus of the a priori analysis is on the learner's construction of knowledge. The a priori analysis reveals hypotheses about constructs that might be observed during the construction process. For AiC, processes of abstraction are inseparable from the context in which they occur. The notion of context is very wide in AiC. The context has many components. For example, as mentioned earlier, the task, the computer, the teacher, the social interaction between students are considered as part of the context. The AiC a priori analysis with its focus on the learner's processes of construction of knowledge cannot take explicitly into account all the contextual factors. In a later phase, the researchers will analyze the influence of the context on the construction processes that were observed in the analysis of data. For example, Kidron & Dreyfus (2010) analyzed the influence of the computer on the construction processes observed in the analysis of data. They describe how instrumentation led to cognitive constructions and how the roles of the learner and the CAS intertwine. But there is an essential difference if the researcher analyzes her data taking explicitly into account in advance the contextual factors or if she first analyzes the data and the processes of construction of knowledge and only then she analyzes the influence of the contextual factors on the construction processes.

For TDS, the situation is different. The focus is on didactical systems. TDS observes the entire situation and not only the student and the mathematical activity. For example, TDS is interested in relations between systems and the teacher is an element of the system. Consequently, TDS considers already in the a priori analysis the role of the teacher and how he extends the results of the a-didactical situation. Because of the different foci between TDS and AiC, context is not theorized and treated in the same way in the different theories. This fact has an important consequence on the differences of the a priori analyses.

AiC a posteriori analysis might be influenced by the fact that some contextual factors are not taken explicitly into account in the a priori analysis. As a consequence, some excerpts which might add direct knowledge in the analysis of the cognitive processes might be missed if one focuses first on the cognitive processes and only then analyzes the influence of other parts of the context.

Kidron et al. (2014) refer to a networking case that links three theories. The issue of context is compared and contrasted in the three theories. The analyses from the different perspectives refer to a set of data from a video recording that show a session from the group-work of two students, during a teaching experiment on the exponential function in secondary school.

In Kidron et al. (2014, p. 175), the authors noted that

> An interesting, and also revealing, point is the fact that, in the analysis, AiC researchers focus on the autonomous work of the students, while TDS researchers pay more attention to the episode where the students interact with the teacher...

The a posteriori analyses of the two theories are influenced by the differences in the a priori analyses and their different priorities in their focus of analysis. Different units of analysis are considered and as a consequence of the focus of analysis, as demonstrated in the a priori analysis, each theory shapes the kind of data that is appropriate to this focus. As pointed by Radford (2008):

> ...it is through a methodological design that data is first produced; then the methodology helps the researcher to “select” some data among the data that was produced but also helps the researcher to “forget” or to leave some other data unattended.

As a consequence, the different a posteriori analyses conducted within the two theories complement each other. Each analysis highlights a specific view which reflects the focus of research of the given theory. AiC analysis, with its specific tools, offers a fine-grained analysis of the students constructing processes. TDS, with its different focus, analyses the entire situation and, in particular, the interaction between the teacher and students. For example, in Kidron et al. (2014, p. 172) we read how TDS analyses the role of the teacher:

> TDS complements the AiC analysis in analyzing how the teacher extends the outcomes of the a-didactical interaction. The TDS analysis seems to start where the AiC analysis stops.

The different a priori analyses result from the different priorities of the theories with regard to the focus of analysis. Investigating these differences in the a priori analyses might lead to a better understanding of the different a posteriori analyses and to the insights offered by one theory to the other one in the networking process. I will extend this comment in the concluding remarks of the paper.

A PRIORI ANALYSES IN DIFFERENT THEORIES: CONNECTED WORKING SPACES (CWS) and ABSTRACTION in CONTEXT (AiC)

Psycharis et al. (2021) describes a research study in which students experience functions in a plurality of settings: physical context, geometry, measures, algebra. Two frameworks are used: Connected Working Spaces (CWS) (Minh & Lagrange, 2016) and AiC. CWS builds on the idea of “Mathematical Working Spaces” (MWS). The MWS theory is well described in Kuzniak et al. (2016) and more recently in Kuzniak et al. (2022).

Psycharis et al. (2021) wrote

> Work in a MWS is organised around three dimensions: semiotic (symbol use, graphics, concrete objects understood as signs); instrumental (construction using artefacts, such as geometric figure, graphs etc.) and discursive (justification and proof using a theoretical frame of reference). CWS builds on this idea of MWS by considering that in activities involving mathematics and other settings, students have to work in several working spaces and to coordinate the semiotic, instrumental and discursive dimensions of these spaces.

In this specific research study, the students had to work in several working spaces and to coordinate the semiotic, instrumental and discursive dimensions of these spaces. This is an example of networking that begins already in the networking of the a priori analyses. The a priori analyses are different in the sense that they complement each other. CWS a priori analysis identifies the different working spaces and the three dimensions (semiotic, instrumental, discursive) in each of the working spaces as well as the opportunity for the students to make connections between the working spaces. AiC a priori analysis relies on the structure offered by CWS. CWS alone could not offer, in its a priori analysis, maximal assumptions on learning. AiC a priori analysis is an effort in this direction, especially which constructs might be observed during the process of knowledge construction. In Psycharis et al. (2021) we read:

> CWS alone allows merely minimal assumptions regarding learning, but it provides a reliable structure for more precise hypotheses by AiC.

In this example of networking between CWS and AiC, we observe how comparing a priori analyses might enable to support the communication between the two different theoretical approaches. The explicit differences between the two a priori analyses demonstrate how CWS combines well with AiC and how connections between working spaces contribute to conceptualization.

A PRIORI ANALYSES IN DIFFERENT THEORIES: THEORY of DIDACTICAL SITUATIONS (TDS) and APC (ACTION, PRODUCTION, and COMMUNICATION)

Artigue (2021) describes how the two theoretical frames TDS and Action, Production, and Communication (APC) (Arzarello, 2008) were used in the thesis of Michela Maschietto (Maschietto, 2002).

Arzarello and Sabena (2014) offer a short introduction to the APC theory, especially the fact that APC provides a frame for investigating semiotic resources in the classroom. The authors describe the importance of gestures for communication and thinking. They underline that the main components of the APC-space are the body, the physical world, and the cultural environment and cite Arzarello (2008, p. 162):

> The APC-space is built up in the classroom as a dynamic single system, where the different components are integrated with each other into a whole unit. The integration is a product of the interactions among pupils, the mediation of the teacher and possibly the interactions with artifacts. The three letters A, P, C illustrate its dynamic features, namely the fact that three main components characterize learning mathematics: students' actions and interactions, their productions and communication aspects.

We also read that "space" should not be considered as a physical entity, but rather in an abstract way.

Artigue describes the tensions that appear comparing the different a priori analyses offered by TDS and APC. The theories belong to different cultures but, as in the case of TDS and AiC, for both the epistemological dimension is important. The main problem was how to create a DE that suits both theories, TDS and APC. I translate freely how Artigue (2021, p. 37) points to the main source of tension between the two a priori analyses:

> In a didactic engineering consistent with APC theory, the gestures of the students expressing cognitive constructions, the way in which these gestures will be taken up and exploited by the teacher through semiotic games, are essential ingredients of the cognitive dynamics of the class. They escape the potential for anticipation and control of the trajectories of a priori analysis.

The interesting part is that, in this specific context, to overcome this source of tensions and in order to be consistent with the Italian culture, the notion of a priori analysis was revisited, especially for what concerns the role and interaction of the teacher in semiotic games with the students. After performing these revisions, the DE that resulted was perfectly successful for this research.

CONCLUDING REMARKS

Considering each specific theory alone, we observe the role of reference of the a priori analysis while comparing with the a posteriori analysis. Comparing the different a priori analyses offered in different theories, we better understand the role of a priori analysis. In the present paper, the focus on comparing a priori analyses highlights the important role of a priori analyses in the networking of theories.

Kidron et al. (2018) wrote that in the last fifteen years:

> Different aims in the efforts to network theories were differentiated. In some cases, the researchers were interested in the complementary insights that are offered when given data or an empirical phenomenon is analysed with different theories. In other cases, the interest in the rich diversity of theories was to explore the insights offered by one theory to the other.

This idea appeared already in Arzarello et al. (2007, pp. 1625-1626) in which examples of different profiles of networking were presented.

Comparing a priori analyses, the aim is to explore the insights offered by each theoretical lense to the other from the very beginning. The networking of theories begins already in the stage of a priori analysis and not only for comparing a posteriori analyses.

We wrote that the different a priori analyses result from the different priorities of the theories with regard to the focus of analysis. In the a priori analyses these priorities and their differences are made explicit.

Reflecting on the role of a priori analysis in both theories, TDS and AiC, we realize its importance and why it is necessary towards a better understanding of the a posteriori analysis of the collected data. For both theories, the a priori analysis plays a role of reference while comparing the a priori and a posteriori analysis. For TDS, it plays the role of revealing the didactic phenomena and helps to deeply understand the functioning of the "situation". For AiC, it offers a structure of intended constructs that are required in a given task as well as possible paths of thinking. We also realize the importance that each theory keeps the specific characteristics of its a priori analysis. I wrote in a previous section that there is an essential difference if the researcher analyzes her data taking into account in advance the contextual factors or if she first analyzes the data and the processes of construction of knowledge and only then she analyzes the influence of the contextual factors on the construction processes. This essential difference is tightly connected to the specific characteristics of the different a priori analyses for AiC and TDS. In Kidron et al. (2008, p. 262), we read that:

> In networking, we want to retain the specificity of each theoretical framework with its basic assumptions, and at the same time profit from combining the different theoretical lenses. What we aim at is to develop meta-theoretical tools able to support the communication between different theoretical languages, which enable researchers to benefit from their complementarities.

Comparing a priori analyses might enable to support the communication between different theoretical approaches:

Realizing some common points in the a priori analyses enables the beginning of a dialogue between the theories. In the case of networking between TDS and AiC, for example, the common points in the epistemological dimension help towards the beginning of the dialogue (as demonstrated in Kidron et al., 2014). This idea could be used for other theories and other cases of networking: Some other common points in the a priori analyses, for example, the social dimension, might help towards the beginning of the dialogue.

Realizing the differences in TDS and AiC a priori analyses, we better understand the choices of data (as well as the “data which was left unattended”) that researchers of each theory select for a posteriori analyses. Sometimes, the data which was left unattended by one theory might add direct knowledge in the analysis of this theory and, as a consequence, some cognitive processes, for example, might be missed. The complementary insights which are missing might be offered by means of networking theories. This situation might happen in different cases of networking theories.

Also, for TDS and APC, the epistemological perspective, as a common point in the a priori analyses, enables the beginning of a dialogue. In spite of the differences and tensions between the two cultures a DE was created that suits both theories and was successful for the research study. In the case of CWS and AiC the explicit differences in the a priori analyses led to the understanding that the a priori analysis of AiC relies on the structure offered by the a priori analysis of CWS. The two theories complement well each other, and the networking experience begun at the stage of comparing a priori analyses.

As we noted in AiC and APC, “a priori analysis” exists outside the French context but with different meanings in different theories.

Moreover, there are theories with which the researchers do not carry a priori analysis. Even so, each theoretical approach, as a research methodology, is used with task sequences that have been designed with well-defined conceptual learning objectives in mind. Sometimes, the data required in order to do the appropriate analysis from the point of view of a specific theory might be different than the data required in another theory. Reflecting on these differences might allow a beginning of dialogue between theories.

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REFERENCES

Artigue, M. (1989). Ingénierie didactique. *Recherches en Didactique des Mathématiques*, *9*(3), 281–308.

Artigue, M. (1990). Épistémologie et didactique. *Recherches en Didactique des Mathématiques*, *10*(23), 241–286.

Artigue, M. (1995). The role of epistemology in the analysis of teaching/learning relationships in mathematics education. In Y. M. Pothier (Ed.), *Proceedings of the 1995 annual meeting of the Canadian Mathematics Education Study Group* (pp. 7–21). Ontario: University of Western Ontario.

Artigue, M. (2015). Perspectives on design research: The case of Didactical Engineering. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 467–496). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-94-017-9181-6_17

Artigue, M. (2020). Didactical engineering. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. *Second Edition* (pp. 202-206). New-York: Springer. https://doi.org/10.1007/978-3-030-15789-0_44

Artigue, M. (2021). Méthodologies de recherche en didactique des mathématiques : Où en sommes-nous? *Educação Matemática Pesquisa*, *22*(3), pp.25-64. https://doi.org/10.23925/1983-3156.2020v22i3p025-064

Artigue, M., Haspekian, M., & Corblin-Lenfant, A. (2014). Introduction to the Theory of Didactical Situations. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories in mathematics education* (pp. 47–65). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-3-319-05389-9_4

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: Perceptions, languages, theories. In E. Emborg & M. Niss (Eds.), *Proceedings of the 10th International Congress of Mathematical Education* (pp. 158-181). Copenhagen: ICMI.

Arzarello, F., Bosch, M., Lenfant, A. & Prediger, S. (2007). Different theoretical perspectives in research. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of CERME 5* (pp. 1618–1627). Cyprus University and ERME. http://erme.site/cerme-conferences/cerme-5/

Arzarello, F. & Sabena, C. (2014). Introduction to the Approach of Action, Production, and Communication (APC).. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories in mathematics education*. (pp. 31–45). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-3-319-05389-9_3

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des Mathématiques*, 1970–1990 (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Ed. & Trans.). Dordrecht: Kluwer.

Davydov, V. V. (1990). *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula* (Soviet studies in mathematics education, Vol. 2, J. Kilpatrick, Ed., J. Teller, Trans.). Reston: National Council of Teachers of Mathematics. (Original work published 1972.)

Dreyfus, T., Hershkowitz, R., & Schwarz, B.B. (2015). The nested epistemic actions model for Abstraction in Context: Theory as methodological tool and methodological tool as theory. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 185–217). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-94-017-9181-6_8

Dreyfus, T., & Kidron, I. (2014). Introduction to abstraction in context. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories in mathematics education*. (pp. 85–96). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-3-319-05389-9_6

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer.

Kidron, I. (2016). Epistemology and networking theories. *Educational Studies in Mathematics 91*(2), 149–163. https://doi.org/10.1007/s10649-015-9666-3

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M. & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *ZDM – Mathematics Education, 40*(2), 247–264. https://doi.org/10.1007/s11858-008-0079-y

Kidron, I., & Dreyfus, T. (2010). Interacting parallel constructions of knowledge in a CAS Context. *International Journal of Computers for Mathematical Learning 15*(2), 129–149. https://doi.org/10.1007/s10758-010-9166-6

Kidron, I., Artigue, M., Bosch, M., Dreyfus, T, & Haspekian, M. (2014). Context, milieu and media-milieu dialectic - A case study on networking. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories in mathematics education* (pp. 153–177). Dordrecht, The Netherlands: Springer. https://doi.org/10.1007/978-3-319-05389-9_10

Kidron, I, Bosch, M., Monaghan, J., & Palmer, H. (2018). Theoretical perspectives and approaches in mathematics education research. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger & K. Ruthven (Eds.), *Developing research in mathematics education. Twenty years of communication, cooperation and collaboration in Europe* (pp. 254–268). Routledge. https://doi.org/10.4324/9781315113562

Kuzniak, A., Tanguay, D., & Elia, I. (2016). Mathematical Working Spaces in schooling: an introduction. *ZDM – Mathematics Education,* 48(6), 721-737. https://doi.org/10.1007/s11858-016-0812-x

Kuzniak, A., Montoya-Delgadillo, E., & Richard, P. (2022). Mathematical work in educational context. *Springer International Publishing.* https://doi. org/10.1007/978-3-030-90850-8

Maschietto M. (2002). *L'enseignement de l'analyse au lycée : les débuts du jeu local-global dans l'environnement des calculatrices.* Thèse de doctorat en co-tutelle. Université Paris Diderot - Paris 7 et Universita degli studi, Turin.

Minh, T. K. & Lagrange, J. B. (2016). Connected functional working spaces: a framework for the teaching and learning of functions at upper secondary level. *ZDM – Mathematics Education,* 48(6), 793-807. https://doi.org/10.1007/s11858-016-0774-z

Psycharis, G., Kafetzopoulos, G & Lagrange, J.-B. (2021). A framework for analysing students' learning of function at upper secondary level: Connected Working Spaces and Abstraction in Context. In A. Clark-Wilson, A. Donevska-Todorova, E. Faggiano, J. Trgalová and H-G. Weigand (Eds.), Mathematics Education in the Digital Age: Learning Practice and Theory. Abingdon, UK: Routledge.

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – Mathematics Education,40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Ron, G., Dreyfus, T., & Hershkowitz, R. (2010). Partially correct constructs illuminate students' inconsistent answers. *Educational Studies in Mathematics, 75*(1), 65–87. https://doi.org/10.1007/s10649-010-9241-x

Schwarz, B.B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B.B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds). *Transformation of knowledge through classroom interaction* (pp. 11–41). London: Routledge. https://doi.org/10.4324/9780203879276

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ETHICS IN THE MATHEMATICS CLASSROOM

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Abstract

Despite some recent progress, ethics remains a peripheral topic in mathematics education research. My intention in the first part of the article is to show that ethics is omnipresent in the mathematics classroom. However, ethics does not always operate in the same way. To better grasp the nature of ethics, in the second part of the article, I discuss two ethical systems that have been influential in Western thought (Hobbes's and Kant's). These systems have largely informed the understanding of the mathematics classroom, even if, more often than not, they remain implicit. Then, I move to a short discussion of ethics in postmodern times and try to pinpoint what is at stake in ethics. The previous theoretical considerations pave the way to approach, in the last part of the article, ethics from an educational viewpoint. I argue that, in educational contexts, such as the school, ethics appear framed by the way in which we understand teaching and learning. I end the article with an outline of the communitarian oriented relational ethics articulated in the theory of objectification—a communitarian ethics whose practice features responsibility, commitment, and care.

Keywords: ethics, theory of objectification, Vygotsky, Spinoza, Kant, Hobbes

1. INTRODUCTION

The first question that might arise is the following: What does ethics have to do with mathematics education? Let me present a short twofold answer.

First answer

Teaching-learning mathematics cannot avoid facing the question of the *legitimation* of particular forms of knowledge and knowing that arise in the classroom. Classroom discussions usually lead to *conflicting views* about what counts as mathematically valid and authentic.

Here is an example. In a Grade 5 class (10-11-year-old children) of a French-speaking school in Sudbury, Canada, the students were invited to write a text for a student from another class explaining how to solve linear equations. To write and solve simple equations, the students had been using an iconic semiotic system (ISS) that comprises four signs:

In the ISS, the small rectangles represent cards; the envelope represents the unknown (as each envelope

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contained the *same* unknown number of cards). The problems with which the students had been dealing involved two individuals (e.g., Claudine and Sylvain) who each had a known number of cards and one or more envelopes. The individuals' total number of cards was the same.[1]

Some students suggested a text based on a concrete example: they used the equation

These students divided the two sides of the equation by a vertical line (see Figure 1, left). But other students suggested a text *without* any concrete example (see Figure 1, right; a translation of the texts can be found in the Appendix).

Which text is better? And if you were the teacher, what would you say to the students?

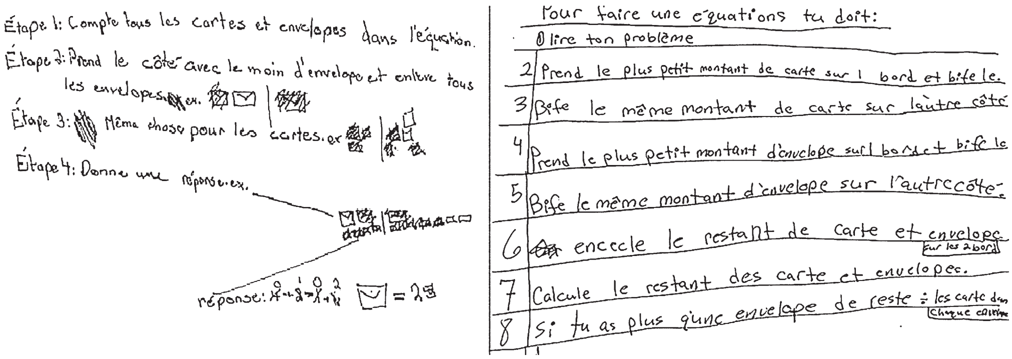

Figure 1. Two mathematical texts. Which one is better?

Since there are plenty of ways in which to think mathematically, taking sides or suggesting something else involves a question of power, and since there is a question of power, there is also a question of subjecting people to a *particular* image of mathematics. This is an ethical matter.

Second answer

My second answer is not about positions taken on questions of mathematical legitimacy but about *relations* between people.

Teaching-learning mathematics in the school involves *interaction* between people. Teaching-learning is based on relations with others, and these relations involve *necessarily* an ethical dimension: in classroom interaction we have, for instance,

- relations of power and subjection,
- relations of *authority* and *obedience*, and
- relations of *solidarity* and inclusiveness.

Here is an example. In a Grade 4 classroom (9-10-year-old children) of a French-speaking school in Ottawa, Canada, the students were working in small groups of three or four trying to solve geometric problems. The

1 A paradigmatic example is the following: "Sylvain and Chantal have some hockey cards. Chantal has three cards and Sylvain has two cards. Their mother puts some cards in three envelopes and makes sure to put the same number of cards in each envelope. She gives one envelope to Chantal and two to Sylvain. Now the two children have the same number of hockey cards. How many hockey cards are inside each envelope?" (Radford, 2017, p. 18). See also, Radford, Demers, and Miranda (2009).

first problem revolved around the classical question of whether squares are rectangles. The pictures in Figure 2 provide a sample of *body positions* of the interaction of a group of four students. In Picture 1, Laura is talking to Sandra, the girl in front of her: “Yes, but they all have four sides.” In Picture 2, Mirna tries to contribute to the group and says: “The squares have same umm … the same edges …” Laura turns to look at Mirna for a short period of time; then, turns back to look at Sandra to continue their discussion. In Picture 3, Híria (front left) tries unsuccessfully to get Sandra’s and Laura’s attention and says: “The squares have parallel faces … because there is … Look!” In Picture 4, after recurring attempts to be heard, Mirna expresses her frustration and utters an anguished “Ahhhhh!!!”

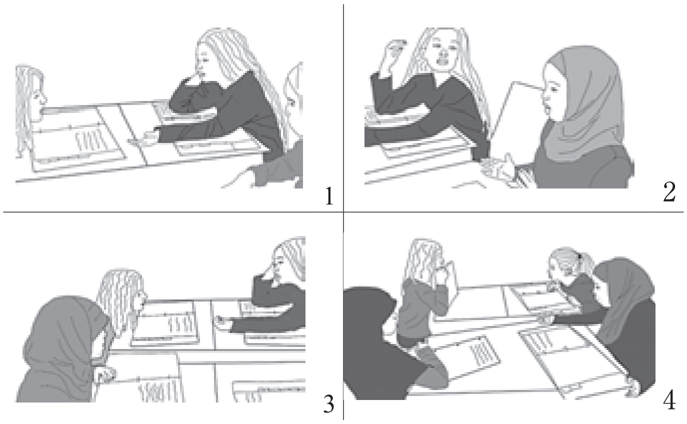

Figure 2. A group of Grade 4 students dealing with a geometry problem.

We are here in the presence of an ethics of exclusion. Híria and Mirna try hard to enter the conversation, to contribute to the discussion, but they are not heard. The practice of an ethics of exclusion impedes a genuine collective engagement in mathematics and raises an invisible, yet important, wall between *us* and *them*.

The previous examples illustrate one of the main ideas of this article: teaching-learning is unavoidably an *ethical* event—and this is so regardless of the pedagogical model that underpins it.

I would like to go a step further and contend that ethics is not only omnipresent in mathematics teaching and learning but is also a crucial component of it for at least two reasons.

First, ethics shapes the manners in which teachers and students *engage* and *assume* (or not) certain *responsibilities* in the mathematics classroom. Consequently, ethics shapes how teachers and students come to understand mathematics and conceive of themselves as practicians of mathematics.

Second, ethics shapes the *students’ and teachers’ relationships with others*—for instance, in the various manners by which the students voice (or not) their values and understandings, and how their voice is heard (or not). In this sense, ethics affects how teachers and students assert themselves as mathematical *subjectivities* (Radford, 2020).

Perhaps what is surprising in the claim that I am making, namely that mathematics teaching and learning is unavoidably an ethical event, is not that ethics is omnipresent in teaching and learning but the fact that, some exceptions put aside (e.g., Atweh, 2014; Atweh & Brady, 2009; Boylan, 2016; Dubbs, 2020; Ernest, 2009; Maheux & Proulx, 2017; Neyland, 2004; Maheux & Roth, 2014; Roth, 2013; Silva D’Ambrosio

& Espasandin Lopes, 2015; Walshaw, 2013), we have not been able to notice this presence before—or at least not with the importance it deserves. I would like to suggest that the root of that omission is entrenched in the long-standing conception of mathematics education as a technical process, a conception that reduces mathematics education to a matter of acquiring knowledge, making the question of being and becoming peripheral aspects of teaching and learning. With this same technicist stroke, we erase from view the fact that the mathematics classroom is not only a site of knowledge production but also a site where subjectivities are produced every day.

The importance of ethics in an encompassing account of learning leads us to ask the question of the kind of ethics that we, implicitly or explicitly, nurture in classrooms and research. This question seems to be important in discussions on the diversity of theories in mathematics education; considering theories from the ethical standpoint they convey (even if they do so only obliquely) may shed new light on the differences, similarities, and complementarities between theories.

In an attempt to better grasp the nature of ethics, in Section 2, I discuss two ethical systems that have been influential in Western thought (Hobbes's and Kant's ethical systems). These systems have largely informed the understanding of the mathematics classroom, even if, more often than not, they have remained implied. Then, in Sections 3 and 4, I discuss what is at stake in ethics. Intended always as a means to better grasp the nature of ethics, Section 5 revolves around some ideas of ethics in postmodern times. The previous theoretical considerations pave the way to approach, in Section 6, ethics from an educational viewpoint. I argue that, in educational contexts, such as the school, ethics appear framed by the way in which we understand teaching and learning. I end the article with an outline of the communitarian oriented relational ethics articulated in the theory of objectification (Radford, 2019, 2021a)—a *communitarian ethics* whose practice features responsibility, commitment, and care.

2. HOBBES AND KANT

All ethical perspectives are directly tied to conceptions of self and the social world. Try to think of what self is and you will quickly realize that it is entangled with how we perceive ourselves and perceive others. This perceiving of the other involves social relations, some to keep, some to discard. Our concepts of self are indeed inseparable from the landscape of ethical issues, "from how one ought to be" (Taylor, 1989, p. 112). These ethical issues change from culture to culture and within a same culture through spans of time. Thus, from the end of the Middle Ages and the dawn of Renaissance, when European feudal structures were being shaken by the rise of mercantile capitalism and the ascending power of the concomitant new economic and commercial relations, the social world appeared more and more as an aggregate of monadic individuals (Elias, 1991). On the social plane, life became increasingly regulated by agreements or contracts between people (Le Goff, 1956; Jeannin, 1957). In the 17th century, in his *Philosophical Rudiments Concerning Government and Society*, Thomas Hobbes conceives of these contracts as means to regulate human conduct and to avoid aggression against each other. In his account, these contracts bring individuals out of the state of nature, which "is nothing else but a mere war of all against all" (Hobbes, 1841, p. xvii). These contracts elevate the individuals to civil society; they embody personal aspirations and interests while trying to safeguard each other's security. Ethics appears here as a contractual mechanism to preserve the social order while affirming the monads' interests in a way that reflects the underpinnings of the emergent bourgeois ideology.

One century later, when Kant articulated his ethical system, capitalism had expanded and become more systematic. Kant's society, of course, is still perceived as an ensemble of monadic individuals, but with an increased complexity out of which the modern concept of self finds a more precise shape.

Kant assumed that individuals are by nature provided with a *same* reason—a legislative Reason. He also assumed that the individuals' reason moves in a world of universal laws that can define the social and moral orders. From those assumptions Kant formulated his famous *categorical imperative*: "*act only in accordance with that maxim through which you can at the same time will that it become a universal law*" (Kant, 2006, p. 31, original emphasis).

As we can see, the categorical imperative answers the question: *What should I do?* Through it, moral content becomes specified not in terms of social ends that the rule would help to attain or achieve. Moral content can only be derived from the form of the rule as expressed and appraised by the individual.

MacIntyre (1966, p. 209) notes that "The eighteenth-century individualist sees the good as the expression of his feelings or the mandate of his individual reason." The categorical imperative is an attempt "by the individual to supply his own morality, and at one and the same time, to claim for it a genuine universality" (p. 208). As I argued in (Radford, 2021a), the categorical imperative makes the Other appear as a *reflection* of the self in a mirror through which the self can see how the actions of the Other might affect itself. It assumes a universal human subject and a universal reason that needs to be understood as a response to "the emergence of a formal law in Europe at that time, the state homogenization that was being imposed [on society], [and] the development of an instrumental rationality" (Bohy-Bunel, 2022, pp. 23-24). This universal human reason is nonetheless an *exclusive* ethnocentric reason in its most intimate nature. As Bohy-Bunel remarks, the Enlightened modern bourgeois individual "arrogates to himself all possible humanity" (p. 24). This individual is

> the Westerner whose social, classist, patriarchal and *colonial* project of domination becomes clearer in the modern age . . . Its "science", which ends up fitting perfectly into this project, postulates that it relies on some universal "human reason," precisely in order to better assign to non-humanity those whose activity and being must be submitted to a total managerial control. (Bohy-Bunel, 2022, p. 25; emphasis in the original)

It is hence under the assumption of one and only one universal reason, the Enlightened managerial reason, that Kant can think that he solves the problem of the tension between the individual and the social world and the tension of human action in that world.

3. WHAT IS AT STAKE

What is hence at stake in conceptions about ethics? The previous discussion provides us with a hint: what is at stake is the social meaning of choices about human conduct. This social meaning refracts, on our dealings with others, the forms of life found in the social world. In the case of Hobbes and Kant, the forms of life were determined to a large extent by the capitalism of their time, which became the "dominant reality in the historical life of European societies" (Foufas, 2020, p. 24), that is capitalism in its commodity form. Confronted with a social understood as an assemblage of monads, Hobbes found in the contractual mechanisms of capitalists the prototype of a juridical form where individuals could negotiate and combine their own interests. Through contractual mechanisms individuals became subject to a new entity, an *assembly*, through which the multitude becomes a state or *civitas*. Pressured by the needs of a more sophisticated social and economic organization and the emergence of a formal law, Kant did not need to have recourse to something extraneous to the individuals, as Hobbes's did through the idea of *assembly*, to which the individuals relinquished the control of their own actions. Kant did not need to do something like that, as he assumed, as we just saw in the previous section, that there was a universal reason legislating choices about human behaviour.

It might not come as a surprise that the growth and implementation of ethics during the 20th century unfolded torn by two opposite forces that we find already in play in Kant's *categorical imperative*: on the one hand, there was an emphasis *on the individual* (we saw that the categorical imperative comes from and is enunciated by the individual). This is the view of the existentialist ethics (Sartre's ethics, for example) where, standing in the debris of WWII, authentic existence was "to be found only in a self-conscious awareness of an absolute freedom of choice" (MacIntyre, 1966, p. 269). On the other hand, there was an emphasis *on the universal validity of the law*, without assuming, however, that the law emanates from the individual itself. The ethical self should be educated through a series of prescriptions. This is what has been called "prescriptionist" ethics; that is, the ethics practiced in particular by the legislators facing the problem of governing the masses. We find modern prescriptionist ethics at work in post-revolutionary France. For example, in 1833, French Minister of Education François Guizot explains that

> The great problem of modern societies is the government of minds… It is not only for the commune and in a purely local interest that the law wants all the French to acquire, if it is possible, the knowledge indispensable to the social life, and without which the intelligence languishes and sometimes becomes stultified... This is because freedom is only assured and regular among people enlightened enough to listen in all circumstances to the voice of reason. (Guizot, cited in Mayeur, 2004, p. 335)

Thus "modern legislators and modern thinkers alike felt that morality, rather than being a 'natural trait' of human life, is something that needs to be designed and injected into human conduct" (Bauman, 1993, p. 6). The result was that, in the footsteps of Kant, modern ethics was mainly understood as deontology; that is, as duty and obligation driven by general rational laws to be applied regardless of the context. At the practical level, and with the push of modernity, this deontology was supplemented with a utilitarian outlook of the world. In the contemporary educational context, Neyland pictures the problem as follows:

> Recent modernist reforms . . . have resulted in teachers increasingly being represented as objects rather than as subjects in policy discourse; professionalism has been replaced by accountability, and collegiality by competition and surveillance; initiative, creativity and teacher-led innovation have been constrained; teaching has become technicized, and learning experiences impoverished. (Neyland, 2004, p. 62)

Here, as Neyland suggests, ethics amounts to proceduralism; that is, the implementation of the dictates of the educational apparatus as how teachers and students should behave and what they have to accomplish.

Postmodernist thinkers have quarrelled with this proceduralist conception of ethics. What bothers many of them is not the subjective aspect involved in ethical contexts. They retain the Kantian idea that morality does not come from outside the subject but from within. Bauman, for example, argues that "The moral call is thoroughly personal; it appeals to my responsibility" (1993, p. 60). What bothers these postmodernists is the deontological dimension. They have reacted suggesting a non-deontological ethics, an ethics that is context-sensitive and hence impossible to frame through a priori laws or principles. As Bauman puts it, "Moral phenomena are inherently 'non-rational'… They are not regular, repetitive, monotonous and predictable in a way that would allow them to be represented as *rule-guided* (1993, p. 11; emphasis in the original) . . . morality is endemically and irredeemably *non-rational*" (1993, p. 60; emphasis in the original).

Now, how can such an ethics possibly work if postmodernist thinkers themselves have been very successful in showing that the individuals are subtly shaped by a multitude of social mechanisms of governance that make them think and act in specified ways? How can we make sure that the ideas with which we ponder a delicate ethical situation are not those prescribed by the system even if they do not appear in a law form? In other words, how can we make sure that it is *me* and not the system through its webs of power and knowledge that is talking through me?

Foucault, the last modern Kantian and one of the first postmodern thinkers, raises the question as follows: How can the subject be the locus of meaning, feeling, intentionality, and ethical decisions if this subject must talk, feel, and intend through thoughts and words that are not its own?

> Can I, in fact, say that I am this language I speak, into which my thought insinuates itself to the point of finding in it the system of all its own possibilities, yet which exists only in the weight of sedimentations [that] my thought will never be capable of actualizing altogether? (Foucault, 1966, p. 335)

Rothenberg articulates this problem asking how the individuals produced within the social modes of power, "subsisting as nothing other than the intersection of various discursive determinants" (2010, p. 26), could be able to act independently of ideological operations. These individuals, Rothenberg argues, "have to transcend their ideological determinations somehow and lift the veil of misrecognition, even though they themselves are nothing other than the expression of ideology and thoroughly blinded by it" (Rothenberg, 2010, p. 25).

What is at stake, I argued before, is the social meaning of choices about human conduct. We now see that, going a step further, what is at stake at a deeper level, is our conceptions of human nature. Are humans capable of transcending their ideological determinations or are they trapped in them forever?

4. SELF AS LACK AND SELF AS EXCESS

Historically speaking, feminist, multiculturalist, psychoanalytic, and dialectical materialist thinkers, and scholars in related fields have contended that human nature is such that ideological determinations can be overcome. We are not trapped. The "trapped forever" position, they argue, is based on a conception of the self as *lack*. It features a self that lacks wholeness or plenitude, subjected forever to the limits of the given structures and confining discourses. They counter this conception of self with a conception of self as *excess*.

In feminist theory the subject is conceived of as having the power "to reflect on the social discourse and challenge its determinations" (Alcoff, 1988, p. 417). Following a Hegelian thread, for Butler (1999), the subversion of the subject is possible because all acts of signification not only restrict the subject's actions but are, at the same time, in their enactment, always located within the possibility of a variation in the "alternative domains of cultural intelligibility" (p. 185).

Rothenberg (2010) approaches the problem from a psychoanalytic perspective and claims that meanings have the peculiar property of being impossible to be totally controlled. Although individuals are socially produced, the producing causes leave "a remainder or [meaning] indeterminacy, so that every subject bears some unspecifiable excess within the social field. Every subject is an 'excessive' subject [and] the excess is ineradicable" (2010, p. 10).

Drawing on Lacan, Žižek refers to a fundamental characteristic of the Symbolic: its "openness."[2]

> The cause of this irreducible "openness" of the Symbolic is not its excessive complexity (we never know in what decentered context our statement will be inscribed), but the much more refined, properly dialectical impossibility of taking into account the way our own intervention will transform the field. (Žižek, 2010a, pp. ix-x)

Through our interventions in the world, Žižek contends, we affect our reality, which means that we change and affect our social determinations. These are not outside an undisturbed reality, nor is our free will.

> When we feel thwarted in our freedom by the constraining pressure of external reality, there must be something in us, some desire or striving, which is thus thwarted, but where should this striving come if not from this same reality? Our "free will" does not then in some mysterious way "disturb the natural course of things," it is part and parcel of this course. (Žižek , 2010a, p. xii)

Dialectical materialist thinkers (e.g., Fischbach, 2014; Macherey, 2008) have drawn on Marx's (1998) work, particularly on the idea of *transformative praxis* quickly sketched in Thesis 3 of *Theses on Feuerbach*. Following Marx's ideas, in dialectical materialism the subject is featured as one that, while being produced by its circumstances, has, inversely, the power to reflect on these circumstances and to transform them (for an enlightening analysis see Macherey (2008) and Fischbach (2015a)). Following Marx, what contemporary dialectical materialists add to the agentic conceptions of self mentioned above (i.e., the self as excess) is that subversion is accomplished in *praxis*, *with others* (Freire, 1998). Subversive praxis is the arena of the emergence of a new form of social consciousness. For consciousness is not only a refraction of reality. Consciousness, through its varying layers of depth, is a concrete relation that, given our biography and cultural background, *propels us* towards the world and leads us to act on/in it and *transform* it (Clot, 2015). In short, what dialectical materialists add to the agentic conception of self mentioned before is that the subversion of self is not the result of the deeds of a solitary being: it is a political and social project strictly tied to the transformation of the individuals' consciousness. There is still hope, then, that in our Grade 4 example, Mirna and her twin sister, Híria, will be heard. However, for this to occur, there must be a transformation of circumstances. The classroom culture must be transformed. This transformation requires a new *praxis*, a classroom praxis, out of which a new form of social consciousness can emerge.

To recap, in each one of these accounts, although for different reasons, there are always possibilities to

2 The Symbolic is the sphere of the Law. Without submitting ourselves to the Symbolic, we would not be able to function in society. The Symbolic includes grammatical rules, social norms, and unconscious prohibitions (see Žižek, 2007).

interrupt the quotidian train of our actions and thinking. It is at this point that Scott's (1990) *The Question of Ethics* as a question of *interruption* of habits and values acquires its whole sense. For Scott, *The Question of Ethics* indicates our capacity to bring forward "an interruption in which the definitive values that govern thought and everyday action lose their power and authority" (p. 4). This interruption makes sense precisely because of our cultural-historical *agentic* nature (i.e., our power to sublate and surpass the cultural and historical possibilities on which we draw when we re-act to, and re-enact, the world).

5. ETHICS IN POSTMODERN TIMES

5.1 Ambiguity

As stated before, one of the main features of postmodern ethics is to consider ethics as having an intrinsically *ambiguous* nature. To be ambiguous means that ethics is neither something contractual (like teachers do this, students do that), nor is it something that works on the basis of rules and abstract principles (like *do your homework!*). Valero and Jørgensen (2021) give an example of the ambiguous nature of ethics in contemporary ethical thought. A student submits an essay that, according to a plagiarism software, shows 21.5% of compatibility between a student's text and a referred text. However, the institutional maximum of compatibility is 20%. For the institution, plagiarism is

> unacceptable and grounds for expulsion. The teacher knows the student and knows that there is a case of terminal cancer in the immediate family that has affected the student's academic ability … For the teacher, ensuring access to education, particularly for poor people, is a fundamental part of his understanding of fairness in education. It is part of his social responsibility and commitment as a teacher. What does the teacher decide to do: follow institutional norms or his personal norms? (Valero and Jørgensen, 2021, p. 271)

As the authors remark, "The issue is to enter into a deep reflection to generate an ethical rule that is applicable in *this* situation" (Valero and Jørgensen, 2021, p. 272; my emphasis).

So, to say that ethics is intrinsically ambiguous means that our acts and relations to others do not have *one* obvious meaning. They are context sensitive. Ethics is seen here as a context-sensitive dynamic and open-ended relational stance that is continuously materialized and assessed as teachers and students engage with each other.

5.2 Ethics as thoroughly personal

Another main feature of postmodern ethics, or at least the one we find articulated in Bauman's work, is, as mentioned before, that ethics is something "thoroughly personal," something that "appeals to my responsibility" (1993, p. 60).

Bauman continues here the Western modern tradition that posits the individual as the origin of intentionality (Husserl, 1982) and ethical concerns (Kant, 2006). This feature of postmodernism is anything but new. It is easily understood once one realizes that postmodernity "one may say, is modernity without illusions (the obverse of which is that modernity is postmodernity refusing to accept its own truth)" (Bauman, 1993, p. 32). In other words, postmodernity is modernity disenchanted; modernity removed of the illusions of the universal abstract reason of the Enlightenment and its overarching grand narratives. Of course, postmodernism, even if it continues the modern tradition of the “new,” the “novel,” the “creative,” is more complex than this (Macherey, 2006). Capital (understood as the general system of production of the individuals’ lives, the system of production that characterized modernity) has not disappeared. It continues to be *the* mechanism through which human life is shaped and quotidianly produced. But capital has been refined and expanded: it has become more complex, more nuanced, and much more oppressive than in modern times. Bauman, hence, is right in seeing in postmodernism the story of an unending end; it is still a modernism, but one that seeks with relentless vigour to get rid of the encumbrances of the Symbolic (the “law” in Žizek’s sense) that has continuously threatened the autonomy of the modern subject. In this sense, postmodernist ethics is still (or again) a subjective ethics, this time pushed to its radical limits—with all the contradictions that such a push entails (Lipovetsky, 1989; Žizek, 2010b). If postmodern ethics recognizes the role of the Other in some qualified Levinas’s sense (as Bauman does), it does so within a monadic understanding of the social featured by modernity, that is, as an arena of “competitive struggle” whose pinnacle is “individual choice” (Bauman, 1993, p. 46).

6. ETHICS IN THE THEORY OF OBJECTIFICATION

6.1 Reconceptualizing the social

Ethics in the theory of objectification is neither modern nor postmodern. Although ethics is understood in a non-deontological, context-sensitive manner, ethics is not considered as coming from within. It is not something “thoroughly personal.” The reason is not because the individual is exempt of all moral responsibility. The reason is that it understands the social not as an agglomeration of monads that, once constituted as humans, enter the social realm. This monadic conception of the individual, where the social is seen as a derivative of the individual’s life, has been one of the chief marks of modernity and postmodernity, from where we have inherited the “acids of individualism” (MacIntyre, 1966, p. 266).

The conception of the social that underpins the theory of objectification is such that there is no separation between the individual and the social. The social is always appearing within the individual and between individuals. One cannot exist without the other. The social is neither a container, nor a flat surface where we live our life. The social is made up of *relations* (*rapports*) between individuals. These relations are not merely connectors; they are susceptible of modifying the individuals they put into relations.

To the extent that social relations express and manifest in their enactment political, economic, cultural and social conceptions about the world, ourselves and others, they affirm certain *forms* of dealing with others. This is what we see in our discussion of Kant’s and Hobbes’s ethical systems: each one of them suggests a social form of self-other dealings (Kant’s ethics turned to a universal reason legislating the transactions between individuals; Hobbes’s one turned to transactions of a contractual nature). Ethics is found precisely in these forms, which are nothing else but *forms of alterity.* Ethics, then, appears as the form of Self-and-Other.

If the social is understood as made up of social relations in movement, what would then be the kind of relations that are targeted in the theory of objectification?

6.2 Ethics and learning

To answer this question, we need first to make explicit the educational project in which this theory is subsumed. The theory of objectification inscribes itself in a Freirean and Vygotskian educational project that posits the goal of Mathematics Education as a political, societal, historical, and cultural endeavour aimed at the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical discourses and practices, and who ponder new possibilities of action and thinking.

As in other educational theories and pedagogical models, ethics in the theory of objectification appears in the relations between students and teachers. Ethics, understood as the form of Self-and-Other that manifests itself in our dealings with other individuals, is directly linked to the understanding of how learning occurs.

Let us consider two examples.

Think of the theory of transmissive instruction. Learning is conceived of as the assimilation, through practice and repetition, of knowledge that the teacher possesses. This theory positions teachers as knowers and the students as lacking knowledge. The ethics of the transmissive instruction, manifesting itself in the form of alterity that it promotes, reflects, but also operationalizes. It manifests itself concretely in the alienating relations of power and subjection it fosters and maintains, relations that are thematized along the obedient lines of superior/inferior, potent/impotent, knower/ignorant, authority/vassal (Radford & Lasprilla Herrera, 2020).

Now think of constructivism. In contradistinction to the theory of transmissive instruction, constructivism is based on an ethics that stresses the freedom of the student: since knowledge is conceived of as what results from the autonomous deeds of the student, and learning is the very process of the student's subjective construction of knowledge, teachers and students are positioned otherwise: the student's freedom and autonomy configure the constructivist ethical space (Radford, 2012). It alienates teachers and students, but in a way that differs from how transmissive instruction does (Radford, 2021a).

Learning in the theory of objectification is neither about transmitting knowledge (as in direct or transmissive teaching), nor is it about the students' constructing their own knowledge (as in constructivist approaches). In the theory of objectification, learning is conceived of as a *collective* and *truly social* embodied and material process through which students critically encounter culturally and historically constituted ways of thinking mathematically. This encounter happens in what the theory terms *joint labour* (Radford, 2021a, 2021b).

6.3 Joint labour

Joint labour is a sensuous, practical, material *activity*—activity understood as driven by *collective* concerns. The German and Russian languages have a specific term for this type of activity: *Tätigkeit* and *deyatel'nost',* respectively. Activity in this sense is opposed to activity as being merely busy with something (as in watching TV). Again, the German and Russian languages have a specific term for this other type of activity: Aktivität or aktivnost'. Unfortunately, in the translation into English (and several other languages), the distinction is lost and both types of activity are rendered as *activity*.

In the case of the theory of transmissive instruction, classroom activity is not oriented towards the satisfaction of collective needs. This activity corresponds hence to Aktivität or aktivnost'. In joint labour, by contrast, students and teachers work hand in hand to *produce* something *together*, what Hegel termed "a common work," in our case, *mathematics*. It is this sense of labouring together (as opposed to simply interacting or exchanging with others) that makes joint labour a truly social activity and learning a collective process.

6.4 Transforming ethics

Classroom research has shown us, however, that for learning to become a truly collective process, radical changes in the classroom culture might need to occur. Often, drawing on experiences shaped by transmissive instruction, the students conceptualize the teacher as the possessor of knowledge and power, and conceptualize themselves as submissive to the teacher and her knowledge, even when the teacher tries to conceptualize herself differently and encourages the students to learn collectively and organizes the classroom into small groups. Often, the students configure small, enclosed groups and erect aggressive or exclusive antagonistic barriers between their group and other groups—they resort to what we may term a *clique ethics* or *gang ethics* (Radford & Lasprilla Herrera, 2020), which is also what we see in the example from the Grade 4 classroom briefly mentioned in the Introduction.

Of course, moving towards a more encompassing, democratic, and inclusive ethics cannot be achieved by imposing new social forms of conduct. "It is not obedience to someone or obedience to something, but the free adoption of those patterns of behavior which will vouchsafe the consonance of all of behavior" (Vygotsky, 1997, p. 233). The educational problem around ethics becomes the problem of the creation of classroom conditions for new ethical relations (new *forms* of alterity) to emerge and to be collectively pondered and discussed against the always contested background of culture and history. The process of emergence of such an ethics has to overcome the dynamic power of capital, which constantly tries to appropriate the social "in order to put it at the service of its own development." It is hence "a question of not letting ourselves be dispossessed: it is necessary to maintain and to develop [the spring of the production of social life] against capital" (Fischbach, 2015b, p. 71). This can only be done by developing a form of democracy that would "increase our power to generate ourselves our own social life" (p. 71).

6.5 A communitarian ethics

So, what are the new ethical relations that we strive to nurture in the theory of objectification? We focus on a classroom mathematics practice featuring a *communitarian ethics* which is consonant with the conceptual historical-cultural bases of the theory and its conception of learning as a collective process.

Communitarian ethics is based on what we call three ethical vectors. They define a dynamic ethical space. Rather than be seen as fixed terms, we see them as *repères*; that is, reference points against which to gauge our actions. These vectors work as "primary orientation of the ethical self" and should not be seen as "a modern uniform code"; they are "shared ethical ideas, priorities and principles that are subject to ongoing and 'antagonistic' negotiation'" (Neyland, 2004, p. 57). These vectors (or "virtues" to use MacIntyre's term) are responsibility, commitment, and care. It is here where we resort to the construct of "voice"—not voice in a linguistic sense exactly; rather we resort to voice as something that brings in the notion of *difference* and the *primacy of the political* (Giroux, 2005). Finding one's voice or having a voice is "moving from silence into speech," it is "a gesture of defiance that heals, that makes new life and new growth possible" (bell hooks, 2015, p. 29), something that "assumes a primacy in talk, discourse, writing, and action" (p. 33).

Coming back to our Figure 1, when Mirna utters an anguished "Ahhhhh!!!" and moves her right hand towards the two other girls who are not listening, she is moving into speech to express her frustration about not being counted and heard. Her voice (which is much more than what she discursively *says*, as it also says things in her body posture, facial expression, pitch, gesture) opens up new possibilities for action (for herself and the other teammates). Mirna's embodied utterance is, indeed, a *call* to the Other.

6.5.1 Responsibility

The call now must be responded to, and it is responded to within a certain node of social relations that tie the students together. Whatever path the teammates' response takes, it is cast in a general ethical attribute that Lévinas calls *responsibility*. For Lévinas, responsibility is "the essential, primary, and fundamental structure of subjectivity … [where] the very node of the subjective is knotted" (1982, p. 101). Since all educational theories put into motion a certain ethics—for, as mentioned before, ethics is the substrate and the form that is continuously materialized in our dealings with the Other—responsibility is a common denominator of all of them. Yet, the *meaning* of responsibility is not the same. In the theory of direct instruction, the students' responsibility consists in assuming a submissive role vis-à-vis the teacher. The teacher's responsibility consists in subjecting the students through the power of knowledge. In the case of the theory of didactical situations (Brousseau, 2005), responsibility follows Hobbes's contractual outlook and appears as a didactical contract; that is, a reciprocal distribution of duties. In the case of the theory of objectification, responsibility means living and acting *with* and *for* others; it means to respond to the call of others as they are on their own terms: in their "existence, in [their] being-for-other[s] . . . as free being[s]" (Hegel, 1978, p. 57).

6.5.2 Commitment

Commitment is both the promise and its realization of doing everything possible to work side by side with others in the course of our joint labour (e.g., trying to understand the process being followed to solve a problem, trying to contribute to the classroom common work).

The classical utterance "I do not understand" of classrooms operating within the transmissive instruction model often conveys the stance of a de-responsabilization; it intimates that it is the teacher who has not done his work properly. Through this utterance, the student invests the teacher with the role of patriarch of knowledge and requests the teacher to deliver knowledge. However, if teachers and students understand that knowledge is produced collectively in the classroom, the same utterance would mean something like: "I do not understand, but let's work together so perhaps things will become clearer." Commitment is this resolution to engage in joint labour, to participate in the creation of the classroom common work (*oeuvre commune*), which can be establishing *collectively* how equations can be solved or theorems proved, for example.

6.5.3 Care for others

Far from being an act of condescending or simply caring for someone, the care for others is a relational involvement entailing the attention to, and recognition of, others and their material and spiritual needs. Although caring for the Other opens up the possibility of seeing ourselves in the Other, of recognizing our vulnerability in the vulnerability of the Other, the importance of caring for the Other is to go beyond ourselves and to be dragged powerfully into the world, to position ourselves there, with-the-Other.

To understand Mirna's "Ahhhhh!!!" within an ethical practice of responsibility, commitment, and care, we need to broaden our conception of language and *recognize* (in its Hegelian sense) this painful expression as *voice*; that is, as something where, as Lévinas suggests, the *saying* (that which I want so much to utter but I will never be able to do) moves beyond the totalizing enclosure of the *said* (that which I manage to say) and becomes rather the possibility of openness to the other (Radford, 2021a). In this conception of voice, power does not disappear since power is not a *thing*, but something imbricated in our relations to others. What we can expect in the transformative movement towards a communitarian ethics is that, through conscious, reflective, and critical stance, power in the classroom goes beyond its own subjecting mechanisms of social order and becomes rather something fluid, dynamic, to be exercised with responsibility *for* the Other.

7. CONCLUDING REMARKS

In the first part of this article, I argued that mathematics teaching and learning is unavoidably an ethical event. There are at least two reasons for this. The first reason has to do with the fact that teaching and learning always involve interaction between teachers and students and this interaction is based on relations (relations of power and subjection, relations of exclusion, or relations of solidarity, inclusivity, etc.). The second reason has to do with what is to be learned, with knowledge. It has to do with the *legitimation* of particular forms of knowledge and knowing. Teachers are confronted in their everyday practice with a variety of ways that students bring about how to solve mathematical problems. Legitimation appears here as what counts as true, as mathematically right, sound, worth pursuing, etc. I presented an example in which students produced two different mathematical texts explaining how to solve linear equations. The question revolved around the epistemological role of examples: should the text include an example? Should it be rather general? What text is better? As soon as we move beyond Eurocentric views of mathematics (D'Ambrosio, 2006; Powell & Frankenstein, 1997), the question of legitimation acquires its full sense.

In the first part of the article, I complained that despite its omnipresent nature in the mathematics classroom, ethics still remains largely unexplored (although it is gaining some traction, as we can see in the No. 38 (December 2021) issue of the *Philosophy of Mathematics Education Journal*, edited by Paul Ernest).

To better understand conceptions of ethics, in the second part of the article, I dealt with two ethical systems—Hobbes's and Kant's—that have been influential in Western thought and have informed the understanding of the mathematics classroom, even if, more often than not, those ethical systems have remained implicit (for instance, Kant's subjectivist ethical stance has influenced constructivism; Hobbes's contractualism has influenced transmissive learning, and both Kant and Hobbes have influenced the theory of didactical situations).

In the third part, I moved to a discussion of what is at stake in ethics. We saw that, on a first level, the answer lies in the social meaning of choices about human conduct. On a second and deeper level, the answer lies in our conception of human nature.

The discussion of ethics in postmodern times, intended always as a means to better grasp the nature of ethics, allowed us to see that, on the one hand, the postmodern project breaks with the modern project, while, on the other hand, continues it. The postmodern project breaks with the modern one in refusing to resort to a universal reason as the basis of the ethical self. At the same time, it continues the modern project in formulating a subjective ethics; that is, an ethics that is at the end of the day "thoroughly personal" (Bauman, 1993, p.60). The postmodern understanding of ethics still rests on the modern conception of the social as an aggregate of individuals, monadic self-constituted agents.

The previous theoretical considerations paved the way to approach, in the last part of the article, ethics from an educational viewpoint. I outlined the ethics in a Vygotskian theory of teaching and learning, the theory of objectification (Radford, 2021a). To do so, drawing on the work of Fischbach (2015a, 2015b), I started offering a reconceptualization of the social as made up of *social relations*, which are much more than simple links between individuals. So, instead of being a mere background, the social appears to be a dynamic totality in perpetual movement and transformation. The relations of which the social is made up express conceptions about ourselves and others. They are always tinged with political and cultural valences. These considerations led me to suggest that ethics can be considered as the *form of alterity* (the form of Self-and-Other) that manifests in our dealings with other individuals. I argued that, in educational contexts, such as the school, ethics appear framed by the way in which we understand teaching and learning.

Against this background, I outlined the communitarian oriented relational ethics that is articulated in the theory of objectification—a *communitarian ethics.* This ethics follows Spinoza in stressing the body's *power* of acting in the world (*agendi potentia*; Spinoza, 1989, p. 210). The communitarian ethics also follows Lévinas's (1982) work which radicalizes previous ethical systems in acknowledging that our actions and deeds are always modulated by the presence of the *Other*—a presence that comes to us in a sentient and fleshy manner: through the *proximity of our bodies*. In this proximity our conceptual epistemological categories and mechanisms are put on hold, and we encounter the Other as *is*. "The Other is appreciated precisely *as* Other, in her radical alterity and irreducible singularity, only when thought renounces its totalitarian hubris and learns to think of the Other on her own terms . . . 'beyond essence'" (Min, 1998, pp. 573–574; emphasis in the original). The self appears here conceived not as the prerequisite of existence of ethical relations. The self is rather seen as the *result* of those relations.

In conceiving of ethics as the form of alterity, the focus turns not to moral precepts but rather to the fluid and content-dependent relationships between subjects as they appear in the immediacy and banality of everyday life. In this view, ethics is continuously *materialized* in *praxis* out of a myriad of possibilities, for, in this view, the individual appears as "full of unrealized possibilities every minute" (Vygotski, 2003, p. 76). If we come back to Figure 2, we see that Mirna's teammates could have opted for other actions. The materialization of the students' *actions* reflects their understanding of the context (consciousness) and the manner in which the context is lived through their unfolding collective affective experience (emotions), as well as the relational stances (ethics) they (consciously or not) adopt towards each other. Through the previous classroom examples, we see that embodied, emotional, contextual, cultural-historical action is related to ethical postures that students assume and show in practice.

What is specific to the communitarian ethics of the theory of objectification is that, braking with the subjectivist stances of modern and postmodern ethics, its practice features responsibility, commitment, and care. This communitarian ethics provides reference points for pedagogical actions in the classroom, where teachers and students explore together new critical spaces that promote engagement, inclusiveness, debate, and respect (for some classroom examples, see Radford, 2021, 2021b). Underneath the communitarian ethics lies the recognition that our historical, cultural, and material origin embeds and refracts dynamic and antagonistic visions and conceptions of the world and of what a good life can mean. Although impossible to posit and describe a priori, the good life and the common good are understood here as ideas (*idéalités*), more specifically, as "*visées éthiques*" (aims), generated in a joint unending project created itself within the dynamic that produces the social relations, a project that will always be paved with tensions, contradictions, and difficulties. "Let us call 'ethical aim' (*visée éthique*) the aim of the 'good life' with and for others in just institutions" (Ricœur, 1990, p. 202). It is the vitality of contradictions that gives substance to social human life, always changing, always challenged, and that makes ethics to always unfold in a context of struggle.

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REFERENCES

Alcoff, L. (1988). Cultural feminism versus post-structuralism. *Signs*, *13*(3), 405– 436. https://doi.org/10.1086/494426

Atweh, B. (2014). Mathematics education and democratic participation between the critical and the ethical: A socially response-able approach. In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education* (pp. 325–342). Sense.

Atweh, B., & Brady, K. (2009). Socially response-able mathematics education: Implications of an ethical approach. *Eurasia Journal of Mathematics Science and Technology Education*, *5*(3), 267–276. https://doi.org/10.12973/ejmste/75278

Bauman, Z. (1993). *Postmodern ethics*. Blackwell.

bell hooks. (2015). *Talking back: Thinking feminist, thinking black*. Routledge.

Bohy-Bunel, B. (2022). *Approche matérialiste de la Critique de la raison pure*. L'Harmattan.

Boylan, M. (2016). Ethical dimensions of mathematics education. *Educational Studies in Mathematics*, *92*(3), 395–409. https://doi.org/10.1007/s10649-015-9678-z

Brousseau, G. (2005). *Theory of didactical situations in mathematics*. Kluwer.

Butler, J. (1999). *Gender trouble: Feminism and the subversion of identity*. Routledge.

Clot, Y. (2015). Vygotski avec Spinoza, au-delà de Freud [Vygotski with Spinoza, beyond Freud]. *Revue Philosophique de la France et de l'Étranger*, *140*(2), 205–224. http://www.jstor.org/stable/43575014

D'Ambrosio, U. (2006). *Ethnomathematics*. Sense.

Dubbs, C. (2020). Whose ethics? Toward clarifying ethics in mathematics education research. *Journal of Philosophy of Education*, *54*(3), 521–540. https://doi.org/10.1111/1467-9752.12427

Elias, N. (1991). *The society of individuals*. Basil Blackwell.

Ernest, P. (2009). What is 'first philosophy' in mathematics education? In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of*

Mathematics Education (Vol. 1, pp. 25–42). PME.

Fischbach, F. (2014). *La production des hommes. Marx avec Spinoza* [The production of men. Marx with Spinoza]. Vrin.

Fischbach, F. (2015a). *Philosophies de Marx*. [Marx's philosophies]. Vrin.

Fischbach, F. (2015b). *Le sens du social: Les puissances de la coopération* [The sense of the social: The power of cooperation]. Lux Éditeur.

Foucault, M. (1966). *Les mots et les choses*. [The order of things]. Éditions Gallimard.

Foufas, N. (2020). *De l'aliénation chez Rousseau: Genèse et dualité d'un concept*. [Alienation in Rousseau: Genesis and duality of a concept]. L'Harmattan.

Freire, P. (1998). *Pedagogy of freedom*. Rowman & Littlefield.

Giroux, H. (2005). *Border crossings: Cultural workers and the politics of education*. Routledge.

Hegel, G. (1978). *Hegel's philosophy of subjective spirit: Vol. 3* (M. J. Petry, Ed.). D. Reider.

Hobbes, T. (1841). *The English works of Thomas Hobbes of Malmesbury: Vol. 2* (W. Molesworth, Ed.). John Bohn.

Husserl, E. (1982). *Cartesian Meditations: An Introduction to Phenomenology*. Martinus Nijhoff Publishers.

Jeannin, P. (1957). *Les marchands au XVIe siècle*. Seuil.

Kant, I. (2006). *Groundwork of the metaphysic of morals* (Mary Gregor, Trans.). Cambridge University Press.

Le Goff, J. (1956). *Marchands et banquiers du Moyen Age*. [Merchants and bankers of the Middle Ages]. Presses Universitaires de France.

Lévinas, E. (1982). *Éthique et infini*. [Ethics and infinity]. Fayard.

Lipovetsky, G. (1989). *L'ére du vide. Essais sur l'individualisme contemporain* [The era of emptiness. Essays on contemporary individualism]. Gallimard.

Macherey, P. (2006). *Qu'est-ce qu'être post-moderne ?* [What does it mean to be postmodern?]. Groupe d'études «La philosophie au sens large». https://philolarge.hypotheses.org/files/2017/09/13-04-2006.pdf

Macherey, P. (2008). *Marx 1845: Les "thèses" sur Feuerbach: Traduction et commentaire*. [Les thèses on Feuerbach]. Éditions Amsterdam.

MacIntyre, A. (1966). *A short history of ethics*. Touchstone.

Maheux, J.-F., & Proulx, J. (2017). Éthique et activité mathématique. [Ethics and mathematical activity]. *Éducation et francophonie*, *45*(1), 174–194. https://doi.org/10.7202/1040726ar

Maheux, J.-F., & Roth, M.-W. (2014). The relationality in/of teacher-student communication. *Mathematics Education Research Journal*, *26*, 503–529. https://doi.org/10.1007/s13394-013-0096-1

Marx, K. (1998). *The German ideology, including Theses on Feuerbach and introduction to The critique of political economy*. Prometheus Books.

Mayeur, F. (2004). *Histoire de l'enseignement et de l'éducation. III. 1789–1930*. [History of teaching and education. III. 1789-1930]. Perrin.

Min, A. (1998). Toward a dialectic of totality and infinity: Reflections on Emmanuel Lévinas. *The Journal of Religion*, *78*(4), 571–592. https://doi.org/10.1086/490291

Neyland, J. (2004). Toward a postmodern ethics of mathematics education. In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 55–73). Information Age Publishing.

Powell, A. B., & Frankenstein, M. (1997). *Ethnomathematics*. State University of New York Press.

Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics*, *80*(1), 101–118. https://doi.org/10.1007/s10649-011-9380-8

Radford, L. (2017). La fenomenología del significado [The phenomenology of signifying]. In Costa dos Santos, M. J., & F. Vieira Alves (Eds.), *Docência, cognição e aprendizagem: Contextos diversos* (pp. 15–29). Editora CRV.

Radford, L. (2019). Une théorie vygotskienne de l'enseignement-apprentissage: La théorie de l'objectivation. [A Vygotskian theory of teaching-tearning: The theory of objectification]. In J. Pilet & C. Vendeira (Eds.), *Actes du* *séminaire de didactique des mathématiques de l'ARDM 2018* (pp. 314–332). IREM de Paris – Université Paris Diderot. https://www.researchgate.net/publication/338584417_Une_theorie_vygotskienne_de_l'enseignement-apprentissage_la_theorie_de_l'objectivation

Radford, L. (2020). Play and the production of subjectivities in preschool. In M. Carlsen, I. Erfjord, & P. S. Hundeland (Eds.), *Mathematics education in the early years. Results from the POEM4 conference 2018* (pp. 43–60). Springer. https://doi.org/10.1007/978-3-030-34776-5_3

Radford, L. (2021a). *The theory of objectification: A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill Sense.

Radford, L. (2021b). Reimaginar el aula de matemáticas: Las matemáticas escolares como praxis emancipadora. [Reimagining the mathematics classroom: School mathematics as emancipatory praxis]. *Revista Chilena de Educación Matemática*, *13*(2), 44–55. https://doi.org/10.46219/rechiem.v13i2.88

Radford, L. (2021c). Mathematics teaching and learning as an ethical event. *La matematica e la sua didattica*, *29*(2), 185–198. https://rsddm.dm.unibo.it/wp-content/uploads/2021/10/5-L-Radford-MD-292-2021.pdf

Radford, L., & Lasprilla Herrera, A. (2020). De porqué la ética es ineludible de considerar en la enseñanza-aprendizaje de las matemáticas. [Why ethics is inevitable to be considered in the teaching-learning of mathematics]. *La matematica e la sua didattica*, *28*(1), 107–128. https://rsddm.dm.unibo.it/wp-content/uploads/2020/05/A4-Radford-Herrera-MD-28-1-2020.pdf

Radford, L., Demers, S., & Miranda, I. (2009). Processus d'abstraction en mathématiques [Processes of abstraction in mathematics]. Centre franco-ontarien de ressources pédagogiques, Imprimeur de la Reine pour l'Ontario.

Ricœur, P. (1990). *Soi-même comme un autre*. [Oneself as another]. Éditions du Seuil.

Roth, W.-M. (2013). Toward a post-constructivist ethics in/of teaching and learning. *Pedagogies*, *8*(2), 103–125. https://doi.org/10.1080/1554480X.2013.767769

Rothenberg, M. (2010). *The excessive subject. A new theory of social change*. Polity.

Scott, C. E. (1990). *The question of ethics: Nietzsche, Foucault, Heidegger*. Indiana University Press.

Silva D'Ambrosio, B., & Espasandin Lopes, C. (2015). Insubordinação Criativa: um convite à reinvenção do educador matemático [Creative Insubordination: an invitation to reinvent the mathematics educator]. *Bolema*, *29*(51), 1–17. https://doi.org/10.1590/1980-4415v29n51a01

Spinoza, B. (1989). *Ethics including the improvement of the understanding* (R. Elwes, Trans.). Prometheus Books.

Taylor, C. (1989). *Sources of the self.* Harvard University Press.

Valero, P., & Jørgensen, K. (2021). ¡A la educación se le perdió la ética! Y… ¿por qué queremos reencontrarsela? [Education has lost its ethics! And... why do we want to find it again?] In L. Radford & M. Silva (Eds.) (pp. 257–275). Ediciones Uniandes.

Vygotsky, L. S. (1997). *Educational psychology*. St. Lucie Press.

Vygotski [Vygotsky], L. S. (2003). *Conscience, inconscient, émotions*. [Consciousness, unconscious, emotions]. La dispute.

Walshaw, M. (2013). Post-structuralism and ethical practical action: Issues of identity and power. *Journal for Research in Mathematics Education*, *44*(1), 100–118. https://doi.org/10.5951/jresematheduc.44.1.0100

Žižek, S. (2007). *How to read Lacan*. W. W. Norton & Company.

Žižek. S. (2010a). With the keen edge of a knife. In M. Rothenberg, *The excessive subject. A new theory of social change*. Polity.

Žižek, S. (2010b). *Living in the end times*. Verso.

APPENDIX

Translation of the texts shown in Figure 1.

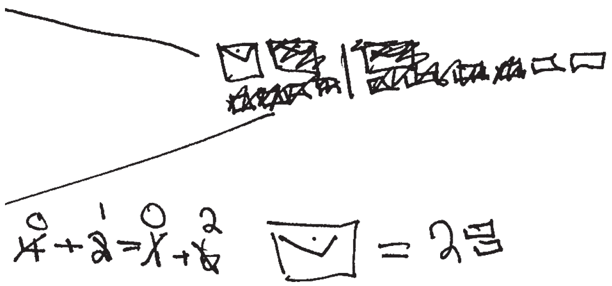
First text:

Step 1: Count all cards and envelopes in the equation.

Step 2: Take the number with fewer envelopes and remove all the envelopes. Ex.

Step 3: Same thing for the cards. Ex.

Step 4: Give an answer ex.



Second text:

To write an equation you have to:

1. Read the problem.
2. Take the smaller amount of card[s] on one side and cross it out.
3. Cross out the same amount of cards on the other side.
4. Take the smaller amount of envelopes on one board and cross it out.
5. Cross out the same amount of envelopes on the other side.
6. Circle the remaining cards and envelopes.
7. Count the remaining cards and envelopes.
8. Divide the cards into each envelope if you have more than one envelope remaining.

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NETWORKING PRAXEOLOGIES AND THEORETICAL GRAIN SIZES IN MATHEMATICS EDUCATION: CULTURAL ISSUES ILLUSTRATED BY THREE EXAMPLES FROM THE JAPANESE RESEARCH CONTEXT

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Abstract

This study aims to identify the theoretical underpinnings of researchers' work on networking theories in terms of the *Anthropological Theory of the Didactic*. The study is based on the concept of *research praxeology*, which allows us to characterise researchers' practice and knowledge of networking. Three research examples, conducted in the Japanese educational context, are used to illustrate a means to characterise the four praxeological elements (*type of tasks*, *technique*, *technology*, and *theory*). The results imply that the notion of theoretical grain sizes (*grand*, *intermediate*, and *local levels*) can be used to deepen our understanding of researchers' work on networking theories. Based on our results, different characteristics of networking theories are discussed according to the cultural issues and specificities of the illustrative examples.

Keywords: Networking theories, networking strategies, research praxeology, logos block, theoretical grain sizes

INTRODUCTION

The growth and success of mathematics education as a scientific research field are marked by the existence of diverse theories. However, such growth also presents a challenge. As Prediger et al. (2008) stated:

> [T]he more the number of theories grows, the more difficult it will be to get an overview of all of them and of small theoretical bricks using different languages with slightly different meanings. (p. 169)

This motivates networking between different theories wherein the researchers work towards creating a dialogue between different theoretical approaches while respecting the identity of the different approaches in mathematics education (Prediger & Bikner-Ahsbahs, 2014). Several studies have worked within the context of networking theories for over a decade, and some have reviewed and reflected researchers' networking endeavours (Bikner-Ahsbahs & Prediger, 2014; Kidron et al., 2018). It appears that the latter is a meta-theoretical study of the former. For instance, Artigue et al. (2011) use the notion of *research praxeology* based on the *Anthropological Theory of the Didactic* (ATD); and Radford (2008) conceives a dynamic cultural semiotic space as a semiosphere introducing a triplet comprising a system of basic principles, methodology, and a set of research questions. In this space, networking of theories takes place and hence may also inform the dynamic development of the semiosphere. More recently, Tabach et al. (2020) elaborated

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on an argumentative grammar for networking theories by adopting Toulmin's model of argument.

While such meta-level studies clarify the nature of networking theories, attempting them can be difficult because characterising researchers' practices and knowledge in networking requires a meta-theoretical framework; however, this is still under development. Research praxeology has been adopted as a meta-theoretical framework in this paper, along with some research examples of networking theories that have taken place in the Japanese educational context, to contribute to the development of a framework for researchers' activities. Many researchers in mathematics education in Japan have worked on different theoretical approaches developed in Western countries for their studies in the local context. However, only a small number has explicitly described their networking endeavours. In this study, we attempt a retrospective analysis, which aims to make the implicit explicit in terms of praxeologies. Thus, the research question in this study is as follows: *How can we characterise researchers' theoretical work on networking endeavours in terms of praxeology?* Through this research question we expect to formulate a method to characterise the praxeologies of networking theories in different types of studies. Additionally, this paper discusses some cultural issues related to the Japanese educational context.

RESEARCH PRAXEOLOGY AND NETWORKING THEORIES

Elements of research praxeology

The concept of praxeology is one of the main constructs of the ATD (Chevallard, 2019). Using this concept, any human activity can be understood as consisting of two blocks: praxis and logos. Each of these blocks has two elements: *type of tasks* (**T**) and *technique* ($\boldsymbol{\tau}$) for the praxis block, and *technology* ($\boldsymbol{\theta}$) and *theory* ($\boldsymbol{\Theta}$) for the logos block. A task (**T**) indicates a problem of a given type; a technique ($\boldsymbol{\tau}$) is a way of performing tasks of this type; a technology ($\boldsymbol{\theta}$) is a way of explaining and justifying the technique; and theory ($\boldsymbol{\Theta}$) explains or justifies the technology. Artigue et al. (2011) extended the concept of praxeology to researchers' practices and knowledge (see also Artigue & Bosch, 2014; Artigue & Mariotti, 2014; Artigue, 2019). Research praxeologies of researchers' activities also comprise four elements ($\mathbf{T}/\boldsymbol{\tau}/\boldsymbol{\theta}/\boldsymbol{\Theta}$) (Artigue & Bosch, 2014). The type of tasks (**T**) of a research praxeology often refers to research questions and problems to be studied; the technique ($\boldsymbol{\tau}$) comprises the research method that addresses the research questions; the technology ($\boldsymbol{\theta}$) corresponds to methodological discourse that justifies the choice of the method and explains the results; and the theory ($\boldsymbol{\Theta}$) includes the main principle and notions of a given theoretical framework.

Networking praxeologies

Networking praxeologies are used to understand the research practices of networking theories (Artigue et al., 2011). The praxis and logos blocks of networking praxeologies can be identified through retrospective analysis of studies (Artigue & Bosch, 2014; Artigue & Mariotti, 2014; Artigue, 2019). However, such analysis is sometimes difficult to practice, because:

> [T]he theoretical block of networking praxeologies is also in the process of emerging, perhaps at the moment it is in a less-developed state. (Artigue & Mariotti, 2014, p. 351)

Our study attempts to make the logos block of networking praxeologies more visible, based on our retrospective analysis of networking strategies and theoretical grain sizes. For this analysis, let us explain here what we can locate as the four elements (**T/τ/θ/Θ**) of networking praxeologies.

The praxis block of a networking praxeology also corresponds to the networking questions or problems (**T**), and method (**τ**). Unlike research practice in general, the networking endeavour refers to two or more theoretical approaches. The choice of theoretical frameworks and their usage are part of the type of tasks and the techniques. The logos block of networking praxeologies is quite different from that of research praxeologies in general. The methodological discourse (**θ**) of networking theories describes, explains, and justifies networking strategies. Prediger et al. (2008) propose a formulation in terms of: *understanding* and *making understandable*, comparing and contrasting, *combining* and *coordinating*, and *integrating locally* and *synthesising*. However, the proposal of Artigue and Bosch (2014) includes other ingredients based on the praxeological structure of research strategies. Finally, the theoretical discourse (**Θ**) explains and justifies the networking strategies. Identifying the elements that qualify as a theory (**Θ**) of networking praxeologies is difficult as "the technological and theoretical discourses are not fully articulated" (Artigue & Bosch, 2014, p. 260) in the current state of networking praxeologies. This is probably because of the scientific challenge of sharing a common meaning of theory and the role of theory in mathematics education and why it is important – or even necessary for researchers – to network theories. Therefore, we attempted to identify technological and theoretical discourses behind the networking endeavour. The theoretical grain sizes can be used to describe the praxis block of networking praxeologies as well as to interpret the implicit logos block for characterizing researchers' discourses.

Theoretical grain sizes

The notion of *grain sizes*[1] has been used to describe, review, and categorize different levels and magnitudes of theories in mathematics education (e.g., Kieran, 2019; Shinno & Mizoguchi, 2021; Silver & Herbst, 2007; Watson & Ohtani, 2015). For examples, Silver and Herbst (2007) distinguished three theoretical levels: *grand*, *middle-range*, and *local* theories. Kieran (2019) also distinguished between three theoretical levels in the context of task design research: *grand*, *intermediate*, and *domain-specific* theories (or theoretical frames). These categories allow us to understand and describe the nature and spectrum of a given theory in terms of different theoretical levels and magnitudes. However, it is not easy to determine the grain sizes of theories in a general way, because one could consider different categorizations from different perspectives.

For Silver and Herbst (2007), grand theories respond "to a need for broad schemes of thought that can help us organise the field and relate our field to other fields" (p. 60); and middle-range theories grow "from the need to inform a discrete variety of practices, including individual mathematical thinking, teaching, and learning in classrooms, or mathematics teacher education" (p. 61). Local theories help mediate connections between *problems*, *research*, and *practices*, in a particular study's context. However, for Kieran (2019), grand theories include the cognitive-psychological, the constructivist, the socio-constructivist, the sociocultural, and other general educational theories. Intermediate theories have a more specialised focus

[1] Someone might interpret the meaning of the word *grain sizes* differently. In this paper, we used this term as a metonymy for describing and understanding different levels and magnitudes of theories in mathematics education. The use of this term is similar to what Watson and Ohtani (2015) mentioned in the reviews of task design research: "Grain size descriptions are intended to be *descriptive tools* [emphasis is added] for thinking in a structural way about task design, rather than being prescriptive" (p. 5)

than grand theories and, as such, can contribute in a more refined way to the design of curricular areas (pp. 271–272). Domain-specific theories "deal with distinct mathematical concepts, procedures, or processes of mathematical reasoning" (p. 272). The list below summarises Kieran's (2019) categorisations with examples of the theoretical grain sizes[2].

- Grand theoretical frames shape the background understanding of research in mathematics education (e.g., the cognitive-psychological, the constructivist, the socio-constructivist, the sociocultural)
- Intermediate level frames[3] are located between the grand theories and the domain-specific frames (e.g., realistic mathematics education theory (RME), theory of didactical situations (TDS), ATD, lesson study, variation theory, conceptual change theory)
- Domain-specific frames specify distinct mathematical concepts, reasoning processes, or tools (e.g., a frame for fostering mathematical argumentation within problem-solving, frame for proof problems with diagrams, frame for learning algebra using technological tools)

Silver and Herbst's (2007) categories are more general than Kieran's (2019), despite both considering three levels of grain size. This is because Kieran aimed to review and discuss different theoretical frameworks which have been used in task design research, but Silver and Herbst argued for the role of theories in mathematics education research in general. More recently, Shinno and Mizoguchi (2021) distinguished three theoretical grain sizes in the context of mathematics teacher education as follows:

- Grand theories are well-established frames for research inside and outside mathematics education that have been developed in a broader context of human activity (e.g., ATD, CHAT)
- Intermediate theories are frames that do not specify a feature, but rather a general aspect of teachers' knowledge or activities (e.g., knowledge for teaching, professional growth, documentational approach)
- Local theories are frames that specify a particular feature of teachers' knowledge or activities (e.g., teacher noticing, teacher design).

In this way, the method of distinguishing the grain sizes of theories cannot be absolute but is rather relative to the research area of the study being undertaken. For the analysis in this paper, we re-categorize Kieran's (2019) distinctions of grain sizes into three levels (*grand*, *intermediate*, and *local*) by generalising the above descriptions of the three levels of Shinno and Mizoguchi (2021). In Table 1, there are a few critical points when re-categorizing the earlier version by Kieran (2019). Despite the *local-level* being almost identical to *domain-specific frames* in Kieran (2019), we reconceptualised the other two levels to distinguish the degree of generality of a given theory in mathematics education. For instance, while both ATD and TDS were categorised as *intermediate-level frames* by Kieran (2019), we acknowledge the differences between the two theories in terms of grain size. Nonetheless, our categorisations cannot distinguish theories for research inside and outside mathematics education (e.g., ATD and variation theory) as we do not consider theories developed in other disciplines in this study. In addition, we excluded "the background understanding of

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[2] In Kieran (2019), what she called the grand theoretical frame is based on Cobb (2007). Examples for the intermediate and the domain-specific frames includes several frames used in the task design research. While Kieran (2019) includes references to each frame or study, we omitted the references in this table.

[3] Kieran (2019) provided an additional explanation of the intermediate-level frames referring to their roots as follows: "In addition, intermediate level frames can also be characterized according to whether their roots are primarily theoretical or whether they are based to a large extent on deep craft knowledge. An example of the former is the Theory of Didactical Situations and the latter, Lesson Study" (Kieran, 2019, p. 272).

research in mathematics education", which might refer to a general philosophical standpoint, from the description of *grand-level* theories. It is because we think background theories (such as constructivism) providing general principles that shape the universe of what can be a researchable object are relevant to all levels of grain sizes [4].

Table 1. (Re)categorizations of different theoretical grain sizes

Grain sizes of theories	Descriptions	Examples
Grand-level	well-established frames for conducting research inside and outside mathematics education, that have been developed in the broader context of human activity	ATD, commognitive framework, variation theory, etc.
Intermediate-level	frames that do not specify any particular knowledge or activities but specify a context rather than a general aspect of the research object to be studied	TDS[5], theory of realistic mathematics education (RME), etc.
Local-level	frames that specify a particular mathematical or didactic knowledge or activity	A model of problem solving, van Hiele levels, etc.

METHOD

Seeking an interface between the logos and praxis blocks is an appropriate way to develop the researchers' discourse of networking praxeologies (Artigue & Bosch, 2014). Three research examples, utilizing multiple theoretical frameworks for different purposes, are used to illustrate the praxis and logos blocks of networking praxeologies. The three case studies involved different types of research (comparative study, empirical study, and curriculum development). While these studies can be considered as cases of networking theory research, the networking strategies used are not explicitly mentioned in these papers. The theoretical grain sizes were also implicit. Therefore, we must interpret these implicit aspects. Thus, we selected research examples from studies that have been conducted in the Japanese educational context (such as lesson study, classroom teaching, and curriculum development). One study (Example 1) was selected because it involved a type of study (lesson study) that is culturally situated in Japanese educational research. The other two studies (Examples 2 and 3) were conducted by the author(s), for which we can analyse their theoretical aspects

4 For example, Shvarts and Bakker (2021) considered the grain sizes of levels in terms of the philosophical and historical roots of theories under considerations. They distinguished six different levels by means of *vertical analysis* for networking theories: such as epistemic presumption, ontological presumption, grand theory, local theory in mathematics education, application for teaching, and application for educational design. From this vertical perspective, *background theories* refer to higher levels of a certain level of theories.

5 Some researchers would also call TDS a grand-level theory as it can be applied to any situation wherein teaching and learning take place. Therefore, it should be noted that the distinctions or categorisations of different theoretical grain sizes are not reserved for the ones mentioned in Table 1.

retrospectively. While the three research studies are analysed as illustrative examples, they cannot be conceived as typical or representative, and the results of the analysis cannot be generalised. Our attention is drawn to the different praxeological models of the research on networking theories. Thus, the primary aim of this analysis is to make implicit discourses in the logos block explicit by focusing on the theoretical grain sizes of each theoretical framework to be networked. It allows us to understand the interface between the logos and praxis blocks of networking praxeologies.

Regarding the praxis block of networking praxeologies, we identify two elements (**T**/**τ**) by analysing the aims, questions, and methods of networking in the papers. For the types of networking tasks (**T**), we especially focus on the theoretical frameworks and/or models being used for networking, rather than practical methods of procedures or analyses in implemented studies (such as data collection or data analysis). Then, the networking strategies used in each study are identified as techniques (**τ**). Concerning the logos block of networking praxeologies, the technological discourses (**θ**) to explain and justify the networking strategies are interpreted by referring to some relevant quotations from each paper. To better understand the theoretical discourses (**Θ**) underlying the technologies in each study, we discuss the role of cultural issues in networking praxeologies; for example, what is a theory in different educational research traditions? What is the role of theory regarding educational practice? Why is it necessary to connect or reduce the number of theories? For such praxeological analyses of research on networking theories, the theoretical grain sizes categorized in Table 1 may play a determinant role for both the praxis and logos blocks.

ILLUSTRATIVE RESEARCH EXAMPLES

Three research examples

Example 1 – Comparing and contrasting

Miyakawa and Winsløw (2009) compared two didactical designs to introduce primary school students to proportional reasoning. The designs were based on two different approaches: *didactical engineering* in France, and *lesson study* in Japan. The comparison is informative as both approaches are used for planning a lesson, but the method of designing is very different. In the former, the lesson was designed according to the theoretical principles proposed in a specific theory; however, in the latter, the lesson was created according to the teachers' experiences in the classroom. Didactical engineering is a methodology for TDS introduced by Brousseau (1997) to gain theoretical insight into the functioning of a didactic system. In contrast, lesson study does not necessarily involve an explicit didactic theory, but often refers to a certain practical or pedagogical approach as a theoretical basis for teachers' practice. Miyakawa and Winsløw (2009) referred to a Japanese teaching approach called *open approach method* by Nohda (2000), which is relatively known as a teaching method to enhance multiple ways of thinking in the process of problem-solving, and contrasted it with the notion of *fundamental situation*[6] in TDS through their analysis and observations. Below, some elements in the process of designing a lesson to introduce students to proportional reasoning are summarised and contrasted according to theoretical basis, design formats, and realised lessons.

[6] A f*undamental situation* for a concept is a mathematical situation for which the concept constitutes a priori an optimal solution (Artigue et al., 2014, p. 49).

- A Japanese case
  - Theoretical basis: Open approach (Nohda, 2000)
  - Design formats: Lesson study
  - Realised lesson: *Hatsumon* (questioning)[7] to enquire about an *open problem* (Nohda, 2000)
- A French case
  - Theoretical basis: Fundamental situation (TDS) (Brousseau, 1997)
  - Design formats: Didactical engineering
  - Realised lesson: *adidactical* milieu[8] in the *puzzle* situation (Brousseau, 1997)

According to Miyakawa and Winsløw (2009), a comparison of the two didactical designs shows crucial similarities and differences in a lesson in the following ways.

> Both of the designs emphasize the social interaction and independent thinking of students. Both formats for design require quite similar kinds of analysis, including anticipating student strategies and revising the design in an experimental cycle. […] In a fundamental situation, they should lead to the personalization and institutionalization of a target mathematical knowledge (*savoir*), consistent with the 'official' mathematical knowledge. […] In the open approach, the aim is for students to apply and test their mathematical knowledge, through two main processes: the process in which some conditions and hypotheses from a 'real world' problem are formulated mathematically and the process of generalization and systematization after solving a problem. (Miyakawa & Winsløw, 2009, p. 216)

Miyakawa and Winsløw (2009) also discussed how didactical engineering and lesson study differ at the level of their objectives. Lesson study provides an opportunity for teachers to develop their teaching professions; the main objective is to improve lessons. In contrast, didactical engineering aims to establish scientific knowledge wherein a lesson confirms or rejects/questions the conditions for learning.

Example 2 – Combining and coordinating

Shinno (2018) aimed to characterise the development of mathematical discourses in a series of lessons in terms of the *semiotic chaining* model (triadic nested model) proposed by Presmeg (2006) and the *commognitive* framework introduced by Sfard (2008). There were two reasons for using two different theoretical frameworks in this study. The first is related to the content-specific aspect of students' difficulties of reification in the learning of square roots. Shinno (2018) argued that the cognitive account of reification can be used to explain this difficulty; however, conceptualising it differently from semiotic and discursive points of view may allow us to arrive at a deeper understanding of the reification phenomenon. The second reason is related to the setting of the mathematics classroom used in this study. Based on the observations of earlier studies on Japanese classroom culture (e.g., Emori & Winsløw, 2006), Shinno (2018) mentioned that "the social interaction between the teacher and the students in a Japanese mathematics classroom can be assumed to constitute a culturally unique discursive community" (p. 279), and that the semiotic and discursive approaches can be suitable for analysing the process through which students become familiar with the usage and meaning of square roots through classroom interactions. For example, Shinno (2018) analysed how a new expression

[7] *Hatsumon* (in Japanese) is a general pedagogical term which refers to a teacher's key questioning in a lesson.

[8] Within the TDS, *milieu* is a component of a didactical triangle (teacher, students, and milieu), which constitutes a didactical situation. In an *adidactical situation*, "students accept to take the mathematical responsibility of solving a given problem, and the teacher refrains from interfering and suggesting the target mathematical knowledge" (Artigue et al., 2014, p. 51). The *adidactical* is referring to "the situation has been temporally freed from its didactical intentionality" (ibid.)

$\sqrt{2}+\sqrt{3}$ becomes a mathematical object rather than a computational process through classroom interactions to justify or reject a teacher's question 'is $\sqrt{2}+\sqrt{3}=\sqrt{2+3}$ true?'

As shown in Table 1, Shinno (2018) attempts to coordinate the commognitive terms[9] with semiotic terms[10] in order to gain a multidimensional insight of the reification phenomenon through the networking of the two theoretical approaches. Shinno (2018) proposed two theoretical benefits of using two theoretical lenses in the strategy of combining and coordinating:

> From the viewpoint of the *commognitive framework*, the implicit meta-discursive rule may become explicitly identified as the third component of 'interpretant' by means of the triadic nested model. From the viewpoint of the triadic nested model, the first and second components, the signifier and signified, can be characterised as distinct features, such as keywords, visual mediators, and endorsed narratives. (Shinno, 2018, p. 302)

Table 2. Coordinating theoretical terms from the two different frameworks

Commognitive framework (Sfard, 2008)	*Triadic nested model* (Presmeg, 2006)
Word use (keywords)	Signifier or signified
Visual mediators	Signifier or signified
Endorsed narratives	Signifier or signified
Routines (meta-rules)	Interpretant

Note. Adapted from "Reification in the learning of square roots in a ninth-grade classroom: Combining semiotic and discursive approaches" by Y. Shinno, 2018, *International Journal of Science and Mathematics Education, 16*(2), p. 302.

In this empirical study, the semiotic chaining model was used to identify the three components (signifier, signified, and interpretant) in the classroom episodes and then characterised by their discursive features from a commoginive point of view. For example, when a signifier $\sqrt{2}+\sqrt{3}$ was reified (objectified), an interpretant 'treating $\sqrt{2}+\sqrt{3}$ as one irrational number' was also understood as a meta-discursive rule. In this way, Shinno (2018) sought to use theoretical terms from the two different frameworks (Table 2) interchangeably for the analysis. Thus, combining and coordinating the two theoretical frameworks allowed him to gain a multi-faceted insight into the reification phenomenon, meaning a change from process-oriented use to object-

[9] The meanings of the commognitive terms are as follows: *word use* refers to mathematical vocabulary, syntax, and ordinary words associated with mathematics; *visual mediators* include physical, diagrammatic, and symbolic mediators of mathematical objects; *endorsed narratives* mean a set of mathematical statements, proofs, rules of calculation, which are accepted within a given community; and *routines* refer to regularly employed and patterned repetitive activities (e.g., calculating, proving, generalising) as well as a set of meta-rules (e.g., how to calculate, how to prove, how to generalise).

[10] The model of semiotic chaining by Presmeg (2006) based on Lacan's inversion of Saussure's dyadic model (signifier and signified) as well as Peirce's triadic model (object/signified, representation/signifier, and interpretant). For this model, the relationship between *signifier* and *signified* is considered to have a nested structure, and meaning-making in the two components is understood by the third component *interpretant*.

oriented use of a signifier[11].

Example 3 – Locally integrating

Shinno et al. (2015, 2018) proposed a theoretical framework for curriculum development of proof in secondary schools in Japan. Unlike the other two research examples, the theoretical framework undertaken in their paper was aimed at being used for developing, designing, or improving a curriculum related to mathematical proof. For this aim, the theoretical framework was constructed by adapting and integrating different theoretical models and concepts as follows:

- The *Mathematical Theorem* as a system, consisting of *statement*, *proof*, and *theory* (Mariotti et al., 1997)[12]
- A model of mathematical proofs, comprising *knowledge*, *formulation*, and *validation* (Balacheff, 1987)[13]
- The concept of *local organisation* and *global organisation* (Freudenthal, 1971, 1973)[14]
- The concepts of *small theory* and *large theory* (Hanna & Jahnke, 2002)[15]

On the one hand, the theoretical framework proposed by Shinno et al. (2015, 2018) was a framework for curriculum development, describing and prescribing mathematical contents and sequences in the Japanese educational context. On the other hand, it seems that the theoretical models and concepts to be integrated have different theoretical natures and are not always considered the ones for educational development. For example, the notion of Mathematical Theorem (Mariotti et al., 1997) is a model to understand mathematical practice wherein a proof is carried out, but originally this model does not prescribe curricular contents and activities to be taught in schools. Therefore, when constructing the framework, Shinno et al. (2015, 2018) reconceptualized some concepts, adapting (or detaching from) their original senses, and integrated them into one framework so that it is relevant to the Japanese curriculum for proof. Let us briefly explain how they adapted and integrated different theoretical concepts.

In their framework, the three elements – *statement*, *proof*, and *theory* – were used as the foreground in shaping the framework. Balacheff's (1987) categorisation of *formulation* and *validation* was used to understand levels of *statement* and *proof*. While the original notion of theory in Mathematical Theorem

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[11] Regarding the implications of using the two frameworks, Shinno (2018) concluded: "[This study] enables to characterize reification as the change in a set of meta-rules, which means the replacement of the process-oriented use with the object-oriented use of a new signifier. Although the model of semiotic chaining itself conceptualizes reification as the act of making signifier-signified couple in each node, as far as the reification in the discursive process is concerned, it may be important to detect a set of interpretants (meta-rules)" (Shinno, 2018, p. 311).

[12] A *Mathematical Theorem* consists of a system of relations between a *statement*, and its *proof*, and the *theory* in which the proofs make sense. "[The] existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense" (Mariotti et al., 1997, p. 182).

[13] Balacheff (1987) proposed a theoretical framework composed of *knowledge, formulation,* and *validation*, which allow us to analyse the complex nature of proof and proving. The four levels (naïve empiricism, crucial experiment, generic examples, and thought experiment) are regarded as the levels of validation.

[14] The local organisation by Freudenthal (1973) is a concept to explain short deduction chains consisting of a statement and its proof in geometry, wherein some properties can be accepted as taken for granted, while the global organisation is a concept to explain an axiomatized system (such as a system of Euclidean geometry).

[15] The distinction between *small theory* and *large theory* is similar to that between local and global organization. Hanna and Jahnke (2002) mention that "instead of building a large theory (namely, Euclidean geometry) in the course of the curriculum, it seems to be more appropriate to work in several small theories" (p. 3). Thus, *small theory* and *large theory* are considered concepts rather than part of an operational framework.

(Mariotti et al., 1997) mainly refers to mathematical theory (such as arithmetic, algebraic, or geometrical theories), Shinno et al. (2015, 2018) reconceptualised it as a component which includes different layers of the system (such as local and axiomatic theory) to consider the wide range of contents and levels in the curriculum. They adapted the notions of Freudenthal (1971, 1973) and Hanna and Jahnke (2002).

Their reconceptualization of the element (i.e., theory) of Mathematical Theorem was needed to account for the axiomatic aspect of geometry curriculum in secondary schools in Japan, as the Japanese geometry curriculum emphasizes Euclidean geometry (Miyakawa, 2017; Shinno et al., 2018). Miyakawa (2017) characterised this nature of the system in secondary school mathematics as quasi-axiomatized geometry with respect to some characteristics such as "the term axiom or postulate is not used; some properties are introduced after observation; some are admitted implicitly and there is no long list of axioms as in Euclid's Elements" (p. 49). This is also related to what Hanna and Jahnke (2002) distinguished as small theory and large theory.

Shinno et al. (2018) called a theoretical framework for curriculum development, the reference epistemological model (REM) and leaned on the elaboration of RME by Bosch and Gascón (2006), which constitutes the basic theoretical lens for researchers to analyse different types of mathematical knowledge among different institutions. Thus, Shinno et al. (2018) attempted to integrate different theoretical constructs regarding proof into one framework (REM), wherein an internal consistency within the framework is created to understand contents and levels in a curriculum, summarised as follows:

> Regarding the proposed REM, we conceptualized that the statements and proofs are interrelated according to the nature of theory and illustrated how this model helps to describe the evolution of each element. This model may help us understand the gaps in the evolution of the formulation of a statement and how different meanings of proof relate to the distinction between local and (quasi-) axiomatic levels of theory. (Shinno et al., 2018, p. 30)

For example, there are three levels of the formulation of a statement, i) drawing, diagrams, manipulation, and gesture; ii) ordinary language and words; iii) mathematical words and symbols. Considering a universal proposition (statement), a universal quantification can be formulated by ordinary words (e.g., *all*, *any*) in schools and by mathematical symbols (e.g., $\forall$) at advanced levels. Based on the proposed framework, Shinno et al. (2018) implied that there is a gap in the levels of formulation in the Japanese curriculum (especially between the ordinary and mathematical language of a universal proposition), and there is a crucial transition between the levels for development[16].

From the perspective of networking praxeologies

Our retrospective analysis of the three case studies in terms of networking praxeologies and theoretical grain sizes can shed light on the principal aspects of networking in each study (as summarised in Table 3), and thus enable the comparison and discussion of the praxis (**T/τ**) and the logos (**θ/Θ**). The types of tasks (**T**) were found in the choice of the theoretical frameworks in the papers. The techniques (**τ**) were identified by focusing on the networking strategies that were most relevant to the studies (based on our interpretations and retrospections). In Example 1, the strategy of comparing and contrasting was chosen and the similarities and

16 Regarding linguistic issues, we also discussed how Japanese language may affect the difficulty and gap in the levels of the formulation of quantifications (Shinno et al., 2019).

differences between the French and Japanese approaches (TDS and open approach) to didactical designs are examined. In Example 2, the strategy of combining and coordinating is used to analyse classroom teaching with a commognitive framework, and with the semiotic chaining model. In Example 3, local integration is used to construct a theoretical framework for the curriculum development of proof in secondary schools in Japan. The strategy in this case also offers a coordination of theoretical concepts (such as the relationship between Freudenthal's local/global organizations and Hanna & Jahnke's small/large theories). We think this is natural since coordinating is a strategy that "can be a starting point for a process of theorizing" (Prediger et al., 2008, p. 173).

Table 3. Networking praxeologies and theoretical grain sizes in the three research examples

	Example 1	Example 2	Example 3
T	To compare *TDS* and *open approach method*	To combine *the model of semiotic chaining* and *the commognitive framework*	To construct a theoretical framework for curriculum development, using the *Mathematical Theorem*
τ	A strategy of *comparing and contrasting* in terms of the theoretical basis, design format, realized lesson	A strategy of *combining and coordinating* in terms of the theoretical terminologies between two frameworks (see Table 2 (Shinno, 2018, p. 302))	A strategy of '*coordinating* and *integrating locally*' in terms of *statement, proof,* and *theory*
θ	Discourses on similarities and differences on the didactic designs based on the two approaches (see the quotation above (Miyakawa & Winsløw, 2009, p. 216))	Discourses on the compatibilities and interchangeabilities between the semiotic and the commognitive terms	Discourses on how to integrate different theoretical elements, levels, and concepts into a common framework
GS[a]	*Intermediate-level* (TDS) and *local-level* (open approach)	*Grand-level* (commognition) and *intermediate-level* (semiotic chaining)	*Local-level* (all theoretical constructs)

[a]GS; grain sizes

To determine the elements of the logos block, we needed to interpret the descriptions explicitly or implicitly written in the papers. We considered the technological discourses (**θ**) by focusing on the descriptions to explain and justify how two or more theories were networked by a certain strategy and why the strategy was most relevant to the study undertaken. In the previous subsection, we tried to show such descriptions by the quotations or elaborations from the papers. Table 3 included summarised technological discourses. However, it is more difficult to determine the theoretical discourses (**Θ**) because they are mostly implicit in the papers.

We considered such theoretical underpinnings by focusing on cultural elements included in each study; for example, what is regarded as a theory in the Japanese educational and research context? What is the role of theories regarding educational practice? What is the influence of the educational system in attempting to answer these questions? We develop the discussion related to the cultural issues in the section titled ‘Discussion and conclusion’.

From the perspective of theoretical grain sizes

To better understand the networking praxeologies in each study (Table 3), different theoretical grain sizes were considered. Based on our categorizations of grain sizes mentioned earlier, TDS from Example 1 is considered an intermediate-level theory, since it is often used to understand and analyse the epistemological processes in mathematical classrooms from a systemic perspective. TDS is not a domain-specific theory which specifies any mathematical knowledge or activity. Rather, it is primarily developed to model the functions and mechanisms between teachers, students, and the didactical milieu in a didactical/adidactical situation (in the classroom). In contrast, the open approach is seen as a local-level framework or instructional principle which does not specify any mathematical content but does specify teachers’ teaching methods. For instance, using an open problem is a characteristic aspect of this approach which is comparable with the notion of fundamental situation from TDS (Miyakawa & Winsløw, 2009). However, TDS and the open approach are not fully comparable because the latter does not provide a tool to analyse teaching and learning of mathematics at a general level (e.g., the approach cannot be applied to a lesson with a closed problem). Therefore, the two frameworks are comparable only if one can focus on a specific notion of one approach that has a counterpart in the other (such as a fundamental situation and an open problem). Thus, the discourses on two didactical designs, as cited above from Miyakawa and Winsløw (2009, p. 216), have focused on some comparable constructs from each framework.

In Example 2, the commognitive framework is viewed as a grand-level theory which is a comprehensive theory of learning from a discursive point of view. The model of semiotic chaining originates from Peirce’s semiotics. However, it was introduced by Presmeg (2006) into mathematics education for analysing the relations between signifier, signified, and interpretant in the teaching and learning of mathematics. This model was used as an intermediate-level framework in Example 2, although one could also consider it as a collection of theoretical concepts. Shinno (2018) combined commognitive and semiotic approaches to provide different accounts of reification in mathematical learning. Since the commognitive framework comprises a much broader range of theoretical constructs, coordination with semiotic chaining is related to only some semiotic parts of this grand theory. This implies that the strategy of coordinating between different grain sizes requires an explicit discourse on complementarity; for example, which part of one theory can be coordinated with another theory.

Networking in Example 3 offers local integration, but the connectivity of different theoretical concepts is rather complex. All theoretical models or concepts used in the study are regarded as local-level frames which have been developed in the context of research on proof and proving. However, this does not mean that all the theoretical constructs have the same theoretical grain size. For example, the triplet (statement, proof, theory) has been used as a foreground frame to integrate other theoretical concepts. There have been many discourses on argumentative connectivity about how to integrate different models or concepts since Shinno et al.'s (2015, 2018) attempt to propose a theoretical framework. A characteristic aspect of such a discussion is that it changes or extends the original meaning of the theoretical terms (statement, proof, theory) to those with a broader sense; this allows the inclusion of different contents and levels in the curriculum in secondary schools in Japan. Additionally, they created new terminologies (for the three different layers of the system; the logic of the real world, local theory, and quasi-axiomatic theory) through the adaptation of existing notions (such as small/large theory)[17].

DISCUSSION AND CONCLUSION

A deeper understanding of researchers' work on networking theories

The praxis block corresponds to how researchers work when they choose different theories and network them. The researchers may consider the grain sizes of theories according to the types of networking tasks (**T**) for the study undertaken. We then considered networking strategies as a technique (**τ**) and explored the technological discourse (**θ**) to explain and justify the strategy through a retrospective analysis of three illustrative research studies as examples. Artigue and Bosch (2014) argue that the characterisation of the logos block (**θ/Θ**) of a networking praxeology has not yet been established. In our study, we offered the notion of theoretical grain sizes to deepen our understanding of the networking praxeologies. As mentioned in the previous section, the networking techniques are affected by the grain sizes of the theories involved. While we have only described the grain sizes (e.g., grand, intermediate, or local levels) of the theoretical frameworks used in the three research examples shown in Table 3, attention was paid to the discourses on networking different theoretical frameworks, which may have different sizes or are of the same grain sizes.

The characteristics of such networking praxeologies may differ according to the study type, such as empirical studies, design studies, or theory development. For example, Bikner-Ahsbahs and Prediger (2014) discussed the notion of *perspective triangulation* (Denzin, 1970), meaning a research practice to increase the validity of empirical analysis. Although this notion does not refer to theories, increasing the validity of empirical studies can bring forth a benefit from such networking. Combining theories in/for empirical studies often provides deeper insights into complex phenomena (Prediger et al., 2008), wherein the results of the empirical analysis can respond to the original theories. Bikner-Ahsbahs and Prediger (2010, 2014) also mentioned other perspectives (explicitness, empirical scope, stability, and connectivity) to discuss the empirical and theoretical benefits of networking theories for different types of studies. Some of these perspectives are related to what we identified as technological discourse (a justification of the strategies involved in each case study). For instance, perspective triangulation or connectivity can support the strategy of combining and coordinating in Example 2; connectivity can explicate the strategy for locally integrating in Example 3; and *explicitness* can be the strategy for comparing and contrasting in Example 1, as well as the strategy for

[17] For example, a geometric statement "the sum of the interior angles of a triangle is 180°" can be explained by a physical approach (such as a measurement or experiment) which is acceptable within *the logic of the real world*. The same statement can be proved by previously accepted properties of parallel lines, based on *local theory* which is often regarding one particular proof. The *quasi-axiomatic theory* can be relevant when considering a system of statements and proofs in school geometry (even though the term axiom or postulate is not introduced). More descriptions of the three layers are mentioned in Shinno et al. (2018).

locally integrating in Example 3. In addition, it is also worth considering how theoretical terminologies may change or preserve their meaning before and after networking attempts when examining linguistic connectivity (Shinno, 2017). As shown in Example 2, by coordinating, the theoretical terms in one framework are interchangeable with those in another framework. In Example 3, some original concepts are locally elaborated and integrated into new terminologies in the new framework for curriculum development; however, such new terms do not preserve their meanings in the original sense.

Cultural issues to be considered

Networking praxeologies can provide a model for understanding researchers' practices and knowledge of networking theories. This implies that it is important to consider the diversity of research activities related to theories of mathematics education. One of the reasons for this diversity is *cultural issues,* which may affect the researchers' work. This is also the case in the three research examples involving Japanese educational contexts. If the theory (**Θ**) in the logos block of networking praxeologies is implicit, its description can only be based on the researchers' conception of a theory in research and practice.

For instance, Example 1 indicated the differences between TDS and the open approach; where the open approach was a practice-oriented theory or a prescriptive theory, which is often used to improve teachers' classroom teaching and facilitate students' problem-solving activities. In fact, both the descriptive and the prescriptive nature of a theory in mathematics education are often discussed in Japan (e.g., Koyama, 2006; Yamada, 2011). This is related to what has been discussed in the previous Topic Study Group in ICME-13 (Dreyfus et al., 2017); for example, "[i]n Japan, 'theory' is always a theory of some practice. A practice develops, somebody notices it, reflects upon it, and constructs a theory of practice" (p. 617). This implies two distinct aspects. One is that a theory emerges and develops from/for some practice. The other is that a theory is very close to practice and the theory often seems to be a pragmatic one[18]. However, in the international context, some researchers might not or cannot accept such a *pragmatic theory* as a theory in mathematics education (cf. Miyakawa & Shinno, 2022). Thus, there is miscommunication among international researchers regarding what is called theory in Japanese educational research (e.g., Zazkis & Zazkis, 2013). This may open a discussion that reveals the cultural specificity of Japanese (or East Asian) educational research, which contrasts with the French (or European) tradition of didactics (Blum et al., 2019). Here, we do not discuss the nature of theory in different research cultures (although such discussion is promising, it lies beyond the scope of the paper), but merely wish to point out that there is a cultural issue that may shape researchers' knowledge, even in the context of networking theories.

In addition, in Example 2, Shinno (2018) considered the cultural specificity of the Japanese mathematics classroom to be related to semiotic activities (Emori & Winsløw, 2006); however, this specificity of classroom is generally attributed to the Japanese teaching pattern (Stigler & Hiebert, 1999). Such a cultural factor may influence the researcher's choice of theory or his/her perspective of using a theory for classroom study. Recently, Funahashi and Hino (2014) have also investigated the interactive classroom pattern, called a guided focusing pattern, which is often observed in Japanese classroom lessons, in terms of a discursive or commognitive perspective for their theoretical and methodological framework. Although other theoretical

[18] Here we use the term *pragmatic* as an ordinary word without any theoretical underpinning.

approaches can be used to analyse a classroom culture, what we would imply is that the researchers' perspective of using a theory (or multiple ones) can be affected by the local educational context and by the researchers' conceptions of how the theory can be useful for investigating such a cultural specificity.

Furthermore, in Example 3, the specificity of the Japanese mathematics (geometry) curriculum in secondary schools was considered when adapting some theoretical concepts that were integrated into a framework for curriculum development. This also implies that the educational contexts (particularly the curriculum) in a country may affect researchers' work on theories. For example, given that the Japanese geometry curriculum is characterised as quasi-axiomatized geometry (Miyakawa, 2017), we could adopt or adapt the related concepts (such as, local/global organization, small/large theory) in their framework. In addition, developmental research, which is often aiming at improving educational practices (including task design, curriculum development, or lesson study) is a dominant type of research and/or study in Japan. Although a theoretical framework is often constructed for such developmental work, what is important for the framework is to be useful for development, rather than for scientific research (Miyakawa & Shinno, 2022). This is one reason why the proposed framework in Example 3 referred to different constructs (models, concepts, frameworks) to be integrated. For this reason, the researchers see all constructs as theoretical underpinnings of the framework for development, even though some constructs are not usually brought forth as theory for research. This may have the implication that one can distinguish between *theory for research* and *theory for practice*.

Finally, we attempted to translocate the cultural issues (such as research culture, classroom culture, and curriculum culture) into the theoretical discourses (**Θ**), as shown in Table 4, which are implicit in the networking praxeologies (Table 3). Furthermore, the logos block of networking praxeologies is also related to the researchers' conception of the field of mathematics education and the nature of mathematics education as a scientific discipline. This raises at least two questions: How can we make such implicit discourses more explicit in networking theories? Why is such a meta-level and a self-referential study important for researchers? While we believe that the cultural issues in different educational research traditions play a determinant role, further research is needed to address these questions and issues.

Table 4. Theoretical discourses (Θ) regarding cultural issues in the three research examples

	Example 1	Example 2	Example 3
Θ	Discourses on the nature of theories in different research traditions (research culture)	Discourses on a culturally unique discursive community in a Japanese mathematics classroom	Discourses on the necessity of a theoretical framework for curriculum development in Japanese mathematics education

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References

Artigue, M. (2019). Reflecting on a theoretical approach from a networking perspective: The case of the documentational approach to didactics. In L. Trouche (Eds.), *The 'resource' approach to mathematics education* (pp. 89–118). Springer. https://doi.org/10.1007/978-3-030-20393-1_5

Artigue, M., & Bosch, M. (2014). Reflection on networking through the praxeological lens. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 249–265). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_15

Artigue, M., Bosch, M., & Gascón, J. (2011). Research praxeologies and networking theories. In M. Pytlak et al. (Eds.). *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2381–2390). Rzeszów: ERME. https://hal.archives-ouvertes.fr/hal-02385558

Artigue, M., Haspekian, M., & Corblin-Lenfant, A. (2014). Introduction to the theory of didactical situations (TDS). In A. Bikner-Ahsbahs, S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 47–65). Cham, Springer. https://doi.org/10.1007/978-3-319-05389-9_4

Artigue, M., & Mariotti, M. A. (2014). Networking theoretical frames: The ReMath enterprise. *Educational Studies in Mathematics, 85*(3), 329–355. https://doi.org/10.1007/s10649-013-9522-2

Balacheff, N. (1987). Processus de revue et situations de validation. *Educational Studies in Mathematics, 18*(2), 147–176. https://doi.org/10.1007/BF00314724

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories – An approach for exploiting the diversity of theoretical approaches. In B. Sriraman and L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 483–506). Springer. https://doi.org/10.1007/978-3-642-00742-2_46

Bikner-Ahsbahs, A., & Prediger, S. (2014). Networking as research practices: Methodological lessons learnt from the case studies. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 235–247). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_14

Blum, W., Artigue, M., Mariotti, M. A., Sträßer, R., & Van den Heuvel-Panhuizen, M. (2019). European Didactic traditions in mathematics: Introduction and overview. In W. Blum (Eds.), *European traditions in didactics of mathematics* (pp. 1–10). Cham, Springer. https://doi.org/10.1007/978-3-030-05514-1_1

Bosch, M., & Gascón, J. (2006). *Twenty-five years of the didactic transposition [ICMI bulletin],* (Vol. 58) (pp. 51–65).

Brousseau, G. (1997). *The theory of didactical situations in mathematics*. Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47211-2

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114. https://doi.org/10.24529/hjme.1205

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–38). NCTM.

Denzin, N. K. (1970). *The research act in sociology*. Aldine. https://doi.org/10.4324/9781315134543

Dreyfus, T., Sierpinska, A., Halverscheid, S., Lerman, S., & Miyakawa, T. (2017). Topic Study Group No. 51: Diversity of theories in mathematics education. G. Kaiser et al. (Eds.), *Proceedings of the 13th International Congress on Mathematical Education*, ICME-13 [Monograph] (pp. 613–617). Springer. https://doi.org/10.1007/978-3-319-62597-3_78

Emori, H., & Winsløw, C. (2006). Elements of a semiotic analysis of the secondary level classroom in Japan. In F. K. S. Leung et al. (Eds.), *Mathematics education in different cultural traditions: A comparative study of East Asia and the West* (pp. 553–566). Springer. https://doi.org/10.1007/0-387-29723-5_33

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3*(3–4), 413–435. https://doi.org/10.1007/BF00302305

Freudenthal, H. (1973). *Mathematics as an educational task*. Reidel. https://doi.org/10.1007/978-94-010-2903-2

Funahashi, Y., & Hino, K. (2014). The teacher's role in guiding children's mathematical ideas toward meeting lesson objectives. *ZDM*, *46*(3), 423–436. https://doi.org/10.1007/s11858-014-0592-0

Hanna, G., & Jahnke, H. K. (2002). Another approach to proof: Arguments from physics. *ZDM – Mathematics Education, 34*(1), 1–8. https://doi.org/10.1007/bf02655687

Kidron, I., Bosch, M., Monaghan, J., & Palmér, H. (2018). Theoretical perspectives and approaches in mathematics education research. In T. Dreyfus et al. (Eds.), *Developing research in mathematics education* (pp. 255–268). Routledge. https://doi.org/10.4324/9781315113562-19

Kieran, C. (2019). Task design frameworks in mathematics education research: An example of a domain specific frame for algebra learning. In G. Kaiser, & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education* (pp. 265–287). Springer. https://doi.org/10.1007/978-3-030-15636-7_12

Koyama, M. (2006). Sugaku rikai no 2-jiku katei moderu ni motozuku jugyo kousei no genri to houhou (Principles and methods for designing mathematics lessons based on the two-axes process model of understanding mathematics). *Bulletin of Japanese Curriculum Research and Development, 28*(4), 61–70. (In Japanese). https://doi.org/10.18993/jcrdajp.28.4_61

Mariotti, M. A., Bartolini, M., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts: From history and epistemology to cognition. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the Int. Group. For the psychology of mathematics education,* (Vol. 1) (pp. 180–195). Lahti, Finland: University of Helsinki.

Miyakawa, T. (2017). Comparative analysis on the nature of proof to be taught in geometry: The cases of French and Japanese lower secondary schools. *Educational Studies in Mathematics, 94*(1), 37–54. https://doi.org/10.1007/s10649-016-9711-x

Miyakawa, T., & Shinno, Y. (2022). Dezain-kenkyu wa 'kenkyu' ni ikani kouken sutuka: purakuseorojii no shiten kara (How design research contributes to scientific research: From a perspective of praxeology). In *Proceedings of the 10th Spring Conference of Japan Society of Mathematical Education*. (In Japanese).

Miyakawa, T., & Winsløw, C. (2009). Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics*, *72*(2), 199–218. https://doi.org/10.1007/s10649-009-9188-y

Nohda, N. (2000). Teaching by open-approach method in Japanese mathematics classroom. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th Conference of the Int. Group. For the psychology of mathematics education*, (Vol. 1). Hiroshima, Japan: PME.

Prediger, S., & Bikner-Ahsbahs, A. (2014). Introduction to networking: Networking strategies and their background. In A. Bikner-Ahsbahs, & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 117–125). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_8

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Presmeg, N. (2006). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 163–182. https://doi.org/10.1007/s10649-006-3365-z

Radford, L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – Mathematics Education*, *40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing*. Cambridge University Press. https://doi.org/10.1017/CBO9780511499944

Shinno, Y. (2017). Meta-theoretical aspects of the two case studies of networking theoretical perspectives: Focusing on the treatments of theoretical terms in different networking strategies. In T. Dooley et al. (Eds.), *Proceedings* *of the Tenth Congress of European Research in Mathematics Education* (pp. 2700–2701). Dublin, Ireland: DCU Institute of Education and ERME. https://hal.archives-ouvertes.fr/hal-01948868

Shinno, Y. (2018). Reification in the learning of square roots in a ninth grade classroom: Combining semiotic and discursive approaches. *International Journal of Science and Mathematics Education, 16*(2), 295–314. https://doi.org/10.1007/s10763-016-9765-3

Shinno, Y., Miyakawa, T., Iwasaki, H., Kunimune, S., Mizoguchi, T., Ishii, T., & Abe, Y. (2015). A theoretical framework for curriculum development in the teaching of mathematical proof at the secondary school level. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the Int. Group. for the psychology of mathematics education,* (Vol. 4) (pp. 169–176). Hobart, Australia: PME.

Shinno, Y., Miyakawa, T., Iwasaki, H., Kunimune, S., Mizoguchi, T., Ishii, T., & Abe, Y. (2018). Challenges in curriculum development for mathematical proof in secondary school: Cultural dimensions to be considered. *For the Learning of Mathematics*, *38*(1), 26–30.

Shinno, Y., Miyakawa, T., Mizoguchi, T., Hamanaka, H., Kunimune, S. (2019). Some linguistic issues on the teaching of mathematical proof. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuiset (Eds.), *Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 318–325), Utrecht University, The Netherlands. https://hal.archives-ouvertes.fr/hal-02398502/

Shinno, Y., & Mizoguchi, T. (2021). Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics textbooks: Two cases from *kyouzai kenkyuu* in Japan. *ZDM – Mathematics Education*, *53*(6), 1387–1402. https://doi.org/10.1007/s11858-021-01269-8

Shvarts, A., & Bakker, A. (2021). Vertical analysis as a strategy of theoretical work: From philosophical roots to instrumental and embodies branches. Paper presented at 14th International Congress on Mathematical Education.

Silver, E. A., & Herbst, P. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–67). NCTM.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. Free Press.

Tabach, M., Rasmussen, C., Dreyfus, T., & Apkarian, N. (2020). Towards an argumentative grammar for networking: A case of coordinating two approaches. *Educational Studies in Mathematics, 103*(2), 139–155. https://doi.org/10.1007/s10649-020-09934-7

Watson, A., & Ohtani, M. (2015). Themes and issues in mathematics education concerning task design: Editorial introduction. In A. Watson, & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study 22* (pp. 3–15). Springer. https://doi.org/10.1007/978-3-319-09629-2_1

Yamada, A. (2011). Sugakuteki mondai-kaiketsu katei no moderu ni tsuite: Mondai-kaiketsuteki na jugyo no dezain ni muketa yobiteki kousatsu (On models of mathematical problem solving process: a preliminary consideration towards problem-solving lessons). *Epsilon*, *53*, 25–38. (In Japanese).

Zazkis, R., & Zazkis, D. (2013). Mathematical thinking: How to develop it in the classroom. *Research in Mathematics Education*, *15*(1), 89–95. https://doi.org/10.1080/14794802.2013.763609

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THE ROLE OF FEEDBACK WHEN LEARNING WITH A DIGITAL ARTIFACT: A THEORY NETWORKING CASE ON MULTIMODAL ALGEBRA LEARNING

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Abstract

How digital feedback supports teaching/learning with a digital tool is not yet well understood. In a networking of theories approach, Activity Theory and the Instrumental Approach are combined to investigate the role of digital feedback for the teaching/learning of integers with the MAL-system, a multimodal algebra learning system. The MAL-system is designed as a multi-touch tangible user interface with feedback functions allowing students to mathematically operate with (negative) numbers represented as virtual tiles. We explore the role of digital feedback by a multi-case study at the grade five level, in which we conducted experimental task-based interviews of four student pairs, each supported by a tutor. Findings show that (digital) feedback mediates the teaching/learning activity in a supportive way. Reflection on the way the two theories are combined reveals that they can be regarded as locally integrated into a layered model enriching the describing of the transformation of teaching/learning mediated by digital feedback.

Keywords: Networking of theories, activity theory, instrumental genesis, digital feedback, negative numbers, transformation

INTRODUCTION

Hattie and Timperley (2007) emphasized feedback to be an essential part of teaching and learning. When learning is supported by digital tools, digital feedback is highly relevant (Fyfe, 2016; van der Kleij et al., 2015). Both immediate feedback as well as feedback variation have been shown to be effective for learning (Bockhove & Drijvers, 2012). A recent study on algebra learning shows that any digital feedback is better than no feedback and the most important digital feedback takes an explanatory form (Fyfe, 2016). However, "… the computer can serve as an effective problem-solving tool only if accompanied by more traditional forms of discourse between pupils and teacher" (Hillel et al., 1993, p. 38). But it is not so clear how feedback may foster processes of teaching/learning, and how teacher feedback and digital feedback differ and interact

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in the use of an artifact to support in-depth learning. Answering these questions depends on the background theory on learning (Evans, 2013). We use a networking of theories approach to investigate this problem through a multi-case study in which students learn about negative numbers with a digital tool specifically designed for this purpose. This paper aims at answering the following research question: *What is the function of digital feedback in the support of the teaching/learning of negative numbers with this specific digital artifact?* To answer this question we will conceptualize the main terms in our theoretical framework, using *Activity Theory* and the *Instrumental Approach* within a theory networking approach.

THEORETICAL FRAMEWORK

Our approach aims at networking two theoretical approaches in mathematics education (Bikner-Ahsbahs & Prediger, 2014), *Activity Theory* (Leontjew, 1987)[1] is used as a background theory (Mason & Waywood, 1996) for teaching/learning coordinated with the *Instrumental Approach* (Artigue, 2002; Trouche, 2020). We will conceptualize the *teaching/learning of negative numbers* using activity theory. The instrumental approach will allow us to embed the *use of the artifact* by the learners into the activity.

The networking of these two theoretical approaches builds on Radford's elaboration of the notion of theory (2008) and how theories can be related. Radford defines a theory as:

> a way of producing understandings and ways of action based on:
>
> - A system, P, of basic principles, which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective.
> - A methodology, M, which includes techniques of data collection and data-interpretation as supported by P.
> - A set, Q, of paradigmatic research questions (templates or schemas that generate specific questions as new interpretations arise or as the principles are deepened, expanded or modified) (2008, p. 320)

He abbreviates this definition by the triplet (P, M, Q). Radford (2012) proposes later to expand this notion of

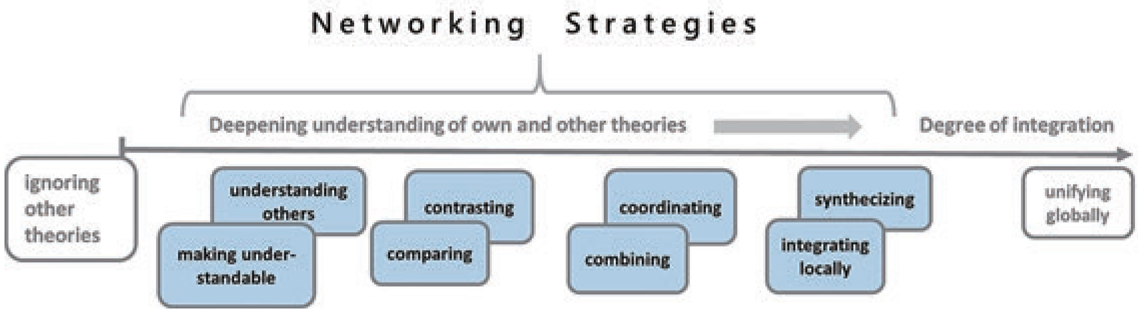

Figure 1: Networking Strategies
(CC BY 4.0, Bikner-Ahsbahs et al., 2016, p. 34, adapted from Prediger et al., 2008)

1 The author's name Леóнтьев is transliterated differently into Roman characters in different articles. Leontjew is the spelling in German literature and Leontjev, Leont'ev, Leontyev and Leontiev are spellings found in English literature.

theory by including results R as a relevant constituent because research results rebound on the theories used in research, thus pointing to the purpose of theories as tools for research. He represents this expanded understanding of theory as a dynamic structure by [(P, M, Q), R].

The networking of theories means to relate theories with respect to their constituents. Four pairs of networking strategies positioned on a scale of increasing integration (Figure 1) have shown to be useful to identify relationships. The two pairs of strategies that are relevant in this study are coordinating and local integration. While combining is a kind of triangulation, i.e., viewing an empirical phenomenon from different theoretical perspectives, coordinating can be considered, when "a conceptual framework is built by fitting together elements from different theory elements for making sense of an empirical phenomenon" (Prediger & Bikner-Ahsbahs, 2014, p. 120). The latter becomes relevant when we complement an already existing way of theoretical understanding with an additional theory element to achieve a more comprehensive understanding of an empirical situation. Local integration means theorizing a phenomenon at the boundary of two or more theories so that the theories themselves are expanded in their local understanding of a specific phenomenon.

Activity Theory

The concept of activity in activity theory is transformative as it is "the specifically human form of activity, of interacting with the world in which man changes it and himself at the same time" (Giest & Lompscher, 2006, p. 27, our translation). According to Leontjew (1987), activity is driven by its *motive*, i.e., the object that initiates the activity on which it is built, in our research this is the activity of teaching/learning of negative numbers. An activity is embedded in the cultural-historical situation of society, its development changes the society as well as society changes the activity, and by "interacting with the world" (Giest & Lompscher, 2006, p. 27, our translation) learners change it and themselves at the same time. Activity (Figure 2) is made and renewed by *actions* directed towards *goals,* which are related to the motive, here for example to add positive and negative numbers. Each goal is pursued by a task, e.g., to manipulate one side of the equality 2 – 3 = 2 – 3 preserving the equality relation. This task could result in sub-goals with respective

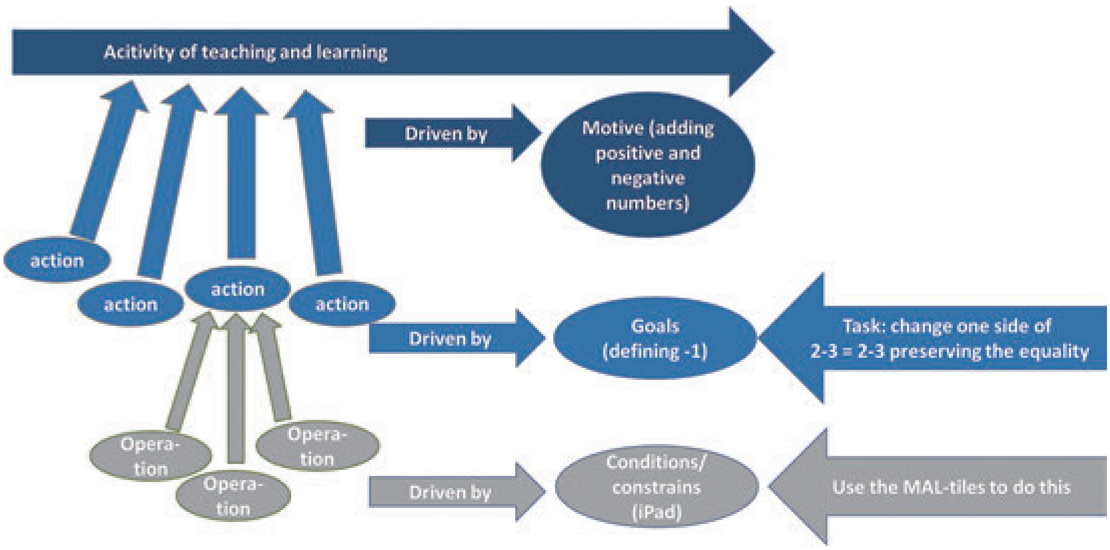

Figure 2: Activity system

sub-actions, e.g., to manipulate the numbers on the right side without changing its result (e.g., 2 – 3 = (2+1) – (3+1)).

The manner in which these actions are performed is determined by the *conditions* of the setting and the *constraints* the goals meet in the specific situation, e.g., by available artifacts (e.g., digital or non-digital tangibles), which allow specific *operations* to be performed but constrain others. For example, symbols used with paper and pencil differ from using a symbolic calculator or our MAL-system in which numbers are represented by virtual tiles, which can be placed, removed or re-arranged (Figure 3), as we will see. Activity can only be observed through its actions on the object that is the motive of the activity. Leontjew (1987) accentuates that activity is in constant transformation, it is (re)constituted by variations of actions developing over time. It even may change into an action, and when routinized it may be transformed into an operation (p. 109). While "[a]ctions becoming ever richer, outgrow the circle of activity that they realize, and enter into a contradiction with motives that engender them" (Leontyev, 2009, p. 175), then new motives may enter the scene, restructure actions and establish a new activity, for example solving algebraic equations including negative numbers after the latter is routinized.

Tools play an essential role in the transformative nature of an activity as they are cultural objects through which people act and interact with the world around them and so change it and themselves (see Leontjev, 2009). Tools mediate an activity and thus, its transformation. However, when new tools like the MAL-system are developed, it is not so clear how they contribute to these transformations as their use in culture is not yet established. This has yet to be explored.

In this paper, we focus on teaching/learning as an irreducible activity (see Shvarts & Abrahamson, 2019) with negative numbers shaping its motive. How students transform themselves and the teaching/learning, in which they interact with others and the tool, is mediated by the MAL-system, a digital artifact designed to support Multimodal Algebra Learning (MAL). For this setting, the MAL-system is elaborated as a digital tool for teaching/learning algebra. Here we address negative numbers (Figure 3) with a specific focus on the role of digital feedback. When the students use the MAL-system, they may transform the artifact into an instrument for a specific situation e.g., for calculating with negative numbers. To grasp this process, we adopt the Instrumental Approach (Artigue, 2002) as a theoretical frame that complements Activity Theory through conceptualizing the use of the tool. As it is rooted in activity theory (Vérillon & Rabardel, 1995) we expect

Figure 3: Teaching/learning setting (left figure), the MAL-system (right figure) with a task (1, text on top), the subtraction zone (2, frame in red), zero-pair (3, red & blue tile), unit-tile (4, blue tile), minus-one-tile (5, red tile), grouping (6, tiles surrounded by a yellow frame), symbolic expression (7) of the tiles equation 2 – 3 = 1 + (-1) – 1 (see Figure 5)

to be able to coordinate the two theories empirically towards a theoretical local integration.

The Instrumental Approach

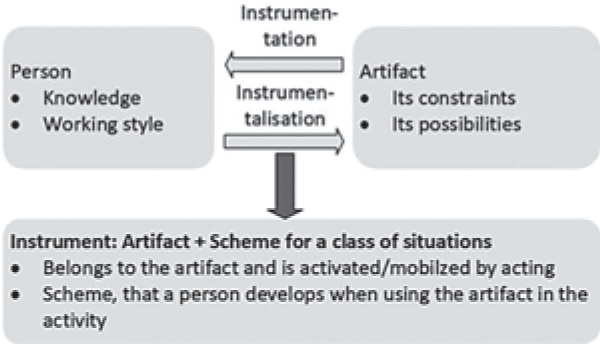

Figure 4: Instrumental genesis (© Springer Nature, permission received, see Abb. 2.12 in Bikner-Ahsbahs, 2022, p. 25; adapted from Fig. 3.1 in Trouche, 2020, p. 395)

In line with activity theory, we consider tools as mediating the dynamic of an activity in the way people use the artifact to achieve a goal. According to this view, instrumental genesis is the process of how a (digital) artifact becomes an individual instrument (Artigue, 2002), where an instrument is the hybrid collection of an artifact with individually and/or socially developed schemes of using the artifact in a specific situation (Trouche, 2020). Instrumental genesis is shaped by two interrelated dialectic processes (Figure 4); by *instrumentation* ("directed towards the subject") and by *instrumentalization* ("directed towards the artifact") (Trouche, 2020, p. 408). These are

> two intrinsically intertwined processes constituting each instrumental genesis, leading a subject to develop, from the artifact, an instrument for performing a particular task; the instrumentation process is the tracer of the artifact on the subject's activity, while the instrumentalization process is the tracer of the subject's activity on the task (Trouche, 2020, p. 409).

Rabardel (2002) emphasizes that "[t]he two processes jointly contribute to the emergence and evolution of instruments, even though, depending on the situations, one of them may be more developed, dominant or even the only one implemented." (p. 103) He further explains that artifact functions "are a characteristic property of the instrumental entity, and because in our perspective this entity is born of both subject and artifact, functions are also mixed in nature. They are rooted in both the artifact and scheme components of the instrument." (p. 104) There are two levels of the evolution of instrumentalization by the attribution of functions to an artifact enriching it with new extrinsic properties, that are acquired momentarily (local level) or durably (global level) (p. 106). This research only addresses the local level of instrumentalization. Figure 3 shows our teaching/learning setting in which the MAL-system is used as the artifact, with which students learn about negative number.

The MAL-system designed for learning about negative numbers

Activity Theory supported the design of tasks for the MAL-system (Reinschluessel et al., 2018), which provides virtual multi-touch tiles that can be arranged on a virtual "mat" to represent key structures of the arithmetic symbol system. In addition to the tiles, there are various features included in the system. An unlimited stock of tiles (Figure 5, 1) and a waste basket (2) on the left and the right side of the mat to allow two learners to work at the same time. The white area on the top (3) is for task descriptions. Pressing an arrow button to the top right (4) either moves on to the next task or restarts the current task. The pen-button on the top left (5) switches the interaction mode between either direct interaction with the tiles or 'painting' mode that allows, drawing or deleting a 'Subtraction Zone' (SZ, the red frame in Figure 5). The general area to place the tiles is an Addition Zone' (AZ).

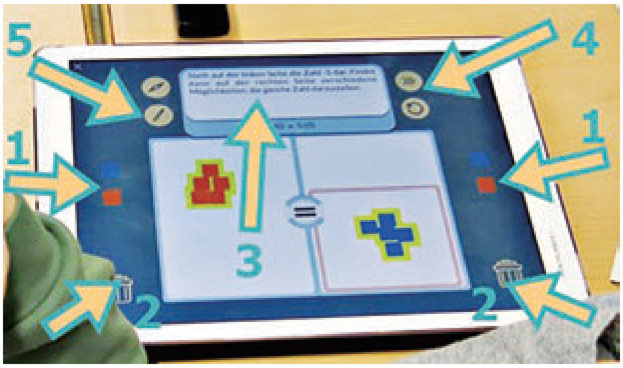

Figure 5: Functionalities for interacting with the MAL-system artifact.

The main representational part of the MAL-system is the mat with two sides (left and right), representing an equation, and an equal or unequal sign in the middle giving feedback on the correctness of the equation represented by the tile arrangements on both sides. Numbers are represented by square unit-tiles. Two different colors indicate the sign of the numbers (red for minus, blue for plus) and mathematical operations

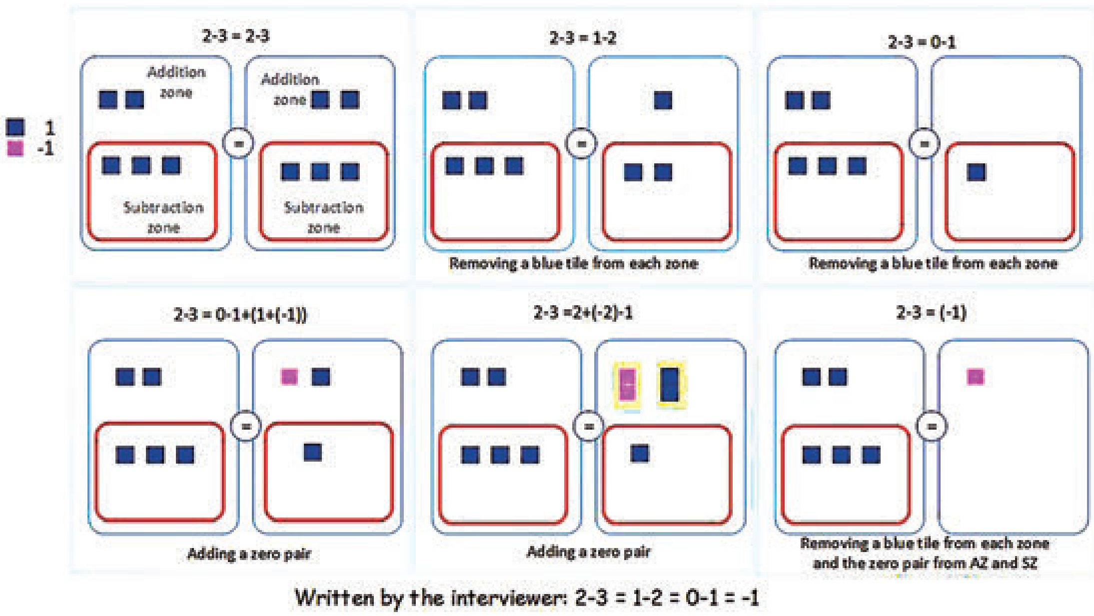

Figure 6: Key features of the MAL-system – defining (-1) as 0 – 1 by 2 – 3 = 0 – 1 = -1 + 1 – 1 = -1

with numbers are represented by practical actions with the tiles to achieve specific goals. The design starts from three fundamental actions: *placing*, *removing*, and *re-arranging* (Reid & Vallejo-Vargas, 2019), which students spontaneously conduct. Placing tiles in the 'Addition Zone' (AZ) (Figure 6) means adding them, removing means subtracting them, and re-arranging produces visual patterns without changing the results but expressed in the symbolic feedback e.g., if the tiles are pushed together in certain ways. A red "blob" drawn by a finger on the mat signals to subtract; it is called a 'Subtraction Zone' (SZ). A SZ shows what is to be subtracted from the AZ that contains the SZ. It is also possible to insert a SZ within a SZ. When tiles are pushed together they are grouped and this is shown by a yellow surround (Figures 5, 6), meaning that the tiles are considered to represent a single number by the system.

The *basic action rule* 'perform a calculation by acting on tiles on one side of the equation while keeping the equal sign' allows for legitimate action schemes e.g., removing two unit-tiles, one from the AZ and one from the SZ (Figure 6), or placing or removing zero-pairs made of a blue and a red tile. Through preserving the equality in this setting, the MAL-system emphasizes the relational notion of the equal sign rather than the operational notion.

Five kinds of digital feedback functions are implemented into the system (Figure 3). The equal or unequal sign produces "balance feedback" showing if the equation is correct or not. In the "symbolic feedback" the system automatically converts the tiles collections into a symbolic arithmetic expression above the mat. When tiles are grouped, this is shown by a yellow frame. If a zero-pair is grouped it vanishes from the mat and if moves are illegal the tiles wiggle or return to their original place.

A layered model and feedback

Learning with an artifact that models operations on negative numbers encompasses both learning to use it and letting it fade away from awareness. We assume that the activity of teaching/learning is layered (see Swidan et al., 2020) and conceptualize this layered process by focusing on four interrelated layers of acting and operating, as Leontjew's conceptualizing of an activity system considers (Figure 2). Layer 1 consists of learning to operate and handle the MAL-system as an artifact, Layer 2 is learning with the MAL-system e.g., how to handle equations based on operations with tiles. Layer 3 addresses learning to link MAL-expressions with their respective arithmetic expressions, and Layer 4 addresses the emancipation from the MAL-system. The third and fourth layers may capture transforming processes of the students as indicated by Leontjew. In Layer 3, the students' learning of negative numbers requires converting and expanding MAL-expressions to more general mathematical expressions by the use of the symbolic system of arithmetic. In Layer 4, the students emancipate themselves from the artifact in that the students' actions with negative numbers do not depend on the artifact anymore but rather are used for other activities. In the latter case, the students become aware of the instrument's limits although they also may use the tool on purpose if necessary. When the activity of learning negative numbers is enriched, handling negative numbers may be transformed from a motive for an activity to an action towards further learning goals, for example for learning to solve algebraic equations with negative numbers. As Figure 2 indicates, the four layers of handling the MAL-system are not separated but rather interrelated with Layer 2 building on Layer 1, and Layer 3 building on both. We conjecture that the layered model can be empirically coordinated with the process of instrumental genesis of the MAL-system when the students pursue goals provided by tasks or by the tutor.

Feedback is an essential part of our digital tool. It is defined as a *response that comes back to an eliciting action of an initial situation* (Bikner-Ahsbahs et al., 2020; Reid et al., 2022), either *by feedback of the MAL-system* or *in speech* (Figure 7). Using the MAL-system, digital feedback is always present as the MAL-system shows the result of the student's moves on the mat. It becomes relevant only if someone in the group, tutor or learner, reacts to it, bringing the feedback into the group's discussion e.g., by identifying a mistake. Only then it may initiate further reactions and contribute to learning. In this way the digital feedback becomes part of a feedback loop, which starts with an *initial situation* from which a feedback *eliciting action* emerges e.g., making a mistake on one side of the mat shown by an unequal sign. The reaction to feedback initiates *retroactive actions* directed to the initial situation, for example reflecting on a mistake and correcting it (Bikner-Ahsbahs et al., 2020). In the case of *tutor feedback,* the tutor orients his/her feedback to a goal to help the students solve a task, thus, tutor feedback is an action, provoking retroactive actions directly, without intermediate reactions.

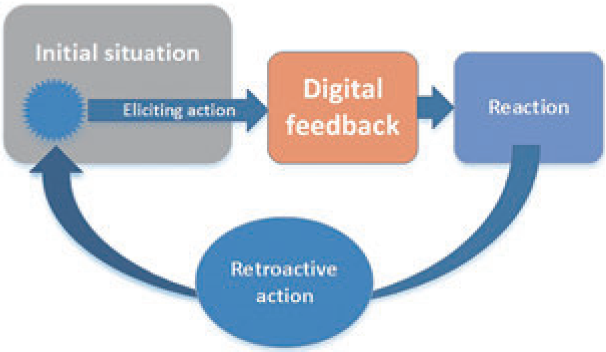

Figure 7: Feedback loop (© Springer Nature, permission received, see Abb. 2.14, Bikner-Ahsbahs, 2022, p. 30; see Bikner-Ahsbahs et al., 2020).

Fyfe (2016) distinguishes between four kinds of feedback, which in our case are not all digital. The MAL-system may offer *verification* feedback (e.g., through balance or symbolic feedback), (indirect) *correct answer feedback* (translating MAL-expressions into arithmetic expressions) and the tutor or the students may offer *explanatory feedback* and *try-again* feedback, and we add *giving-approval* feedback that appeared repeatedly. All these kinds of human feedback are directed to goals in the teaching/learning activity.

METHODOLOGICAL CONSIDERATIONS

We undertook an empirical study to investigate *the function of digital feedback in the support of the teaching/learning of negative numbers with the MAL-system.* The study consisted of four cases of experimental task-based interviews of a task sequence (Maher & Sigley, 2020) of about 110 minutes each. A tutor acting as a teacher conducted the interview that was shaped as a teaching/learning arrangement with two students (Figure 3, left). The student pairs worked collaboratively with one iPad, but distributed their common responsibility: each student was responsible for one side of the equation on mat while collaborating. The interview followed a sequence of 25 tasks (see appendix) that aimed at building knowledge for handling negative numbers. The first five tasks introduced the main functionalities of the MAL-system including the SZ through well-known calculation tasks with natural numbers. Tasks 6–9 introduced negative numbers based on zero-pairs and the SZ. Tasks 10–23 addressed adding and subtracting with negative numbers and reflecting and comparing on these more complex tasks. Tasks 24 and 25 were oral tasks to get an impression of the students' imagery of handling negative numbers.

When a tile is moved from the SZ to the AZ (respectively from AZ to SZ) the sign of the tile changes. This can be modeled by an exchange action (placing a zero-pair in the AZ (SZ) and then removing matching tiles from the AZ and the SZ) or an automatic color change shortcut (when the tile passes the boundary of the SZ). Such a color change happens also to a tile that changes from one side of the equation to the other (not introduced here). Two interviews were conducted with automatic color change and two without it to investigate if shortcuts foster or hinder handling negative numbers. In this paper, we will focus mainly on the two cases without color change. Color change is not the focus here but turned out to be an interesting feedback function in itself.

We conducted all the interviews in one Grade 5 class of a German gymnasium, a school in the two-tier school system with the higher achievement requirements. At this time of the school year, the students were familiar with natural numbers, they had met fractions but negative numbers were not systematically taught before the study, but rather a next step in the mathematics curriculum.

The interviews were video recorded, then paraphrased and images from the videos were extracted. These preliminary analyses showed that one case stood out as it provided specifically rich data because the students worked on all the tasks, addressed all layers with minimal guidance of the tutor and, thus, maximal activation of the students. This case served as a focus case, with which we compared the other cases. For the focus case, we transcribed the video recording until Task 19 verbatim and analyzed this case for each task episode. Tasks 20–25 were analyzed video based. The other cases were included by theoretical sampling i.e., if the analysis of the focus case yielded a conjecture, this was checked by analyzing respective episodes in the other cases for evidence.

In our analyses of the focus case, we investigated turn-by-turn the teaching/learning of negative numbers by reconstructing the reactions and retroactions to digital feedback, identifying the goals and related actions and operations. The findings of the analysis of the focus group are presented. Then they are used to coordinate the Activity Theory with a focus on the activity system with the Instrumental Approach towards local integration.

FINDINGS

Feedback loops of digital and tutor feedback

We first illustrate the loop of digital feedback for task 6.2 (*Place a blue and a red tile on the side where you are performing the calculation.*). Before, the students have experienced what is represented in Figure 6, upper row, where the symbolic feedback says: $2 - 3 = 0 - 1$.

1 Timo: (places a blue tile in the AZ, right side) keeps the same I think (it appears $\neq$, he adds a red tile,

then = appears, see Figure 3, right) yes!

2 Tutor: (translates symbolic feedback into words) mhm yeah, together a red and a blue tile equals zero

3 Timo: because a red is always negative, so -1

4 Simon: that is (refers to the blue tile) 1 and this then (refers to the red one) I think -1

Timo solves the task (#1), where he first conjectures that equality will be kept. His *eliciting actions* are placing the two tiles one by one. Digital feedback shows an *unequal sign* first and then an *equal sign*. His *reaction* is "yes!" directly taking this digital feedback into the group. A second reaction comes from the tutor who refers to the symbolic feedback. *Retroactive actions* follow in two steps by Timo and Simon, who together explain why a red and a blue tile add up to 0 (#3, #4). (Figure 3)

Directly after this scene we can observe a loop for tutor feedback in task 6.3 (*Could you now further calculate as you just have done?*). Timo misunderstands the goal and puts into the same AZ an additional zero-pair (the feedback = confirms). This serves as an *eliciting action* for a tutor feedback, pointing to the symbolic feedback area and saying that just one number is expected to appear on the right side. The tutor says, "*Now at the moment you have* 2 + (–2) – 1" (Figure 6, lower row), providing explanatory feedback to clarify the goal to yield one number.

5 Timo: Then I would push away (refers to the zero-pairs) the reds, and the other tiles, then we would have only –1, hence only one number.

6 Simon: (pushes the blue and red tiles together, symbolic feedback is 0 – 1) this would be two numbers

7 Tutor: Before you have always tried this tile (taps on the blue tile in the SZ, right side) could you do it here, too (puts one zero-pair back)

8 Timo: I have an idea (takes a blue tile from the AZ and the SZ away, = appears), this is still equal, my idea then is to do so (takes the zero-pair away) and then one can do this (makes the SZ disappear), then we only have one number (refers to the red tile)

Tutor feedback indicates what is expected, the goal (one number) initiating directly retroactive actions (#5, #6) by Timo and Simon. Timo does not take 0 as a number, but Simon corrects him, pointing to the two numbers in 0 – 1. The tutor takes #6 as an eliciting action (Figure 6) for the next tutor feedback (#7) and now re-directs the task goal. Timo uses a previously developed instrumented actions to solve the task by showing that taking a blue tile away from AZ as well as from SZ and then pushing a zero-pair together to delete it leaves one red tile only. After deleting the SZ tapping three times on it, (–1) is the only number left (#8). (Figure 6).

What we observe here is a feedback chain of two feedback loops (Figure 7) without reaction. Such a reaction is not necessary because a tutor feedback is already part of the social interaction in the group as tutor feedback is normally directed towards the goal of the task to ensure the task will be solved.

The role of feedback in teaching/learning

By using the MAL-system, Timo and Simon went through different learning experiences. Learning the MAL-system itself (Layer 1, see Figure 9) means acquiring *instrumented action schemes* to meet the affordances of the system (e.g., pushing together tiles to group them standing for adding them). The first phase of acting addressed handling well-known natural numbers, which helped the students to learn to use the MAL-system for performing operations, for example how to place five unit-tiles on the left and the right side and to decompose them to learn the grouping function. Here knowledge about natural numbers is used to explain what grouping means, that a yellow frame around the tiles indicates grouping and that the translation into the symbols serves as symbolic feedback. While the beginning of teaching/learning mainly addresses the instrumentation of acting with the features of the artifact, instrumentalization is locally present, too, through enriching the artifact with attributing to it external knowledge about natural numbers and their decomposition as additional properties.

Layer 1 is also addressed when going beyond natural numbers, for instance, when solving task 6.1 (*Set the expression 2 – 3 on both sides and try to find the result of the calculation. How far do you get with your approach so far?*), and task 6.2 (*Place a blue and a red tile on the addition zone on the side where you are performing the calculation*). The (balance and symbolic) digital feedback supports defining a zero-pair as 'the red and the blue tile making together 0'. Likewise, task 8.1 (*Place together a unit-tile and a minus-one-tile and see what happens*) allows the students to learn how the MAL-system "behaves" after these tiles are pushed together; as Simon reacts with *"they disappear",* he points to the action of 'making a zero-pair disappear', which is being instrumented by the feedback from the MAL-system.

The students also learned with the MAL-system (Layer 2) e.g., Timo explains how this "vanishing" happens: *"the result is also zero and therefore they are only additional tiles which are not needed"*. Timo's explanation is based on how "the result" is experienced on Layer 1 whereas previously it was associated with the idea of having only one number (the result) represented by symbolic feedback. His claim, *"the result is also zero"*, points at least to two different situations, first, it can be read that 0 = 0, and second, after the blue and the red tiles were pushed together on the left side the result is also 0. In this case, Timo's retroaction to the feedback plays the role of an explanation initiated by the tutor. Thus, the fact that the zero-pairs vanished, initiated learning on Layer 1 (when we push a zero-pair together both tiles vanish) and mediated the transition to Layer 2, which resulted in an explanation (because they make together zero they are not needed) and was further used as an idea for treating tiles arrangements shown next.

Making generalizations based on examples are also supported by the MAL-system's feedback and as such constitutes learning with the MAL-system. For example, when solving task 8.2 (*Place additional tiles on the left side only so that the equal sign is still displayed, where the other side shows zero*). Unexpectedly, Timo offers an idea saying "well, simply place zero-pairs" while Simon places zero-pairs on the left side of the mat as eliciting actions. As the equal sign is kept, Timo explains *"it always works with as many zero-pairs as we want",* and the tutor reinforces him (*"when you have the same number of minus-one-tiles and one-tiles then the result is always zero, right?"*), which serves as a *giving-approval* tutor feedback. So balance feedback mediated teaching/learning on Layer 2. In addition, the students' expressing the rule "*it always works with as many zero-pairs as we want"* shows that they have developed an instrument for deleting zero-pairs to solve MAL-tasks. In this instrumental genesis, the following dialectic of instrumentation and instrumentalization is at play: the instrumented action of pushing pairs of tiles together enriched by knowledge about zero-pairs and simultaneously it is Timo's independent choice to place zero-pairs on the mat, check an implicit conjecture and finally infer a rule, thus, expressing an enriched understanding of the task situation by instrumentalization.

Learning to link MAL-expressions and arithmetic expressions (Layer 3) is also supported by the MAL-system's digital feedback. For instance, Timo solves Task 7 (*Place a minus-one-tile on the left and no tiles on the right. Represent a math task by placing tiles on the right side that has the result –1. You may start with red tiles, but in the end you should only see blue tiles on the right side*) by representing 1 – 2 with tiles, placing a unit-tile in the AZ and two unit-tiles in the SZ on the right side. When doing so, balance feedback shows that this is correct and Timo elaborates symbolically saying *"because one minus two is exactly the same as zero minus one*". Additionally, Simon suggests that it is always possible to add two blue tiles, placing one on the AZ and one on the SZ to keep the equal sign. Thus, Timo and Simon learned to consider the number –1 as the result of a mathematical operation e.g., 1 – 2, as well as a set of equivalent subtractions (–1 = 1 – 2 = 2 – 3 = 0 – 1 = …). This generalization of *treatments* of –1 represented in the arithmetic symbol system (see Duval, 2008) is a consequence of the learning with the MAL-system and its feedback on Layer 2 that mediates the transition to Layer 3 in Timo's explanation.

Figure 8: Making sense of the negative number -3.

In Task 8.3 (*Place minus-one-tiles together. How does this affect the symbolic representation above?*), they place three blue and three red tiles, representing 0, on the left and none on the right side. The symbolic feedback shows 3 + (–1) + (–1) + (–1) and Simon reads it loudly (Figure 8). This symbolic feedback makes Timo act on Layer 3 by identifying "*So there appears a double bracket because otherwise it would calculate plus minus three and that would not make sense and therefore there appears the bracket because it is another calculation"* (Figure 8). Simon asks "*will they disappear if they all are placed together?*" His conjecture in this question comes from the perception of how single zero-pairs vanished, an instrumented action learned before, which he now generalizes. Simon tried it out but nothing happened.

As the MAL-system is not programmed to fulfill this conjecture, the expected feedback is not shown resulting in a feedback by non-feedback, which points to Layer 1. However, pushing single zero-pairs together is successful. They disappear as Simon and Timo had experienced earlier. Simon's expectation indicates a step towards emancipating from the MAL-system (Layer 4) considering its limits and overcoming them by another way of acting. This situation shows that, as the students proceed in their process of instrumental genesis their flexibility in working at the three layers increases, interconnecting the layers, continuously supported by feedback.

Reflective summary

Taken together, digital feedback mediates teaching/learning between the four layers (Figure 9). We have shown shifts from Layer 1 to Layer 2 and from Layer 2 to Layer 3, but also to Layer 4. However, it may

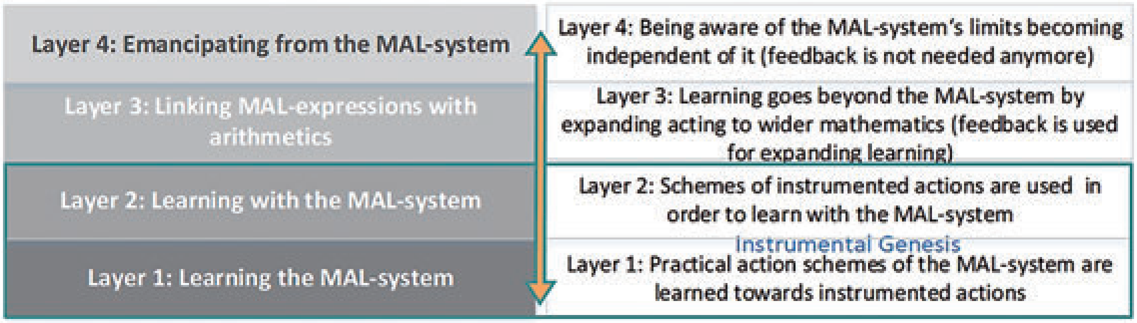

Figure 9: A layered model of teaching/learning mediated by the MAL-system and its feedback (double arrow), to be coordinated with the process of its instrumental genesis (green rectangle)

happen that symbolic feedback or symbolic tasks initiate translating the task into a MAL-expression moving tiles on the mat (shift from Layer 3 to Layer 2) and then coming back to the symbolic expression (Layer 3), thus mediating acting forth and back between Layer 3 and Layer 2. Balance feedback may keep the students acting on Layer 2, for example when they try to solve a problem with the tiles. An example is Task 9 (*Represent the number –5 on the left side. Then, on the right side, find different possibilities for representing the same number*.) In this task, the students discover the exchange-strategy i.e., Timo places a red tile on the AZ and removes a blue tile from the SZ. A bit later, Simon varies this strategy, he removes a blue tile of the SZ first and then places a red tile on the AZ instead and finally Simon removes a red tile from the AZ and places a blue tile on the SZ. In each case, balance feedback confirms the correctness of these exchange strategies. The students provide several explanations addressing Layer 2 by referring to their experience with tiles, such as Timo: "*Yes that would work too because it results in zero, and then we can exchange them.*" However, finally Timo converts tile expressions into symbolic language (see Duval, 2008) to explain: "*That is because a blue one in the subtraction zone is also minus 1 and the red one is already minus 1 and that is why one can simply exchange them because they are both the same.*" Here, first the balance feedback mediates the students' actions on Layer 2, but then an explanation is produced with symbolic terms linking Layer 2 and Layer 3 *without support by symbolic feedback*. Although the students cannot manipulate symbols as these only appear as symbolic feedback, they expand their symbolic knowledge.

Working through the 25 tasks (see appendix) the students become more and more fluent in changing and bridging layers, translating between them, and using them in entwined ways. Thus, their actions become richer and richer, but they also struggled when subtracting a negative number, for example in the task 2 – (–5). The students in the end choose tile expressions and symbolic expressions depending on their own needs, adding verbal explanations when necessary. Symbolic language was more often used for actions of conjecturing and explaining whereas operations with tiles of the MAL-system were often used for actions of checking, showing, testing and exploring. Within this single session, we only found one indication for emancipating from the MAL-system by going beyond its limits. However, the fact that the students became able to decide, which representation system they wanted to use as an instrument for a specific situation shows that the students have passed through a transformation process as described by Leontjew (1987).

THE TEACHING/LEARNING ACTIVITY PASSING THROUGH THE LAYERS

For teaching/learning negative numbers the MAL-system is used as an artifact with digital feedback. Task 6 (*Set the expression 2 – 3 on both sides and try to find the result of the calculation*) addresses acting with the artifact practically on Layer 1. For this, students must draw a blob to get a SZ placing tiles in the AZ and SZ (Layer 1). *How far do you get with your approach so far?* is the subsequent task aiming at reflecting and explaining, hence, learning with the MAL-system in Layer 2 about the current limit of calculating 0 – 1, since the result is not just one number as in other cases where the difference is positive. This results in defining a new number, (–1), by 0 – 1, which means practically placing a red tile in the AZ (Layer 1). (–1) then substitutes 0 – 1, practically and symbolically (Layers 2, 3). Hence, in this process learning on Layer 1 and Layer 2 are intermingled by instrumental genesis. Specifically, when the students proceed in working with negative numbers, more and more features are attributed to the MAL-system on Layer 1, addressing learning with the system on Layer 2 that even goes over into Layer 3. This makes clear that Layer 1 and Layer 2 are neither separated nor can they be regarded as ordered stages during the teaching/learning. Through the dialectic duality of instrumentation and instrumentalization they become entwined impacting the students' further personal development.

Task 7 (*Place a minus-one-tile on the left and no tiles on the right. Represent a math task by placing tiles on the right side that has the result –1. You may start with red tiles, but in the end you should only see blue tiles on the right side*) addresses a goal on Layer 3, to substitute a whole class of tile pairs, resulting in 2 – 3 = 1 – 2 = 0 – 1 …, by (–1) or practically by placing a red tile, but also by other suitable tile pairs (Layer 1). Thus, *operations* on Layer 1 realize actions on Layer 2, on which epistemic insight on Layer 3 is built-up and expressed by generalizing, supported by the symbolic feedback and transforming the learners' abilities towards the use of symbolic expressions.

Feedback assists in bridging between and within the layers. When the students proceed to act in an additional layer, the activity of teaching/learning is transformed in that it changes its nature. If Layer 2 adds to Layer 1, then the students use the MAL-system to explore a situation in their own way. However, this already happens on Layer 1 when the students begin to make sense of the actions they conduct and use the tiles in new ways not expected by the researchers. Layer 3 is already at stake in the symbolic feedback, when tile representations are converted into symbolic expressions by the system. In this way, the students may enlarge their capacity to act by using the symbolic language to explain, whereas on Layer 2 they instead can show how a goal can be achieved or explain – based on tiles – why the goal is achieved or not achieved. When Layer 4 is added, the MAL-system does not constrain acting any longer; the MAL-system can even be left aside.

LINKING TWO THEORIES TOWARDS LOCAL INTEGRATION

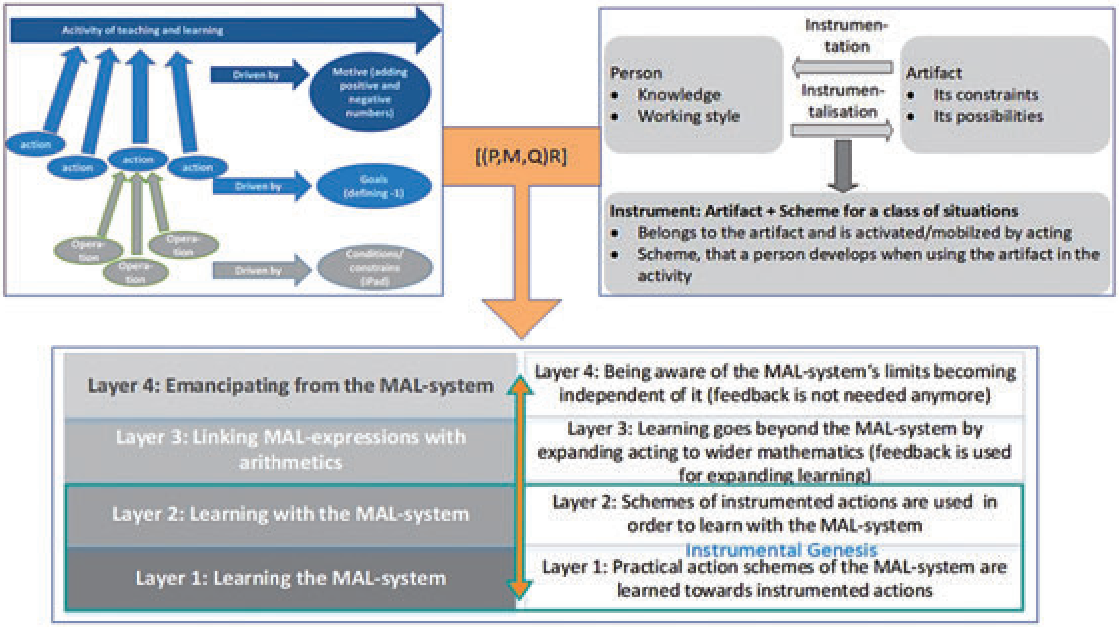

Figure 10: Local integration of activity theory and instrumental approach into a phenomenon of entwining the layers and thus transforming the activity (see Bikner-Ahsbahs, 2022, p. 25)

Activity Theory was empirically coordinated with the Instrumental Approach to explore the support function of digital feedback in the activity of teaching/learning negative numbers with the MAL-system (Figure 10). We assumed that this activity is layered and we elaborated theoretically a four-layer model related to the structure of the activity system. Our findings brought about the new phenomenon that *these four layers became continuously more entwined by digital as well as tutor feedback and the instrumental genesis of the MAL-system in the course of the teaching/learning process while the activity develops and through that the students develop.*

We could empirically confirm four layers of teaching/learning with the MAL-system (Layer 1: Learning the MAL-system, Layer 2: Learning with the MAL-system, Layer 3: Linking MAL-expression with arithmetic-algebraic symbols, Layer 4: Emancipating from the MAL-system) and thus confirming our assumption of a four-layer model. We showed that digital feedback and tutor feedback foster the accumulation of students' learning passing through these layers and thereby becoming more flexible while finally indicating emancipation from the MAL-system.

The MAL-system provides the condition for operations that shape the actions with negative numbers. These conditions are given but the students had to learn how to handle the MAL-system as well, while transforming this artifact into an instrument in a process of instrumental genesis. Analyzing the use of the artifact from the perspective of the Instrumental Approach made us identify instrumentation processes leading to various (sub-) instruments. These included instruments for:

- keeping the equality by *adding zero-pairs*, by simultaneously *taking the same tiles out of the AZ and SZ on the same side* or by *exchanging a blue tile by a red tile* (respectively a red tile by a blue tile) e.g., when the tile is shifted from the AZ to the SZ or vice versa,
- *transforming the action of taking a tile away into representing this by inserting the tile into a SZ and vice versa*
- expressing *–1 by n blue and (n+1) red tiles (while n is a natural number) or placing n blue tiles in the AZ and (n+1) blue tiles in the SZ.*

These processes involved Layers 1 and 2, but also Layer 3 when symbolic expressions were used to explain these processes.

After coordinating the two theories, Activity Theory and the Instrumental Approach, we will now justify that they can be considered being locally integrated by the new phenomenon in that we base our argumentation on Radford's theory concept [(P, M, Q), R] (Figure 11).

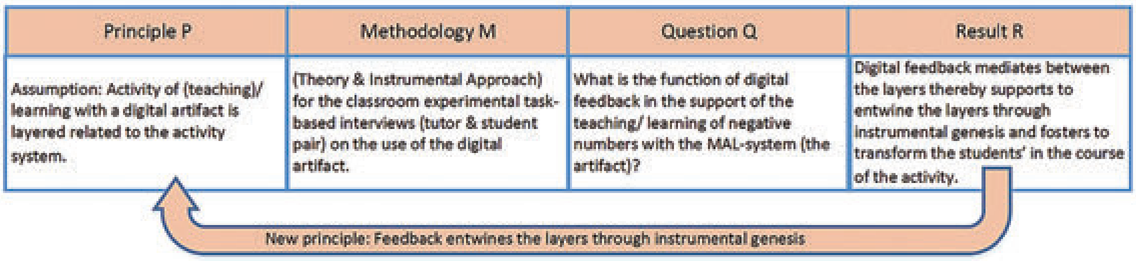

Figure 11: [(P, M, Q), R] for local integration

We have added the principle P of assuming the four-layer model for the activity of teaching/learning. We have elaborated that feedback mediates between the layers and that the instrumental genesis of the MAL-system entwines these layers. This is empirically shown by the methodology M of the dual-theory approach for experimental task-based interviews, conducted in the students' school. The goals of the tasks were exactly those that could come next according to the curriculum. Thereby, we could study an experimental situation of using the artifact in the natural setting for which the MAL-system was developed. The paradigmatic research question Q we have explored was: *What is the function of digital feedback in the support of the activity of teaching/learning of negative numbers with the MAL-system?* This question links the activity with the digital artifact that was not provided by the natural cultural class environment but that is a new tool. Our main results are that the digital feedback of this artifact mediates between the layers, thereby supporting the process of instrumental genesis that – as shown empirically – entwines the layers and assists the students' passing through the four layers in their transformation to emancipate from the MAL-system.

Figure 10 summarizes the local integration we have achieved by adding a principle P, a methodology M and a research question Q (see Figure 11). With the results a new linking principle on the role of feedback becomes apparent, that feedback entwines the layers by instrumental genesis. Thus, the results R rebound on the principles by adding to Activity Theory a better understanding of Leontjew's observation that activity is in constant transformations becoming "richer and richer" (1987, p. 200, our translation) thus pointing to the accumulative nature of teaching/learning.

DISCUSSION

As previously investigated in another setting (Bikner-Ahsbahs et al., 2020) also in this study we found evidence that digital feedback is embedded in a feedback loop shaped by the tutor and the students. Feedback does not speak by itself, the effect of feedback is determined by the way students retroact to the feedback and this may happen in a pragmatic way or in an epistemic way (Artigue, 2002; Reid et al., 2022). Feedback is considered to play the pragmatic role if the student reacts by (re)performing a practical action, for example drawing the blob on Layer 1 to subtract or tipping three times with two fingers on the SZ to make it disappear. It is epistemic when students feel invited to express or approach understanding e.g., with an explanation or a generalization expressed with symbols on Layer 3. The MAL-system's feedback normally does not show any preferred function. The function arises through the students' reactions and retroactive actions, when they just act practically, reflect about their actions or try to explain the feedback. Tutor feedback however is an action, directed to a teaching goal. It may show a pragmatic or epistemic function related to the goal that indicates what is expected from the students.

Digital feedback functions, like the wiggling of tiles in cases of illegal moves, are artifacts themselves. They need to be instrumented by the students. It is interesting to note that the students also used digital feedback for their own purposes. For example, some students used the balance feedback to explore legal moves on both sides of the mat as for an algebraic equation. Others provoked feedback to test conjectures and pushed three red and three blue tiles together to see if they vanish. As this did not happen, they learned to take this missing feedback as a new kind of feedback for what the system is not programmed to do. These and other explorations the students undertook are clear indicators for instrumentalization processes.

Applying Leontjew's (1987) view on the transformative nature of activity to the teaching/learning with the MAL-system the students' personal transformation happened within this activity through feedback that mediates between layers of learning. Instrumental genesis of the MAL-system is the basic process nested in the more comprehensive learning process in which tiles of the artifact are transformed into arithmetic symbols to be used as instruments in future. This transformation in learning advances the students' human ability to act. More generally, Leontjev (2009) explains:

> For man a tool is not only a certain object with some external shape and certain mechanical properties; he sees it as an object embodying socially developed ways of acting with it … An adequate relation between man and tool is therefore primarily expressed in his appropriating … the operations fixed in it, by developing his own human abilities to be developed and used as instruments. (p. 266).

CONCLUDING REMARKS

As the Instrumental Approach is rooted in Activity Theory, our local integration is not a surprise. What is new, however, is the fact that the data provide evidence that the transformations of the activity can be considered as layered and that the digital feedback of the artifact impacts on this transformation by mediating between the layers in the teaching/learning where instrumental genesis plays a key role. The networking of the two theories allowed us to observe the process of intertwining of the layers in the instrumental genesis and how the students developed in this process. A similar result – but not based on a layered model – was attained in our study on dividing algebraic equations with the MAL-system when students transcended integers (Janßen et al., 2020, 5.1.3).

Leontjew (1987) has emphasized that transformations may entail actions that are transformed into operations and activities that are transformed into actions. We could not identify these kinds of transformations. The reason is probably that the transformations we have observed within an activity consist of incremental transformative steps whereas we can expect that a transformation going beyond an activity consists of a stronger shift of view, for example, the shift that is necessary from arithmetic equations to algebra equations. Such transformations go beyond just one session, they must be considered as long-term developments.

A glance into the layered model draws our attention to a third theory: Duval's cognitive theory of representations (2008) warrants the relevance of transitions, within the same register (treatments) as within the MAL-system and transitions between different registers (conversion) here between the registers of the arithmetic symbol system and the MAL-system. Our results indicate that students use these registers for different things. For example they *explore* how to generalize with the MAL-system within Layer 2, but they *explain* perceiving generalizations with tiles using arithmetic symbols in Layer 3. The problem we are faced with in the networking of theories when we consider a cognitive approach is its incompatibility with the cultural-historical approach. Due to the contrasting principles and the lack of appropriate data, Duval's cognitive theory of representations could not be integrated into our theorizing process. But the ReMath project has shown that networking of it with an activity theory perspective could be conducted in the design studies (Artigue & Mariotti, 2014).

As transitions between registers were essential in our study, our layered model calls for further research on transitions between registers and the role of digital feedback there, in the MAL-system and with other artifacts. Our results raise the question whether such transitions may also lead to an emancipation from the artifacts by becoming more flexible in its use and how it is possible to simultaneously improve proficiency in the use of symbolic systems when going beyond the artifact.

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Note This article is an extended version of the paper "Role of feedback when learning with an artifact" presented by Angelika Bikner-Ahsbahs, Estela Vallejo-Vargas and Steffen Rohde in the Topic Study Group 57on the Diversity of Theories of the 14th International Congress on Mathematics Instruction in Shanghai (2020, postponed to 2021). It encompasses more information about the digital resource (the MAL-learning system) and more in-depth elaboration from the perspective of the networking of theories.

REFERENCES

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245–274. https://doi.org/10.1023/A:1022103903080

Artigue, M., & Mariotti, M.A. (2014). Networking theoretical frames: the ReMath enterprise. *Educational Studies in Mathematics, 85*, 329–355. https://doi.org/10.1007/s10649-013-9522-2

Bikner-Ahsbahs. (2016). Networking of theories in the tradition of TME. In Bikner-Ahsbahs, A. et al. (Eds.), *Theories in and of mathematics education. Theory strands in German-speaking countries* (pp. 33-42). Springer. https://doi.org/10.1007/978-3-319-42589-4_2

Bikner-Ahsbahs, A. (2022). Mathematiklehren und -lernen digital. Theorien, Modelle, Konzepte. [Digital teaching and learning mathematics. Theories, models, concepts.] Florian Schacht, Guido Pinkernell, Frank Reinhold and Daniel Walter (Eds.), *Digitales Lehren und Lernen von Mathematik in der Schule [Digital teaching and learning mathematics in school.]* (pp. 7–36) Springer.

Bikner-Ahsbahs, A., & Prediger, S. (2014, Eds.), Networking Theories Group (2014). *Networking of theories as a research practice in mathematics education.* Springer. DOI 10.1007/978-3-319-05389-9

Bikner-Ahsbahs, A., Rohde, S., & Weissbach, A. (2020). Digitales Feedback: ein mächtiger ‚Akteur' im Lernprozess? [Digital feedback: a powerful 'actor' in the learning process?] In H.-S. Siller et al. (Eds.)*, Beiträge zum Mathematikunterricht* (pp. 133-136). Münster WTM.

Bokhove, C., & Drijvers, P. (2012). Effects of feedback in an online algebra intervention. *Technology, Knowledge and Learning*, *17*(1/2) 43–59. https://doi.org/10.1007/s10758-012-9191-8

Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In Luis Radford, Gert Schubring, & Falk Seeger (Eds.), *Semiotics in mathematics education. Epistemology, history, classroom and culture* (pp. 39–61). Sense Publishers.

Evans, C. (2013). Making sense of assessment feedback in higher education. *Review of Educational Research, 83*(1), 70–120. https://doi.org/10.3102/0034654312474350

Fyfe, E. (2016). The benefits of feedback on computer-based algebra homework. In Wood, M, Turner, E., Civil, M., & Eli, J. (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the Int. Group for the Psychology of Mathematics Education* (pp. 581–588). Tucson, AZ: The University of Arizona.

Giest, H., & Lompscher, J. (2006). *Lerntätigkeit – Lernen aus kultur-historischer Perspektive. Ein Beitrag zur Entwicklung einer neuen Lernkultur im Unterricht.* [Learning activity – Learning from a cultural-historical perspective. A contribution to the development of a new learning culture in the classroom]. International Cultural-Historical Human Sciences (ICHS), vol. 15. Lehmans Media.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research*, *77*, 81–112. https://doi.org/10.3102/003465430298487

Hillel, J., Kieran, C., & Gurtner, J.-L. (1989). Solving structured geometric tasks on the computer: the role of feedback in generating strategies. *Educational Studies in Mathematics, 20*, 1–39.

Janßen, T., Vallejo-Vargas; E., Bikner-Ahsbahs, A., & Reid, D. A. (2020). Design and investigation of a touch gesture for dividing in a virtual manipulative model for equation-solving. *Digital Experiences in Mathematics Education, 6,* 166–190. doi.org/10.1007/s40751-020-00070-8

Van der Kleij, F., Feskens, R., & Eggen, T. (2015). Effects of feedback in a computer-based learning environment on students' learning outcomes: A meta-analysis. *Review of Educational Research Month, 20*(10), 1–37. https://doi.org/10.3102/0034654314564881

Leontjev, A. N. (2009). *Development of mind. Selected Works of Aleksei Nikolaevich Leontyev.* Erythrós Press.

Leontjew, A. N. (1987, 3rd ed.). *Tätigkeit. Bewusstsein. Persönlichkeit.* Volk und Wissen.

Leontyev, A. N. (2009). Activity and consciousness. Marxists Internet Archive https://www.marxists.org/archive/leontev/works/1978/index.htm

Maher C.A., Sigley R. (2020, 2nd ed.). Task-based interviews in mathematics education. In Lerman S. (Ed.) *Encyclopedia of mathematics education.* Springer (pp. 821–824). https://doi.org/10.1007/978-94-007-4978-8_147

Mason, J., Waywood, A. (1996). The role of theory in mathematics education and research. In: Bishop, A.J., Clements, K., Keitel, C., Kilpatrick, J., Laborde, C. (Eds.), *International handbook of mathematics education. Kluwer International Handbooks of Education*, *4* (pp. 1055–1089). Springer. https://doi.org/10.1007/978-94-009-1465-0_29

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM-International Journal on Mathematics Education, 40*(2), 165-178. https://doi.org/10.1007/s11858-008-0086-z

Rabardel, P. (2002). *People and technology: a cognitive approach to contemporary instruments.* Université paris 8. https://hal.archives-ouvertes.fr/hal-01020705

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM - The International Journal on Mathematics Education, 40*(2), 317–327. https://doi.org/10.1007/s11858-008-0090-3

Radford, L. (2012). *On the growth and transformation on mathematics education, Paper presented at the International Colloquium The Didactics of Mathematics: Approaches and Issues.* A Homage to Michèle Artigue. Université de Paris VII. May 31 to June 1, 2012. http://www.luisradford.ca/pub/Connecting theories - Paris Oral Presentation rev2014.pdf

Reinschluessel, A., Alexandrovsky, D., Döring, T., Kraft, A., Braukmüller, M., Janßen, T., Reid, D., Vallejo, E., Bikner-Ahsbahs, A., & Malaka, R. (2018). Multimodal algebra learning: From math manipulatives to tangible user interfaces. i-*com. Journal of Interactive Media, 17*(3), 201–209. (Au-thors: 10). https://doi.org/10.1515/icom-2018-0027

Reid, D., Bikner-Ahsbahs, A., Janßen, T., & Vallejo-Vargas, E. (2022). In U.T. Jankvist, R. Elicer, A. Clark-Wilson, H.-G. Weigand, & M. Thomsen (Eds.), *Proceedings of the 15th International Conference on Technology in Mathematics Teaching (ICTMT 15)* (pp. 105–114). Aarhus University. https://doi.org/10.7146/aul.452.

Reid, D.A., & Vallejo Vargas, E.A. (2019). Evidence and argument in a proof based teaching theory. *ZDM-Mathematics Education, 51*, 807–823 (2019). https://doi.org/10.1007/s11858-019-01027-x

Shvarts, A., & Abrahamson, D. (2019). Dual-eye-tracking Vygotsky: A microgenetic account of a teaching/ learning collaboration in an embodied-interaction technological tutorial for mathematics. *Learning Culture and Social Interaction, 22,* 100316. https://doi.org/10.1016/j.lcsi.2019.05.003

Swidan, O., Sabena, C., & Arzarello, F. (2020). Disclosure of mathematical relationships with a digital tool: a three layer-model of meaning. *Educational Studies in Mathematics, 103*, 83–101. https://doi.org/10.1007/s10649-019-09926-2

Trouche, L. (2020). Instrumentalization in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 392-403). Springer. https://doi.org/10.1007/978-3-030-15789-0_100013

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

APPENDIX

Table: The list of tasks

Task 1	1.1	Move individual tiles to one side of the mat and observe what appears at the top.
	1.2	What does each tile stand for? What happens when you push tiles together?
Task 2	2.1.	Represent the number 5 on the left side of the mat.
	2.2.	So far, you see an unequal sign in the middle. Move tiles from the stock to the right side and observe what happens. What do you have to do to make the unequal sign an equal sign?
	2.3.	Find as many different representations as possible for the number 5 on the right.
Task 3	3.1.	So far, everything on the mat has been added up. If we want to subtract, we can draw a circle with our fingers.
	3.2.	A subtraction zone is created. Everything that is placed here is subtracted from what is outside. Try it!
Task 4	4.1.	Set the expression 3 – 2 on both sides.
	4.2.	Make a guess and justify it: What will happen if you remove a tile from each zone from one side of the mat?
	4.3.	Try it: Remove a tile from each of the two zones on one of the two sides of the mat.
	4.4.	Use the procedure to get the result of the task.
Task 5	Determine in the same way the result of the arithmetic task 5 – 3. Explain your approach.	
Task 6	6.1.	Set the expression 2 – 3 on both sides and try to find the result of the calculation. How far do you get with your approach?
	6.2.	Place a blue and a red tile on the addition zone on the side where you are performing the calculation. When added together, they give 0, so an equal sign is still displayed.
	6.3.	Find out how the red tile is represented in the symbolic equation.
	6.4.	Now you can complete the task.
Task 7	7.1.	Place a minus-one tile on the left, no tiles on the right.
	7.2.	Represent a math task by placing tiles on the right side that has the result -1. You may start with red tiles, but in the end you should only see blue tiles on the right side.
Task 8	8.1.	Place a unit-tile and a minus-one tile together and see what happens.
	8.2.	Place additional tiles on the left side only so that the equal sign is still displayed.
	8.3.	Place minus-one tiles together. How does this affect the symbolic representation above?
Task 9	Represent the number -5 on the left side. Then, on the right side, find different possibilities for representing the same number.	
Task 10	Calculate the result of 2 + 5 with the tiles.	
Task 11	Calculate the result of –2 + (–5) with the tiles	
Task 12	Calculate the result of 3 + (–2) with the tiles	
Task 13	Calculate the result of –2 + 3 with the tiles	
Task 14	Calculate the result of 3 + (–5) with the tiles	
Task 15	Calculate the result of –5 + 3 with the tiles	
Task 16	Calculate the result of 3 – 2 with the tiles	
Task 17	Calculate the result of 3 – 5 with the tiles	
Task 18	Calculate the result of –2 – 5 with the tiles	
Task 19	Calculate the result of 2 – (–5) with the tiles	

Task 20 Link all tasks with the same solution

Task 21 How would you calculate 3 – (2 – 1) without tiles?

Task 22 22.1. Represent the solution of 3 – (2 – 1) with tiles and calculate the result.

22.2. Carlos believes that you can just calculate 3 – 2 – 1 = 0. Explain by comparing with your solution why this is wrong.

Task 23 Four students have solved the task 2 – (–3 – 1) in the subsequent way:

a. Alex: 2 – 3 – 1 = –2
b. Kim: 2 + 3 – 1 = 4
c. Karina: 2 – (–3 – 1) = 2 – (–4) = 2 + 4 = 6
d. John: 2– (–4) = 2 – 4 = –2

23.1. Explain with tiles the correct solution path.

23.2. Explain in each step what you have done.

Task 24 24.1. What does –5 mean to you?

24.2. How would you explain the number –5 to your classmates?

24.3. How would you explain –5 to a classmate, who only knows about positive numbers (such as the number 5)?

Task 25 25.1. Do you have an idea of what –4 – 2 means and how you can calculate it?

25.2. Do you have an idea of what –4 + (–2) means and how you can calculate it?

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