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AIMS AND SCOPE

Hiroshima Journal of Mathematics Education (HJME) is the official English-language journal of the Japan Academic Society of Mathematics Education (JASME), and is devoted to research that seeks to improve the mathematics education. It is an annual peer-reviewed and open-access international journal that publishes original papers written in English. HJME was founded by the Department of Mathematics Education at Hiroshima University. Articles have been published by JASME since Volume 12 was issued in 2019.

The journal is dedicated to the dissemination of research findings on mathematics education at all levels (e.g., from preschool to university, professional development, lifelong education) on a variety of issues related to the teaching and learning of mathematics (e.g., students' understanding, classroom teaching, curricula, policy, teacher education). It is open to any type of research (e.g., theoretical, empirical, methodological, qualitative/quantitative). Papers dealing with Japanese mathematics education issues or issues of special interest to the Japanese mathematics education community are particularly welcome.

Paper submissions are welcome from contributors from different sectors (e.g., researchers, educators, and practitioners) who are interested in contributing to the development of mathematics education and its research. HJME is expected to promote reflection on mathematics education and communication among international researchers, educators, and practitioners in mathematics education. Papers are published in either of two HJME sections: original paper and lecture notes. The original paper reflects the main topics of the journal. All papers submitted to this section are peer-reviewed, with an emphasis on papers of an excellent level. Lecture note includes records or materials of scientific events (e.g., invited talks, symposia) organized by JASME

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Special Issue (2): Mathematics for Non-Specialist/Mathematics as a Service Subject at Tertiary Level, Contributions from the Topic Study Group 45 of ICME 14

Special Issue (2):

Mathematics for Non-Specialist/Mathematics as a Service Subject at Tertiary Level, Contributions from the Topic Study Group 45 of ICME 14

Guest editorial

CONTRIBUTIONS FROM THE TOPIC STUDY GROUP ON MATHEMATICS FOR NON-SPECIALIST/MATHEMATICS AS A SERVICE SUBJECT AT TERTIARY LEVEL AT ICME-14 IN SHANGHAI

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At the tertiary level, mathematical education is not only present in mathematical study courses but can also be found as part of general education or in form of special service classes tailored to the needs of an application study course like economics or engineering. It is quite obvious that the goals and forms of mathematical education depend very much on the audience and hence on the study course within which it takes place. This was recognized by the International Commission on Mathematical Instruction (ICMI) more than thirty years ago when the first ICMI study on mathematics as a service subject was launched resulting in two volumes (Howson et al., 1988; Clements et al., 1988). The studies dealt with the questions "why", "what" and "how" addressing the goals, contents and forms of mathematical education. Since then, this topic played a minor role at ICME conferences which are organised by ICMI every four years to summarize and discuss the state of the art in many areas of mathematical education (see (Alpers 2020) for more information on the historical development). At ICME 13 in Hamburg 2016, some aspects of service mathematics were discussed in a more general group on mathematics at the tertiary level (Biza et al., 2016). The reports by Biza et al. (2016) and Alpers (2020) show that mathematics education at the tertiary level has been of growing interest in educational research for the last two decades but the mathematical education of non-specialists is still a widely under-researched area (Artigue 2016). Therefore, for ICME 14 in Shanghai ICMI installed a dedicated topic study group on "Mathematics for non-specialist/Mathematics as a service subject at tertiary level".

At ICME 14 which took place only in 2021 because of the Covid-19 pandemic, there were nine paper contributions and three posters within this topic study group. Although the majority of contributions dealt with mathematics education within engineering study courses where mathematics undoubtedly plays a very important role, there were also contributions on suitable forms of mathematical education in less mathematics-based study courses or within general university education. The four papers which appear in this special issue of the Hiroshima Journal of Mathematics Education represent the spread of contributions in the group very well. The paper by Kawazoe on "A Practice Report on Mathematical Modelling Education for Humanities and Social Science Students" describes a concept of making mathematics relevant for students of social sciences and psychology by using a mathematical modelling approach and engaging students in group activities. This concept has been in use for about ten years now such that experience on what turned out to be viable in the long run can be presented. The other three contributions deal with mathematics in engineering study courses and address different aspects of this setting. The paper by Takagi, Hadano and Yamaguchi on "Teaching Materials on Calculus as seen from the Application to Engineering" also tackles the problem of making mathematics relevant to students and enhancing their understanding by experiencing the practical meaning of mathematical concepts. They suggest to motivate and explain these concepts by first providing a relevant application problem and giving the formal definitions and theoretical developments later on. They illustrate their approach by giving three examples. In the paper by Viirman and Pettersson on "A Small-scale Implementation of Inquiry-based Teaching in a Single-variable Calculus Course for First-Year Engineering Students", the authors introduced group sessions where students investigated application problems where the currently studied mathematical concepts are of relevance. This way, students can better understand the meaning of the concepts and recognize additional aspects like the practical numerical computation of interesting quantities. They interpret their observations within the framework of commognitive theory. Finally, in the paper by Peters and Hochmuth on "Sometimes Mathematics is Different in Electrical Engineering" the authors address the problem that mathematical presentations and practices in the mathematical education of engineers can be different from the practices and usages which can be found in application subjects of an engineering study course. They analyse the differences in practices using the Anthropological Theory of Didactics (ATD) and develop suggestions for dealing with resulting problems in students' understanding.

All in all, the four papers presented in this issue provide a good impression of the themes and methods of current educational research on mathematics for non-specialists. We hope that they will provide inspiration for additional research and development in this under-researched area which is important for many students who need mathematical understanding without intending to major in mathematics.

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SOMETIMES MATHEMATICS IS DIFFERENT IN ELECTRICAL ENGINEERING

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Abstract

In this contribution we will present an ongoing research project on mathematical practices in electrical engineering. Starting with interesting phenomena we have encountered in our research regarding the relationship of mathematics and engineering, we provide some general thoughts on the notions application and modelling. We then present our own vantage point: Using the Anthropological Theory of the Didactic (ATD), we take an institutional point of view on mathematical practices. This allows us to conceptualise two ideal type mathematical discourses corresponding to different epistemological constitutions of mathematical knowledge in mathematics courses for engineers and in advanced courses in electrical engineering, respectively. We will enrich our presentation with short vignettes of our latest research results to illustrate the kind of insights that the institutional point of view enables us to gain particularly regarding educational issues.

Key words: Anthropological theory of the didactic, mathematical practices, electrical engineering, application and modelling

INTRODUCTION

The study of engineering mathematical practices is an important topic in engineering mathematics education (Alpers, 2020; Winsløw et al., 2018). Explicitly focusing on the specific content related needs of engineering mathematics for didactic analyses enables a deeper understanding of practices and potential learning difficulties related to them. A deep analysis of teaching materials and students' works can then also open up new ideas for teaching design. In an ongoing research project on mathematical practices in Signal Theory, we refer to the Anthropological Theory of the Didactic (ATD) (see Bosch et al., 2019; Chevallard, 1992; Chevallard et al., 2022) and, besides other, its understanding of praxeology to model mathematical practices in electrical engineering to address those issues. Other recent ATD related studies on mathematical practices in engineering are done by Bartolomé et al. (2019), Florensa et al. (2018), González-Martín (2022), Palencia (2022), Rønning (2021) and Schmidt and Winsløw (2021). In our research project we developed three foci: First, with a focus on subject specific mathematical practices, we introduced an extended praxeological model to reconstruct the mathematical discourse that justifies the mathematical practices in signal theory (Peters & Hochmuth, 2021). In (Hochmuth & Peters, 2021) we show, how students' solutions to a signal theory exercise can be analysed and understood on the basis of our previous analyses. The second focus

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considers the epistemological relationship between mathematics and electrical engineering (Hochmuth & Peters, 2020; 2022). We showed that epistemological relations between mathematics and engineering can be important for a detailed description and analysis of mathematical practices. Considering epistemological aspects of mathematical practices within the framework of the ATD makes those aspects accessible for didactical analyses and design. The third focus highlights the potential of our ATD analyses for teaching design (Peters, 2022). Here the emphasis is on possible connections between mathematics as taught in higher mathematics courses and mathematics in engineering courses. Based on previous work, we develop an idea for teaching design to foster such connections without the need for the introduction of application examples or the complete restructuring of the course. In an early phase of our research project, when we analysed teaching materials and students' works and had mainly the first focus in mind, we came across two interesting phenomena: First, we repeatedly encountered a deficit-orientation towards mathematical practices of engineers when discussing data and corresponding intermediate analysis results with colleagues. The mathematical practices we studied generally did not follow socio-mathematical norms of academic mathematics[1]. From the standpoint of academic mathematics, mathematical practices in electrical engineering courses like Signal and System Theory (SST) seemed to be sometimes wrong, incomplete and sketchy. This was ascribed to limitations in engineering studies, but was nevertheless seen as a (necessary) deficit. Second, from the perspective of academic mathematics some engineering mathematical practices were difficult or impossible to understand. For example, in our analyses, we could identify arithmetic transformations but could not explain their significance and reasons from the standpoint of academic mathematics (cf. our analysis vignette).

Both phenomena seemed also to be connected. Engineering mathematical practices, that were difficult or impossible to understand, were often simply framed as deficits from the mathematicians' point of view. As fruitful as analyses are that reveal such differences, it is not satisfactory to interpret them sweepingly as deficits. The question is also which deficits from the perspective of academic mathematics are part of an adequate electrotechnical mathematical practice and which are not? Which of the identified deviations from academic mathematics hinder the learning of mathematical concepts in engineering and which deviations are necessary for the adequate teaching and learning of engineering mathematical practices? However, the same question also arises for non-deficit mathematical practices in engineering. Is any access to a mathematical concept that is adequate from an academic mathematical point of view also beneficial for the learning of mathematical practices of engineers? We found these questions important specifically for our analyses, but also relevant to engineering mathematics education in general. Only a detachment from the deficit-oriented view of engineering mathematics practices makes these questions accessible. We could see that those mathematical practices were pragmatic and also necessary to solve specific epistemological problems related to physics or engineering, respectively. In our studies, we later, when we developed the second focus, were able to relate some apparent mathematical shortcomings of engineering mathematics to epistemological issues, which cannot be clarified by inner-mathematical considerations alone and which sometimes underlie

[1] We speak of academic mathematics when we refer to mathematical research institutes and institutes of mathematics at universities. We distinguish academic mathematics from other mathematical institutions, such as mathematics courses for engineers that are part of engineering study programs. In our studies we also use the ATD concept of didactical transposition to connect the relationship of different mathematical institutions with our analyses (cf. Peters, 2022).

the conceptualisation of mathematical knowledge in electrical engineering (Hochmuth & Peters, 2020; 2022). We realised that those positive aspects of the engineering way of doing mathematics were difficult to acknowledge and analyse from the perspective of academic mathematics alone.

In this contribution we want to proceed from this and move on to some general reflections on the concepts of *application* and *modelling*, which are often used in studies of engineering mathematics education to capture the relationship of mathematics and engineering. Thereby, the epistemology of this relationship generally remains implicit and unquestioned. On the other hand, applications of mathematics and modelling problems are often present as important design aspects to improve the teaching of mathematics to engineers (e.g. Alpers, 2020). After bringing up some critique on the standard concepts of application and modelling, we present our stance which enables us to avoid some of the difficulties inherent in application and modelling. After introducing central concepts of ATD and illustrating our stance with two vignettes from previous work, we come back to the notions of application and modelling and show further possibilities of alternative conceptualisations from the viewpoint of ATD.

ON THE CONCEPTS OF APPLICATION AND MODELLING AND AN ALTERNATIVE VANTAGE POINT TO BETTER UNDERSTAND ENGINEERING MATHEMATICAL PRACTICES

There are various definitions and understandings of application and modelling (e.g. Blum et al., 2007), most of which separate between an extra-mathematical world and mathematics. Often application of mathematics and mathematical modelling are seen as related to each other. Niss, Blum, and Galbraith (2007) summarises this relationship and their understanding as follows

> During the last one or two decades the term '*applications and modelling*' has been increasingly used to denote all kinds of relationships whatsoever between the real world and mathematics. The term 'modelling', on the one hand, tends to focus on the direction 'reality → mathematics' and, on the other hand and more generally, emphasises the *processes* involved. Simply put, with *modelling* we are standing outside mathematics looking in: 'Where can I find some mathematics to help me with this problem?' In contrast, the term 'application', on the one hand, tends to focus on the opposite direction 'mathematics → reality' and, more generally, emphasises the *objects* involved - in particular those parts of the real world which are (made) accessible to a mathematical treatment and to which corresponding mathematical models already exist. Again simply put, with *applications* we are standing inside mathematics looking out: 'Where can I use this particular piece of mathematical knowledge?' (p. 10f)

These widely held understandings of the relationship of mathematics and engineering can be very fruitful, especially in teaching design. But it is also regularly noted that the realisation and implementation in everyday teaching is problematic. Barquero et al. (2013) give a survey of literature illustrating the difficulties and barriers as a general problem for the dissemination of modelling activities. They use their own projects to identify and categorise difficulties and barriers. Besides other, they

> focus on describing some constraints related, in the first place, to what may be called the *dominant*

> *epistemology*, that is, the way our society, the university as an institution and, more particularly, the community of university teachers and students, understand what mathematics is and what its relation is to natural sciences. (p. 316)

Regarding the meaning of applications Barquero et al. (2013) reconstructed an epistemology of *applicationism* in the relationship of mathematics to other sciences and identify it as a restriction on the notion of mathematical modelling:

> One of the main characteristics of applicationism is that it greatly restricts the notion of *mathematical modelling*. Under its influence, modelling activity is understood and identified as a mere application of previously constructed mathematical knowledge or, in the extreme, as a simple exemplification of mathematical tools in some extra-mathematical context artificially build in advance to fit these tools. (p. 317)

Regarding the two worlds of mathematics and "the rest of natural sciences" they note that "it is furthermore supposed that both 'worlds' evolve with independent logic and without too many interactions" (p. 318). Also, they note that "in general, the mathematics taught present a highly stereotyped and crystallized structure that does not mingle with the systems that are modelled and, moreover, the mathematics taught are never 'modified' as a consequence of being applied." (p. 319). We can now ask whether an unquestioned academic mathematical perspective on mathematical practices in engineering, where mathematics is also seen as "never modified", can be linked to applicationism? From the applicationsm point of view, it is suggestive to understand mathematical practices in engineering only as more or less deficient applications of previously constructed academic mathematical knowledge.

Regarding the notion of mathematical modelling Bissell and Dillon (2000) note

> Mathematical modelling forms an important part of engineering education and practice. Yet precisely what is meant by the term 'modelling' is often extremely unclear - and, moreover, much of what students are told about the subject is considerably problematic from both a philosophical and a pedagogical point of view. (p. 3)

In their study they look at mathematical modelling from the practicing engineer's perspective. From this perspective the usual modelling cycles are too simplistic to capture mathematical activities of engineers. Also, they note that instead of creating mathematical models in engineering, it is much more important for practicing engineers to use already existing mathematical models (Bissell & Dillon, 2000, p. 4). This shift of perspective on mathematical modelling in engineering, enables an understanding of mathematical models without a necessary separation of mathematics and the rest of the world: here engineering is not only the context for application or the source of the modelling problem. Engineering itself is already mathematised. They also characterise necessary skills for using models: *manipulation* is "the ability to modify the form of the basic model, using algebraic and other skills; essentially 'mechanical'", *interpretation* is the "ability to interpret the modified form of the model in a way relevant to the situation; essentially 'reactive'", and application is the "ability to apply the interpretation and make appropriate recommendations; essentially 'proactive'" (p. 4). Note that they speak of applying the interpretation with respect to the relevance of the situation. Here the applied mathematics does not necessarily remain unchanged as it is the case in applicationism. By connecting their considerations about mathematical modelling also with the general question of "the position of mathematics in engineering" they state that "there is clearly a significant difference between what a mathematician calls ‘doing mathematics’ and what an engineer calls ‘doing mathematics’.” (p. 6).

In our research project we found that to better understand these different ways of doing mathematics and to analyse mathematical practices in engineering without restrictions to applicationism or a deficit-oriented view, we needed a different vantage point: A vantage point that enables us to understand the engineering specific justifications and explanations of mathematical practices and allows for deeper content specific analyses than the considerations by Bissell and Dillon. The ATD enables this, among other things, through the principle of *institutional dependence of knowledge*: In different subject specific institutional contexts mathematical practices are justified, substantiated, validated and constituted differently than in academic mathematics. Also, Castela (2015) emphasises[2] the advantages of an institutional approach to research on mathematical knowledge in different contexts that is fundamental to the ATD. This approach “provides a powerful tool to investigate the mathematics dimension of human social activities in any context, without referring to academic mathematics.” (Castela, 2015, p. 18). This can contribute to counteracting a deficit-oriented view of mathematical practices and provide a deeper understanding of mathematical practices in other siences.

INSTITUTIONAL POINT OF VIEW AND MATHEMATICAL DISCOURSES

Building on Castela’s work, we have introduced a specific extended[3] praxeological model (Peters & Hochmuth, 2021) that allows us to analyse mathematical practices in electrical engineering, particularly taking into account the engineering-specific institutional conceptualisation of mathematical practices. We also showed how institutional analyses of engineering mathematical practices can be related to and help understand individual students’ solutions to exercises (Hochmuth & Peters, 2021). In the following we will present vignettes from both studies to illustrate our approach.

Alongside the institutional dependence of knowledge, *praxeology* is another ATD concept that is important for our work. In ATD a praxeology is a basic epistemological model to describe institutional knowledge in the form of two inseparable, interrelated blocks: the praxis block (know-how) consists of types of problems or *tasks* (T) and a set of relevant *techniques* (τ) used to solve them. The logos block (know-why) consists of a two-levelled reasoning discourse. On the first level, the *technology* (θ) describes, justifies and explains the techniques and on the second level the *theory* (Θ) organises, supports and explains the technology. In short praxeologies are denoted by the 4T-Model $[T,\tau,\theta,\Theta]$.

We illustrate how the concept of praxeology, especially our extended praxeological model, is able to produce (in the sense of a phenomenotechnique, cf. Bosch et. al., 2019) an understanding of the engineering-specific conceptualisation of mathematical practices. We will introduce our extension to the praxeological model in the course of the following exemplary analysis at the step, where the need for an extension concretely arises. For this, we consider an exercise of an SST course at a German university that is taught in the second year

[2] She addresses the relationship of academic mathematics and mathematics in vocational contexts.

[3] With respect to the standard praxeological model of ATD. In our extension we differentiated techniques and technologies according to two mathematical discourses, see below.

of an electrical engineering study program. First, we focus on the lecturer's sample solution, i.e. the taught knowledge in SST. The context of this exercise is amplitude modulation. The exercise under consideration is[4]:

Graphically display $x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t)$ in the complex plane as a rotating phasor with varying amplitude using the relationship $\cos(2\pi f t) = \Re\{\exp(j2\pi f t)\}$.

The lecturer sample solution is:

One first writes

$$x(t) = A\cos(2\pi f_0 t) + \frac{Am}{2}\cos(2\pi f_0 t + \Omega t) + \frac{Am}{2}\cos(2\pi f_0 t - \Omega t) \quad (1)$$

$$= A\,\Re\{\exp(j2\pi f_0 t)\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t + \Omega t))\} + \frac{Am}{2}\Re\{\exp(j(2\pi f_0 t - \Omega t))\} \quad (2)$$

$$= \Re\left\{ \exp(j2\pi f_0 t) \underbrace{\left[A + \frac{Am}{2}\exp(j\Omega t) + \frac{Am}{2}\exp(-j\Omega t)\right]}_{A(t)} \right\} \quad (3)$$

and interprets the expression in the square bracket as a real-valued time-dependent amplitude *A(t)*, which modulates the carrier phasor $\exp(j2\pi f_0 t)$ rotating at frequency f_0 in Figure 1.

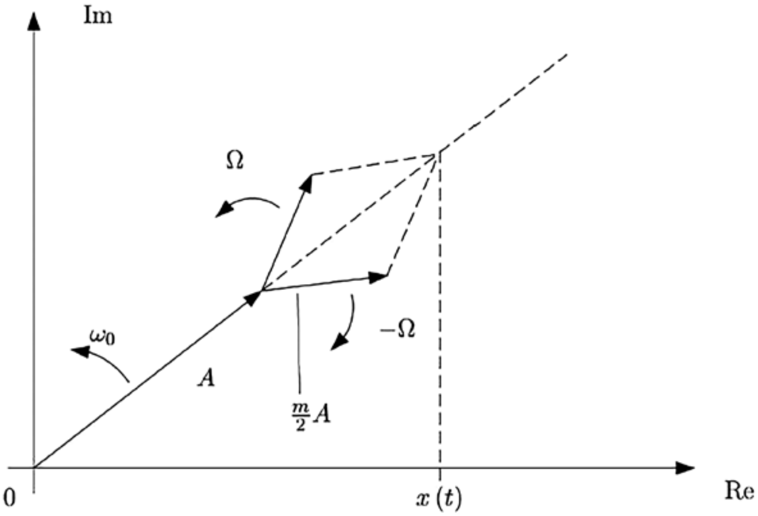

Figure 1: Representation of $x(t)=A[1+m\cos(\Omega t)]\cos(2\pi f_0 t)$ as the real part of a rotating phasor $A(t)\exp(j2\pi f_0 t)$ with $\omega_0=2\pi f_0$.

The sample solution of the exercise represents institutional knowledge of signal and system theory. We now can assign the praxeological components to the steps in the solution. Since this is one exercise, the reconstruction of types of tasks is not relevant here. In the full analysis, as presented in (maybe this is an

[4] Literal translation from the German exercise sheet by the author. The number of the figure is adjusted to the figure counter in this publication.

inconvenient pagebrake 2021) and (Hochmuth & Peters, 2021), we use the method of considering subtasks to further structure the analysis. For this analysis vignette, we focus on techniques as part of the praxis-part and on technologies as part of the logos-part of praxeologies. Tasks and theory will not be further considered and will therefore be understood as SST-tasks and SST-theory, i.e. tasks relevant in the institution SST and theory as the second level of the praxeological reasoning discourse according to the institution SST.

From line (1) to line (2), in the sample solution, using the relationship $\cos(2\pi ft) = \Re\{\exp(j2\pi ft)\}$ is a *technique* τ. Here, connections between the representations of a complex number in polar form and in exponential form are relevant for the justification, the *technology* θ . Both, technique and technology, for this step are known by the students from Higher Mathematics courses earlier in the study program. Here we can see, that for this SST-exercise techniques and technologies from Higher Mathematics courses are relevant. The institutional knowledge of signal and system theory therefore has connections to knowledge from a different institution.

Before we go deeper into details here, we come back to the already introduced idea of institutional dependence of knowledge to clarify what this means for our analysis in particular. Following (Castela, 2015), an institution is

> a stable social organisation that offers a framework in which some different groups of people carry out different groups of activities. These activities are subjected to a set of constraints, - rules, norms, rituals - which specifies the institutional expectations towards the individuals intending to act within the institution I. […] Institutions tend to constrain their subjects but conversely they provide the resources (material and cultural) necessary for activities to take place. (p. 7)

Institutional conditions, norms and aims constitute the technological-theoretical discourse and the practices available. This means that different types of tasks are relevant in different institutions, different solution techniques are adequate, different reasoning discourses are acceptable, and different reasons to study a subject occur. Thus, if one focuses on a specific mathematical subject in different institutions, different praxeologies could emerge. In the following we will show that in the context of our research, the institutional knowledge in SST shows references to other relevant institutions and corresponding institutional mathematical discourses[5]. Our analysis so far showed a praxeology concerning techniques and a technological discourse of dealing with complex numbers that can be assigned to an institution Higher Mathematics (HM). We denote this praxeology by $[T, \tau_{HM}, \theta_{HM}, \Theta]$. In our work, we used the textbook by Strampp (2012), students' lecture notes, and exercises from a course on Higher Mathematics for Engineers based on this textbook to characterise the mathematical knowledge associated with this institution, i.e. the HM-discourse: It is characterised by an internal mathematical conception without concrete references to reality, an orientation towards a generalising rational of academic mathematics, a concentration on calculation rules, and the inclusion of school mathematics concepts. The reason why complex numbers are studied in HM-courses is because they are useful for solving polynomial equations and they are important objects of calculation. Arrows in the Gauß-diagram are used to graphically illustrate calculation rules and properties.

[5] The term discourse refers to the logos part of praxeologies: In ATD, logos is considered as a discourse on praxis (reasoning discourse), but since praxis and logos are dialectically interrelated, every aspect of praxis (i.e. tasks or techniques) is also related to the institutional discourse. Reasoning discourses are institutionally dependent, and so are the respective techniques and technologies. The notion of institutional discourse enables us to differentiate analytically between techniques and technologies that could be associated to different institutional discourses respectively.

When we now look at the solution step from line (2) to line (3) also techniques from the HM-discourse occur: The real parts of the summands are factored out, calculation rules for the exponential function are applied, and the resulting common factor $\exp(j2\pi f_0 t)$ is factored out. But it is difficult to understand the reasons for this transformation from the standpoint of Higher Mathematics. Why make a clearly structured expression more complicated? Also, from the way complex numbers are taught in the HM-course, drawing three phasors associated each with one of the summands in line (2), seems much more obvious than drawing phasors associated to the more complicated expression in line (3). To understand why this transformation is carried out, we have to look for the engineering reasoning that is not part of the HM-course: $x(t)$ is transformed in a specific way to graphically represent principles of amplitude modulation, that could not be represented by a graphical representation of line (1) or (2) (see also our second vignette of an analysis of a student solution to this exercise below). The cosine representation in line (1) does not allow to separate the different frequencies or angular velocities of the carrier-signal, $\omega_0 = 2\pi f_0$, and the message signal, Ω . This is, however, the core of both the representation in line (3) and the graphical representation in Figure 1 in the sample solution. There is no justification within our reconstructed HM-discourse, that gives the reason for the step from line (2) to line (3). So, the technological discourse underlying the step from line (2) to line (3) differs from the HM-discourse. This other mathematical discourse belongs to a different institution. In our analyses we denoted this other mathematical discourse as an electrotechnical mathematics-discourse (ET). In our work, we most notably use studies by Bissell and Dillon (Bissell & Dillon, 2000; Bissell, 2004; 2012) and the electrical engineering textbook by Albach (2011) to characterise this mathematical ET-discourse. In contrast to the HM-discourse, the ET-discourse has references to reality. The degree to which this reference to reality is made explicit can vary greatly, along with a different degree of formalisation and abstraction. In addition, it is characterised by a "linguistic shift" (Bissell & Dillon, 2000, p. 10) in the way of talking about mathematics and mathematical practices and an electrotechnical-typical way of "system-thinking". One reason why complex numbers are studied according to the ET-discourse is that they allow oscillating signals to be described algebraically in a very suitable way and visualised graphically as phasors. This visualisation does not serve to illustrate calculation rules or properties of complex numbers but represent important analysis tools, e.g. for AC circuits (cf. Albach, 2011) or amplitude modulation. For a more comprehensive description of the discourses see (Hochmuth & Peters, 2021; Peters & Hochmuth, 2021; Peters, 2022).

The step from line (2) to line (3) can be associated to a praxeology $[T,\ \tau_{HM}, \theta_{ET},\ \Theta]$. In the next step, the expression in line (3) has to be interpreted in order to draw the Gauß-diagram, cf. Figure 1. The part denoted by $A(t)$ must be interpreted as a modulation process (τ_{ET}). This is justyfied because the phasor which varies in length with $A(t)$ represents a general periodic signal (θ_{ET}). In this praxeology technique and technology are from the mathematical ET-discourse, $[T,\ \tau_{ET}, \theta_{ET}, \Theta]$.

This brief illustrative insight into our analysis of the sample solution shows that the solution of this task can be linked to different praxeological configurations ($[T,\ \tau_{HM},\ \theta_{HM},\ \Theta]$, $[T, \tau_{HM}, \theta_{ET}, \Theta]$, and $[T,\ \tau_{ET},\ \theta_{ET},\ \Theta]$) drawing on the two different institutional mathematical discourses. We could observe that both institutional discourses are interrelated and show up in different combinations of techniques and technologies. Transitions or shifts between the two mathematical discourses constitute epistemological ruptures in the sense that they each follow a different rational. These ruptures often remain implicit, although they represent important aspects. They indicate places that are not accessible from a single mathematical discourse and its techniques, and thus mark something additional to be learned. Neither the modelling and application-point of view nor the standard praxeological model are sufficient for this kind of analyses: Application and modelling both entail a conceptualisation of electrical engineering knowledge as consisting of inner-mathematically justified mathematical practices[6] and extra-mathematical engineering knowledge[7]. The praxeology $[T, \tau_{HM}, \theta_{HM}, \Theta]$ could be interpreted as a purely inner-mathematical, as the HM-discourse has strong relations to academic mathematics and no references to reality. So this part of the analysis could be related to the application- or modelling view. But there is no accompanying extra-mathematical engineering knowledge where this praxeology (i.e. the knowledge that is modelled within ATD with this praxeology) is applied to. The mixed praxeology $[T, \tau_{HM}, \theta_{ET}, \Theta]$, where a technique from the HM-discourse gets a new ET-discourse meaning, does not fit this viewpoint at all. The standard praxeological model that would allow to take different institutional origins of practices into account and to describe the knowledge in form of HM- and ET-praxeologies does not allow to shed light on the interrelatedness of the two mathematical discourses.

As a second vignette we present a short analysis of a student solution to this exercise[8] from (Hochmuth & Peters, 2021). This is an example of a solution, where the necessary shifts between the two mathematical discourses do not occur (cf. Figure 2).

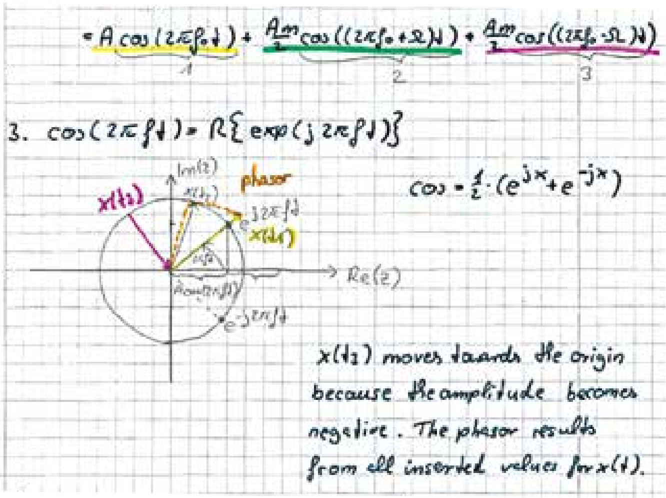

Figure 2 A student solution to the exercise (Hochmuth & Peters, 2021)

At the top we see each of the three cosine-terms separately underlined and each given a number. Each term is thus interpreted individually as something to be drawn. These numbers can also be found in the diagram; the respective phasors are marked accordingly ($x(t_1)$, $x(t_2)$, and $x(t_3)$). While underlining mathematical terms is a technique neither specific to the HM-discourse nor to the ET-discourse the idea represented in this technique, that each term is something to be drawn individually, is a technology of the HM-discourse (θ_{HM}): Each cosine-term stands for a complex number that could be drawn as an arrow starting at the origin of the Gauß-diagram. The sum of three complex numbers then could be drawn as the geometric sum of the

[6] Learnt in mathematical courses and applied later in different contexts.

[7] Providing the context for the application of the mathematical knowledge or the modelling problem.

[8] To protect the student's privacy, we have rewritten the student solution. We omitted the assistant's marking.

respective arrows. The student tried to graphically add the three arrows (dashed line in the diagram in Figure 2). Additionally, the diagram also contains elementary properties of complex numbers: the connection between cosine and the complex exponential function and the complex conjugate. This student solution reproduces the HM-discourse by drawing a diagram similar to diagrams from the mathematics service course where the Gauß-diagram and the unit circle are used to illustrate properties of complex numbers (τ_{HM}). Aspects indicating a connection to amplitude modulation are missing and transitions to the ET-discourse do not occur. This student solution does not produce a diagram that is capable of illustrating aspects of amplitude modulation.

SOMETIMES MATHEMATICS IS DIFFERENT: A DISCUSSION

This short analysis vignettes show the relevance of taking into account the specific mathematical ET-discourse for a deeper understanding of mathematical practices of electrical engineers. Reconstructed mathematical practices from a lecturer sample solution of a signal and system theory-task contain aspects of both discourses, the HM-discourse and the mathematical ET-discourse. An analysis vignette of a student solution to this exercise showed that referring only to the HM-discourse is both a possible student action and not sufficient to solve the task.

We already argued that the standard conceptualisations of the relationship of engineering and mathematics, modelling and application, are not able to capture the complex nature of engineering mathematical practices in this way. From our ATD perspective, we see this relationship not as a relation between independent fields of knowledge. At the core of our approach is the acknowledgement of mathematical practices of engineers as institutional mathematical practices in their own right and with engineering specific conceptualisations of mathematical knowledge. Relations to academic mathematics are present, e.g. in our analysis in the HM-discourse. Also, relations to mathematics developed within the engineering institutions are present, e.g. in our analysis in the ET-discourse. In (Peters, 2022), these relations are discussed in more detail. The mathematical discourses interact in complex ways and are not understandable from the standard modelling and application point of view.

Nevertheless, there are conceptualisations, that are connectable to our stance. Concerning an understanding of application we would like to mention the work by Schmidt and Winsløw (2021). Using an analysis of didactical transpositions between institutions Mathematics and Engineering, they develop a method to design Authentic Problems from Engineering (APE). In their approach the idea of applying mathematical knowledge to engineering starts with engineering. From there they look for possibilities to let the engineering knowledge interact with the mathematical concepts. This approach is specifically capable of counteracting the problem of applicationism. Concerning modelling, we already mentioned the perspective of Bissell and Dillon (2000) who shift the focus to the *use* of mathematical models and show how in the historical process engineers developed mathematical practices specific to their needs and aims (see also Bissell, 2004; 2012). Another important line of development is the reformulation of modelling from an ATD perspective as it is presented for example can we change this to: by Garcia et al. (2006). This reformulation seems particularly appropriate to us here, of course, because we share the same framework of ATD. But apart from that, we also consider the approach fruitful because it takes a decidedly epistemological and institutional perspective on modelling. A first important interpretation of modelling within the ATD is "that modelling is [not] just one more aspect or dimension of mathematics, but that mathematical activity is essentially a modelling activity in itself" (p. 232). Two statements are then important. First mathematical modelling is not restricted to "'mathematization' of non-mathematical issues" (p. 232). Also, inner-mathematical activities are understandable as modelling activities. Second, they highlight the meaning of modelling activity from the standpoint of ATD:

> In the framework of the ATD, what is relevant is not the specific problem situation proposed to be solved (except in 'life or death' situations), but what can be done with the solution obtained –that is, with the constructed praxeology–. The only interesting problems are those that can be reproduced and developed into wider and more complex types of problems. The study of those *fertile problems* provokes the necessity of building new techniques and new technologies to explain these techniques. In other words, the research should focus on those *crucial questions* that can give rise to a rich and wide set of mathematical organizations. Sometimes, those *crucial questions* have an extra-mathematical origin, sometimes they have not. (p. 233)

They summarise the proposed understanding of the modelling process as

> a process of reconstruction and interconnection of praxeologies of increasing complexity (*specific → local → regional*). This process should emerge from an initial question that constitutes the rationale of the sequence of praxeologies. From this questioning, some *crucial questions* to be answered by the *community of study* should arise. (p. 233)

Within ATD this approach was further developed under the notion of study and research paths (SRP) (e.g. Bartolomé et al., 2019; Chevallard, 2006; Florensa et al., 2018). From our research perspective especially, the focus on a crucial question that guides the research or learning process is relevant here. This question is not only a question from a specific context but a question that also has the potential for questioning the content specific institutional rationales. In our analysis, in the context of complex numbers, two different rationales, each within a specific institutional mathematical discourse, occurred. We can connect those rationales with the idea of different ways of doing mathematics from Bissell and Dillon (2000) and shed more light on the meaning of the "significant difference between what a mathematician calls 'doing mathematics' and what an engineer calls 'doing mathematics'." (p. 6).

So, sometimes mathematics is different but the question appearing now from our perspective is: when is which discourse relevant? Our analyses show that both discourses show up in electrical engineering exercise solutions: The HM-discourse with its orientation towards academic mathematics is important, as well as the engineering specific mathematical ET-discourse. To be able to successfully solve exercises as our example, students have to know when which discourse is adequate and when to switch. However, this switching itself is often not made explicit in teaching. Analyses of student solutions of this exercise, like the one above, show that students difficulties could be connected to this question of switching between mathematical discourses (Hochmuth & Peters, 2021). Our analyses can therefore help to identify hurdles for students and to achieve subject-related clarifications. Lecturers can use this to explicitly address difficulties when discussing sample solutions.

In addition, the acknowledgement of mathematical practices of electrical engineers as a justified institutional discourse in itself can prevent a deficit-oriented view and open up new possibilities for teaching, task design and student guidance. In our analyses, for example, we refer to different reasons for studying complex numbers. In the Higher Mathematics course, complex numbers are important because they allow to solve every polynomial equation. In electrical engineering, the general problem of the solvability of polynomial equations is not the main interest. Here, complex numbers are particularly important because they allow to describe oscillating signals. Such reasons to study a mathematical concept are part of the logos block, thus especially part of the institutional mathematical discourses, and may not be explicitly addressed in teaching. So, students may implicitly learn one reason to study complex numbers in one course and a different reason in other courses. The different reasons may then not fit together or even contradict each other. This could convey the impression that mathematics as taught in Higher Mathematics courses is not useful for or disconnected from engineering. In (Peters, 2022) we present a concrete teaching development idea for mathematics service courses based on our research findings. In doing so, we illustrate how the difference between the two identified mathematical discourses can be used constructively in teaching development.

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A SMALL-SCALE IMPLEMENTATION OF INQUIRY-BASED TEACHING IN A SINGLE-VARIABLE CALCULUS COURSE FOR FIRST-YEAR ENGINEERING STUDENTS

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Abstract

We report from the first iteration of a small-scale project introducing elements of inquiry-oriented education in a first-year engineering Calculus course. In four of the exercise sessions we introduced problem solving in groups, using problems designed to provide alternative viewpoints on central topics of the course, for example limits, differentiation and integration, and containing elements of modelling and numerical methods. The theoretical perspective underlying the design was commognitive theory. We discuss some of the problems used in the intervention, focusing particularly on the numerical differentiation and integration problems. We also report some observations made during the first two iterations of the project, and how these have fed into the continued evolution of the project.

Key words: Inquiry oriented teaching, calculus, engineering education, commognition, numerical differentiation, numerical integration

BACKGROUND

It is fast becoming an established fact that so-called "active learning" is beneficial for students when compared to traditional lecturing (Freeman et al., 2014). At the University of Gävle, where both authors worked at the time of the intervention described in this paper, for a few years now the first-year mathematics courses for the engineering programs have been run using a format called PLUSH (Preparation, Lecture, Unsupervised study, Seminar, Homework). The PLUSH format was designed to encourage students to engage more actively, without involving radical changes in the teaching, changes that might risk losing those aspects that have been highlighted as the strengths of the lecture format, such as motivating students and modelling mathematical practices (Pritchard, 2010). In the PLUSH format, before each lecture students are expected to watch short videos covering the essential aspects of the topic. This removes some of the pressure to "cover the content", making room for the lecturer to include more student-centered and conceptually oriented activities, such as clicker quizzes and small group discussion. After the lecture, students have two hours of unsupervised study in which to prepare for the afternoon seminar. However, these seminars were

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not working as well as we would have hoped. Rather than developing into discussion sessions revolving around the recommended exercises, due to student expectations and demands, these seminars typically turned into either continuations of the unsupervised study, but now with a teacher at hand for answering questions; or traditional exercise sessions, where the teacher does exercises at the board while students take notes.

In light of this, in the spring semester of 2019, for the Single Variable Calculus course we decided to implement aspects of an inquiry oriented (IO) teaching approach. There are successful undergraduate mathematics courses run entirely using an IO approach (e.g. Rasmussen & Kwon, 2007). However, not wanting to wholly remodel the course, we were instead interested in seeing if we could gain some of the benefits of the IO approach working on a smaller scale. We did this through devoting some of the seminar time to problem solving in small groups of 4-5 students each. Instead of working on selected exercises from the textbook, the students were asked to work on one or two larger problems for a longer period of time, about 45 minutes. These problems were designed to support and deepen their understanding of the central topics of the course, while also allowing them to practice working collaboratively, and presenting and arguing their mathematical ideas. In this paper, we present the ideas and theoretical considerations underlying the choice and design of problems, giving particular attention to one specific problem concerning numerical integration. Furthermore, we discuss some empirical observations, and how these have fed into the future iterations of the intervention. First, however, we give a brief overview of the context of the intervention.

CONTEXT – THE COURSE, THE STUDENTS, THE TEACHERS

The intervention took place in the context of a 7.5 ECTS credit course in Single Variable Calculus, aimed at first-year engineering students and covering the usual introductory calculus material: elementary functions, limits and continuity, derivatives, integrals, Taylor series, etc. The course has no prerequisites apart from the usual requirements on upper secondary mathematics, but due to its positioning in the study program, most students enrolling in the course will have taken courses in introductory and linear algebra. In addition to approximately 120 engineering students, a small number of prospective upper secondary teachers normally take the course every year. The course is taught over a nine-week period, with two days of teaching per week. These days are structured according to the PLUSH format, with a whole-class lecture in the morning followed by unsupervised study and four parallel seminars. At the time of the intervention reported here, four teachers were involved in the course: two research mathematicians (one of which is the second author of this paper) responsible for lectures and one seminar group each; a mathematics education researcher (the first author) teaching one seminar group, and a teaching assistant running one seminar group. Students self-selected to the different seminar groups, and were encouraged to try out different groups.

LEARNING AS ROUTINIZATION

Theoretically, the intervention was grounded in the commognitive theory of learning (Sfard, 2008), and particularly the notion of *routinization* (Lavie, Steiner & Sfard, 2019). From a commognitive perspective, mathematics is conceptualized as a discourse, and learning as the increased ability to participate in this discourse. The discourse of mathematics (indeed, any discourse) can be distinguished through four characteristics (Sfard, 2008, p. 133-135):

- *word use* – words specific to the discourse or common words used in discourse-specific ways, for instance Riemann integral, function, etc.;
- *visual mediators* – visual objects operated upon as a part of the discursive process, for instance diagrams and special symbols;
- *narratives* – sequences of utterances speaking of the description of objects, relations between and/or processes upon objects, subject to endorsement or rejection within the discourse, for instance theorems, definitions and equations;
- *routines* – repetitive patterns characteristic of the discourse, for instance methods of proof, of performing calculations and so on.

Recently, the notion of routine has been further developed and operationalized. "A routine performed in a given task situation by a given person is the task, as seen by the performer, together with the procedure she executed to perform the task" (Lavie et al., 2019, p. 161). Here, a *task situation* denotes "any setting in which a person considers herself bound to act" (p. 159), and a *task* "as understood by a person in a given task situation, is the set of all the characteristics of the precedent events that she considers as requiring replication" (p. 161). Thus, in problem design and analysis, we have paid particular attention to the kinds of routines students need to engage in when working on the problems, but also how these tie in to students' previously established routines.

PLANNING THE INTERVENTION

In addition to the underlying theoretical perspective, when designing problems for the sessions we were guided by several overarching principles. The sessions should complement the lectures and other teaching activities, taking different perspectives on central topics. Given that the course caters to engineering students, we wanted problems that included elements of modelling and numerical methods, which are not part of the curriculum in any of the obligatory mathematics courses in the three-year engineering programs at the University of Gävle, but which are very useful for future engineers (Alpers et al., 2013). At the same time, we wanted the problems to be flexible and doable with limited resources – we did not want to have to rely on access to computer labs or specialized software, for instance. Importantly, the problems should be grounded in research on the teaching and learning of calculus. One basic aspect of this was designing tasks that require creative rather than imitative or algorithmic reasoning, something that has been shown not to be the case for most standard calculus textbook tasks (Lithner, 2004). Moreover, we wanted problems that connected different parts of the course by making use of settings that could be revisited in later sessions focusing on other topics.

Given that one of our aims was to have the problem-solving sessions contribute to students' understanding of the central concepts of the course, we decided to focus the IO sessions on limits/continuity, differentiation, graph sketching, and integration. The sessions on differentiation and integration were designed to contain elements of modelling and numerical methods. The group work was intended to facilitate students' adaptation to the demands of the new form of teaching, and allow them to develop their collaborative and argumentative skills. In order to trace the progression in learning, we recommended the students to attend all IO sessions to benefit the most from the intervention. For the first trial, we decided to run the intervention in just two of the four seminar groups, to enable students to choose not to participate.

EXAMPLE PROBLEMS

The main problem used in the first IO session concerned the behavior of a function with an irremovable singularity at x=0, and involved estimating the limit of the function along different sequences converging to 0. Since we expected students to have limited experience of group problem solving, we designed the problem to provide a relatively high degree of scaffolding, dividing it into a series of sub-problems. In later sessions, to allow students greater independence, we gradually decreased the scaffolding. The main problem in the session on differentiation, following a brief warm-up problem introducing numerical differentiation and the notions of forward and central differences, concerned the travails of the Svensson family travelling to the airport and forgetting their passports. The problem involved numerical differentiation and the construction of a velocity-time graph from a given position-time graph of a function not given through a formula. This problem was later returned to in the session on integration, where students were asked to recover positions at certain times from the velocity-time graph. The session on graph sketching included a "Chinese whispers" problem on graph analysis, where a student verbally described a graph to a fellow student, who in turn drew a graph according to the description given. This new graph was then given to a third student, and the process was repeated. After a few iterations, the resulting graph was compared to the original, and similarities and differences were discussed. Apart from the follow-up to the differentiation problem, the main problem for the integration session involved numerical estimation of the volume of a vase. In what follows, we will present the numerical differentiation and integration problems in more detail, followed by some empirical observations.

The "Svensson's vacation" problem

The problem was formulated as follows:

> The Svenssons were going on vacation to Thailand. They packed their bags, got into their car and drove off for the airport. After a while, they started worrying that they might have forgotten their passports. They drove off at the next exit, and started rummaging through their bags. Sure enough, the passports were nowhere to be found. They turned around and drove back home. By now, they were in a bit of a hurry. When you're under stress, finding what you're looking for takes longer, but it turned out alright in the end, and the Svenssons got to the airport on time. Attached you see a graph (Figure 1) showing the time in minutes and the position of the Svenssons' car at the corresponding time, measured in kilometers from their house. Please answer the following questions:
>
> - What was the velocity of the car at times t=3; 10; 22; 55 minutes?

- What was the average velocity of the car in the time intervals [10,15] and [20,25] minutes?
- Sketch a graph showing the velocity of the car as a function of time. Does the graph look reasonable? Does it agree with the story of the Svenssons?

As briefly mentioned above, the session on definite integrals included a follow-up to the "Svenssons' vacation" problem. The students were given a graph of the approximation for the instantaneous velocity (Figure 2) and were asked to estimate the distance from home for Svenssons after 15 and 25 minutes respectively.

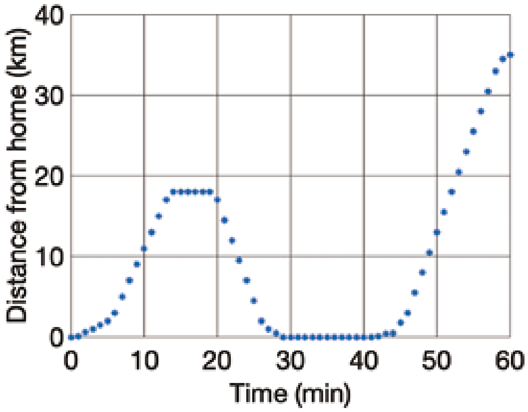

Figure 1: Position of the car at time t

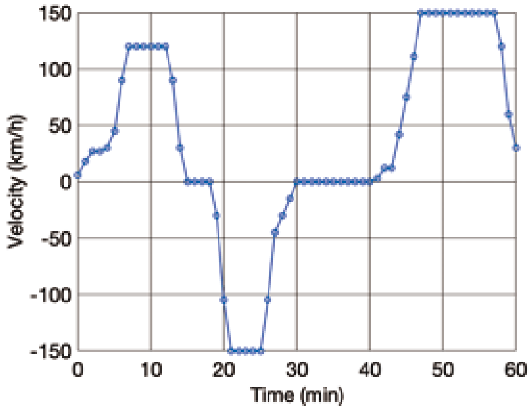

Figure 2: Velocity of the car at time t

The problem aims at engaging students with differentiation (and integration) routines in a setting where functional relationships are not given by formulas, meaning that the familiar analytic differentiation routines are not readily applicable. The problem also creates a potential for students to form connections between the objects of derivative and definite integral, through exploring the relationship between rate of change (RoC) and accumulation (Thompson & Silverman, 2008). In solving the main problem, we expected the students to sketch the velocity graph using either the geometrical interpretation of the derivative as the slope of a tangent line or a numerical approximation of the RoC of the position $s(t)$. For the follow-up problem, we expected the students to reconstruct the distance from the graph of the instantaneous velocity $s'(t)$ either by estimating the area under the curve or by using Riemann sums for the approximation of a definite integral.

The problem uses an everyday setting and everyday language, and contains one visual mediator, the position-time graph, providing a partial mathematization of the problem. The same holds for the follow-up problem, which is also built around a single graph. The questions are closed, but phrased so as not to suggest particular solution methods, thus providing students with the opportunity to engage with various mathematical routines, both familiar and less familiar. We deliberately chose a setting where the function was not given algebraically, to emphasise the need for numerical techniques for handling even very simple real-world problems (Kaput, 1994). To answer the first two questions, students first need to interpret the graph and extract the information needed, routines that should be well-known to them from their previous studies. They then need to use this information to first estimate the instantaneous velocity, which, as already mentioned, can be done either numerically, building on the notion of RoC and using the numerical differentiation routine introduced in the warm-up task, or geometrically, by estimating the slope of the graph at the indicated times. Whichever method the students choose, it will require them to engage with routines less familiar to them. Second, they are asked to estimate average velocity, a routine familiar to them from upper secondary mathematics and physics.

The task allows students to reflect on the relationship between average velocity and instantaneous velocity, in terms of the position-time graph. Since we expect both numerical and geometrical routines to be used for estimating the velocity, there is opportunity for discussion about the relation between these two methods. This resonates with the observation made by Hauger (2000) that students often make sense of instantaneous RoC in terms of average RoC. In the last part of the problem, students then need to build on their work on the first problem in order to find the information needed to engage in a graph construction routine (Viirman & Nardi, 2021), namely constructing the velocity-time graph, another topic where student difficulty is well documented (e.g. Berry & Nyman, 2003). Numerically estimating the instantaneous velocity at several points and plotting them against time, students can see the functional character of the relationship, even though it is not given by a formula.

The follow-up problem allows students to engage in numerical integration routines, discussed in the next section. The problem also allows students to see how the Fundamental Theorem of Calculus connects RoC and accumulation (Thompson & Silverman, 2008), by realising that the accumulated position, that is, the distance, at time *t* is given by the integral of its RoC, that is, the velocity, from the starting point to *t*. Doing this for different values of *t*, the students can gain a sense of the integral as an accumulation function of the quantity whose RoC we know. This is something that students often struggle with, since they are not used to thinking of the upper limit as varying (ibid.). The problem also requires students to interpret the negative area under the velocity curve in terms of accumulated position, something that research has shown to be a challenge for students (Bressoud et al, 2016). For further detail on the "Svenssons' vacation" problem, see (Viirman & Pettersson, 2019).

The "Vase" problem

For this problem, each group of students were given a glass vase, and were asked to estimate its volume. The main aim of the problem was having the students engage in integration routines in an everyday setting where no mathematization of the problem was given. The lecture preceding the session had dealt with, among other topics, solids of revolution, and it was expected, although not explicitly stated, that the students would use this idea to consider the vase as generated through rotation around a central axis, thus in essence reducing it to a two-dimensional problem. Like the "Svenssons' vacation" problem, this problem is formulated completely in everyday language, but it is also open in the sense that it does not suggest any particular solution strategy. Additionally, the formulation contains a quite concrete physical object – the vase itself. The mathematization of the problem then requires the students to construct their own visual mediators representing this concrete object by projecting it onto a planar surface. Since we ourselves found producing a reasonably accurate projection of the contour of the vase in two dimensions time-consuming, we had prepared a handout (Figure 3a) to make available to the students if needed. Indeed, all groups turned out to need the handout. The problem as such does not require the students to formulate any mathematical narratives, since it only asks for a numerical estimate of the volume of the vase. However, we of course expected the students to be able to formulate mathematical narratives describing the method they had used to come up with the estimate. As for mathematical routines, the openness of the problem allows for several solution strategies, allowing students to make use of both routines established earlier in their studies, and routines introduced during the course. However, similarly to the “Svenssons’ vacation” problem, we intentionally designed the problem in such a way that no explicit formula was available. In addition to emphasising the need for numerical techniques (Kaput, 1994), this also led the students away from relying on the standard algorithmic integration routines familiar to them from previous studies. For many students, integration is synonymous with finding the anti-derivative, inserting the endpoints of the interval and subtracting, and working with numerical integration can help support a broader understanding of the definite integral as the limit of a sum (Blomhøj & Hoff Kjeldsen, 2007).

We expected the numerical estimation routines employed to be of one of two main types: either the students would make use of some form of geometrical estimation routine, trying to approximate the area by geometric means, or they would use a Riemann sums routine of the kind previously discussed during the course. The former routine would be familiar to them already from upper secondary school, whereas the latter, although not new to them, could be expected to be less firmly established. Letting students work with Riemann sums in this kind of numerical setting has a further advantage in that, as pointed out by Thompson and Silverman (2008), the role of the variable of integration is often a mystery to students, but by considering Riemann sums this role can be made clearer. When students have to decide how to divide the interval of integration, the integration variable becomes experientially real to them.

EMPIRICAL OBSERVATIONS

We report from the first two iterations of the intervention[1], building on field notes, made by the authors during the problem-solving seminars run by them, and on students’ written productions. A general observation was that only a small number of students, approximately 15[2] out of 120, chose to participate in the first trial. An explanation given by students who chose not to participate was that the IO sessions would deprive them of the usual tutorials, seen as important for the examination. As this was a pilot, the limited participation was perhaps not entirely a bad thing, but it was still somewhat disappointing, particularly since those few students who chose to participate reported that they mostly found the sessions helpful for their understanding and providing variation in a quite demanding mathematics course. Hence, for the second iteration of the course, we made the problem-solving sessions a regular part of the work of all four seminar groups. In this way, we could avoid the sense that the activities were optional and thus of less relevance.

As for the problems we designed, some worked better than others. The first session, on limits, turned out to be too demanding for the students, who were not able to make any progress without support from the teacher. As we realized, the main difficulty stemmed from the fact that in the lectures, the notion of limit had been introduced for functions rather than for sequences, making the basic premise of the problem difficult for the

[1] Since both authors changed affiliations during or after the second iteration, we have no insight into the later iterations.
[2] Precise numbers varied between the different problem-solving sessions.

students to understand. Indeed, in contrast to the situation in, for instance, France, in Sweden limits of sequences do not appear at all in school calculus, and play a less significant role in introductory university calculus (Viirman, Vivier & Monaghan, under review). For the second run of the course, we made further modifications and simplifications to the problem, but it still worked quite poorly. Hence, it was decided to replace it for the third iteration. The “Chinese whispers” graph sketching problem in the third session worked reasonably well in both iterations, but the students’ work on it has not yet been analyzed in any detail.

Students’ work on the “Svenssons’ vacation” problem

The session on derivatives, on the other hand, worked very well. The formulations of the problems were not as abstract as for the session on limits, and to our satisfaction, the students employed a variety of different routines for solving the tasks. The introductory problem introduced a technique for numerical differentiation, and indeed some of the students used this for dealing with the main problem. However, some students chose to adapt previously established routines for graphically estimating derivatives to the situation at hand. This was in accordance with our aims and expectations in designing the problem, where we wanted students to have the opportunity to employ previously establishes routines as well as to engage in “thoughtful imitation” (Sfard, 2008, p. 251), adapting the recently learned numerical differentiation routine to the task at hand. Further observations made included, for instance, difficulties with interpreting and plotting negative velocities, with some students choosing to plot them as positive (cf Berry & Nyman, 2003). Interestingly, one group of students anticipated the follow-up problem, saying that it should be possible to get the distance out of the graph for the velocity by computing the area under the curve. That is, at least to some students the connection between numerical differentiation and integration suggested itself. An insightful comment was made by one student: “It feels like the graph should be smooth, but I do not know why. All functions in physics are smooth”. This prompted a quite useful discussion concerning the behaviour of acceleration and deceleration.

Somewhat surprisingly, when the problems were used for the second iteration, there was less variety in approaches taken. All groups (at least in the sessions run by the authors) used the technique introduced in the introductory problem. Why this was the case we can only speculate, but it is possible that the students participating in the first iteration, having volunteered to participate, were perhaps more interested in problem solving, and thus more willing to try different methods. There might also have been subtle differences in how the problems were introduced, causing the students in the second session to view them as more closely connected.

Students’ work on the “Vase” problem

For this problem, the student groups were handed one physical copy of the vase each and were asked to estimate its volume. Once they had an estimate they were satisfied with, they were to report to the teacher, who wrote the estimated value on the blackboard. As new groups reported their estimates, in some cases the faster groups decided to reconsider their work and revise their estimates. When all the groups had finished working, we let the students check their estimates by filling the vase with water from the sink that was available at the back of the classroom.

Since the formulation of the problem gave no indication of the choice of method, the routines employed were more diverse than for the “Svenssons’ vacation” problem. All groups treated it as a case of a solid of revolution, and as expected they struggled with constructing a plane projection of the vase, causing us to provide them with the handout we had prepared. Once the groups had this handout, the methods they employed for estimating its area, and thus the volume of the vase, differed considerably. As previously mentioned, we had expected two main avenues of approach, and we did indeed see examples of both in both iterations. Taking the session led by the first author in the first iteration as an example, among the groups employing geometrical approximation routines, one tried approximating the vase by a cylinder, drawing a straight horizontal line across the outline of the vase in such a way that the areas created between the outline and the straight line were approximately equal. This is a type of approximation routine familiar to the students already from secondary school, where it is used to calculate the area of polygons. The second group approximated the shape of the vase by a sequence of straight lines, thus creating a polygonal shape whose area could easily be calculated (Figure 3b), again a routine familiar from previous studies. Both these approaches yielded reasonable, but not particularly precise, estimates of the volume. The third and the fourth group, on the other hand, used Riemann sum routines, dividing the interval into subintervals, treating the curve as constant on these intervals, and summing up the areas of the resulting rectangles. The third group used a large number (about 25) of subintervals of equal length, while the fourth group made use of the fact that the slope of the shape of the vase differed along its height to use fewer intervals of unequal length. The students in these two groups discussed actively what height for each rectangle they should use in order to get a better approximation, which is a step towards different formulas for numerical integration. A third approach

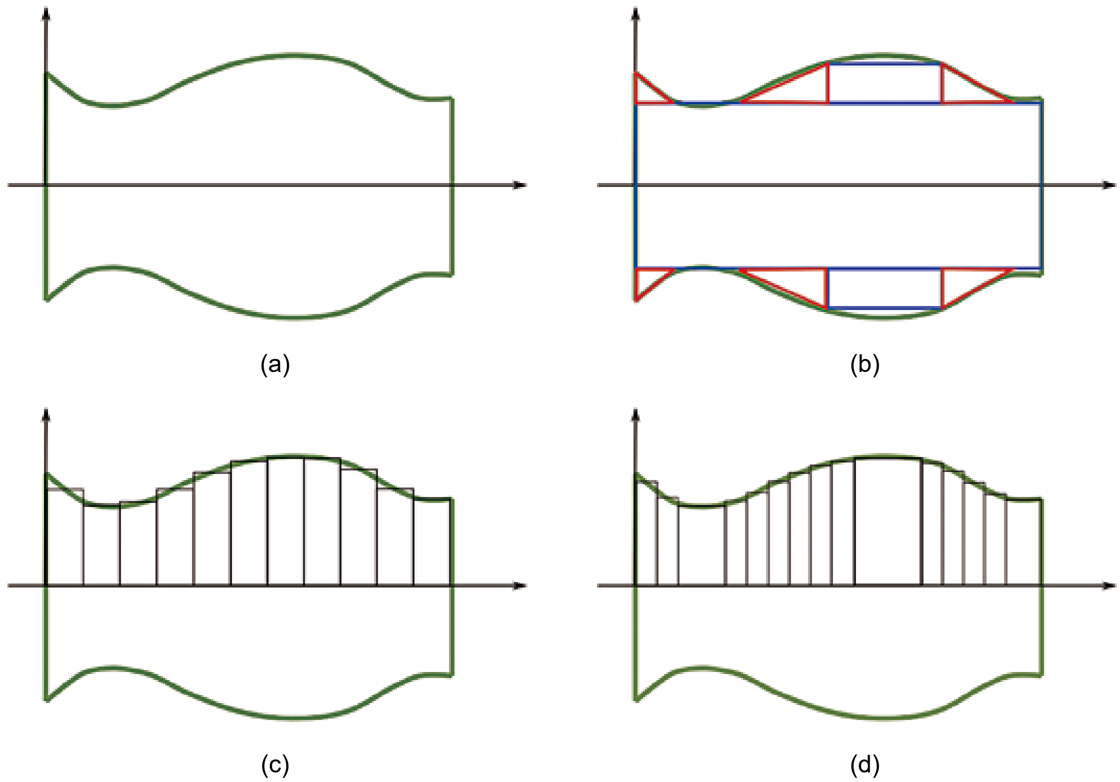

Figure 3: Examples of strategies for estimating the volume of the vase.

was taken by the last group who, somewhat to our surprise, chose to employ least-squares approximation in order to find an explicit function describing the shape of the vase, which they could then integrate using familiar integration routines in order to find an estimate of the volume. The students in this group took a quite fine partition of the interval and then used fourth- and fifth-order polynomials to approximate the shape. It turned out that this approximation was the best one. Least squares approximation had been discussed in the previous linear algebra course, yet another example of how previously established routines were employed in new contexts. We were particularly glad to see that the students applied the least-squares technique, since this technique is widely used in other areas and is a standard method to fit functions to a set of data.

CONCLUDING REMARKS

Despite the difficulties with the session on limits, the first iteration of the intervention worked well enough to warrant a full-scale implementation for the second iteration. In particular the problems on differentiation and integration conformed well with our intentions, confronting the students with the need for numerical techniques, and allowing them to engage in a variety of routines, well-established as well as more recently learnt. Judging from conversations with students, and from course evaluations, the intervention was well-received, although so far, we have not made any systematic efforts to evaluate to what extent the problem-solving sessions have contributed to student learning.

Apart from the need to redesign some of the problems, one important lesson we bring to the planning of future iterations is that more care needs to be taken when orchestrating the sessions, to get students to actively discuss their work with other groups and sharing their solutions. In the sessions that worked well, we managed to get students engaged in problem solving, but the kind of collaborative atmosphere where students can learn from each other's work did not quite develop. Most successful in this regard was the session on integration, possibly (and perhaps somewhat paradoxically) because of the element of competition involved. Still, we need to think further on how to enable students to benefit from the solution strategies of others, without the sessions becoming too time-consuming, or deteriorating into "show-and-tell" (Stein, Engle, Smith & Hughes, 2008). One option might be to use peer-review during the sessions, having the groups provide critique of each other. It is our hope that future iterations of the project will include further steps in this direction.

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A PRACTICE REPORT ON MATHEMATICAL MODELLING EDUCATION FOR HUMANITIES AND SOCIAL SCIENCES STUDENTS

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Abstract

This is a practice report on our teaching practices of mathematical modelling education for humanities and social sciences students at a Japanese university. Our practices were initiated in order to respond to a request from social sciences and psychology departments, and to the growing social demand for such education in Japan. It has been a challenging task because many of these students are not good at mathematics, and some have math anxiety. In this report, we will reflect on our teaching practices over a nine-year period, including the preparation phase, and report our findings.

Key words: university mathematics education, mathematical modelling education, humanities and social sciences students, math anxiety

INTRODUCTION

Mathematics supports our lives in various areas. Mathematical models and statistics are becoming important tools in many research fields. From the beginning of this century, there has been a growing emphasis on fostering students' ability to use mathematics in problem-solving situations in the real world. The Programme for International Student Assessment (PISA) study by the Organisation for Economic Co-operation and Development (OECD) is the most famous and has affected primary and secondary school curricula worldwide. Although the term "mathematical literacy" is used in PISA's assessment framework (OECD, 2017), the emphasis is on the mathematical modelling process and its activities. At the university level, mathematics education for non-specialists has received considerable attention in recent years. As such, mathematics education for engineering students has reveived attention mainly in the last century. Recently, mathematics education for students in other majors has also gained attention. In European countries, mathematics in other disciplines, such as engineering, physics, biology, economics, etc., is being studied (Alpers et al., 2013; Feudel, 2018; Gonzalez-Martin & Hernandes-Gomes, 2019; Viirman & Nardi, 2019, etc.). In the United States and Japan, such education is often discussed in the context of general education; hence, mathematics education is discussed not only for students in science, technology, engineering, and mathematics (STEM) majors but also for students in social sciences and humanities majors. In the United States, the term "quantitative literacy" (QL), which is also called "numeracy" or "quantitative reasoning" (QR), is used to refer to such education. It is defined as "a "habit of mind," competency, and comfort in

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working with numerical data" (Association of American Colleges and Universities [AAC&U], 2009) and the AAC&U (2009) provides the Quantitative Literacy VALUE Rubric for the assessment of QL. There are reports on QL and QR education (e.g., Rocconi et al., 2013), some of which deal with math anxiety among students in social sciences and humanities majors (e.g., Henrich & Lee, 2011). In Japan, it has often and long been claimed that all of university students, including humanities and social sciences students, should have mathematical abilities at a certain level. In the discussion on mathematics education for non-specialists, the term "mathematical literacy" has been used in Japan since the 1980s (Nagasaki & Abe, 2007). Subsequent to the OECD's PISA, it is used in the sense embodied in that framework, but it includes university mathematics education in the Japanese context. Very recently, the Japanese government launched a program to promote tertiary-level education in artificial intelligence, mathematical sciences, and data sciences, with the aim of increasing the number of experts in those fields and promoting a beginner-level education in those areas for all university students. Although university education systems differ among countries, they share the same emphasis on mathematical modelling activities in mathematics education for non-specialists.

In the contexts of the United States and Japan, mathematics education for students majoring in social sciences and humanities fields has been recognized as a challenging task. This is because many of these students are not good at mathematics, and some have math anxiety, making it imperative in mathematics education to reduce their dislike and fear. Fujii (1994) reported that social sciences and humanities students experienced more math anxiety than STEM major students did. There have been various approaches to address the issue of teaching mathematics to math anxious students. For example, in a study on the subject, Henrich and Lee (2011) incorporated a service-learning component into a QR course primarily taken by humanities majors. During our teaching practices from 2011 to 2019, we attempted to address this issue by developing design principles focused on mathematics education for humanities and social sciences students. In our previous studies (Kawazoe et al., 2013; Kawazoe & Okamoto, 2017; Kawazoe, 2019), we demonstrated that a course based on our design principles could increase students' interest in mathematics and their self-confidence in their mathematical thinking skills, and could change their view of mathematics. However, we have not investigated the course design from a long-term perspective. In this paper, we would like to extract knowledge about the design of mathematics education for humanities and social sciences students by reviewing our teaching practices over the past nine years. More precisely, we address the following research questions: (1) Is each item comprising our design principles sufficiently robust for long-term use?; (2) Regarding any modifications to the design principles, what changes were made, and why?

DATA AND METHODOLOGY

The mathematics course investigated by the present study consists of two successive one-semester mathematics classes, a spring semester class and a fall semester class, for humanities and social sciences students in their first year at a Japanese university. Both are two-credit classes that meet for 90 minutes each week for 14 weeks, followed by an examination period. The course began in the 2012 academic year, after a one-year pilot course in the 2011 academic year. We stored various data concerning the course during the period 2011 to 2019, including syllabi, lesson plans, paper-based worksheets used in each lesson, homework (paper-based, online), students' self-reflections after each lesson and at the end of each semester, questionnaires administered at the beginning and end of each semester, and the log data of e-learning materials. We conducted a retrospective analysis with these data, chronologically describing our teaching practices over the past nine years, including the preparation phase.

REFLECTION ON TEACHING PRACTICES INCLUDING THE PREPARATION PHASE

In 2010, our university decided to offer mandatory mathematics and statistics courses to humanities and social sciences students beginning in the 2012 academic year. Course development started in 2010. The author contributed to the development of the mathematics course. In the following subsections, we reflect on the mathematics course teaching practices, including the preparation phase.

Course preparation: Oct. 2010–Mar. 2012

In the preparation phase, the author, as a mathematician, partnered with a cognitive psychologist to develop the course. We developed a mathematics course consisting of a spring semester class and a fall semester class. Each class was designed as a two-credit class that would meet for 90 minutes each week for 14 weeks, followed by an examination period.

Determining the course objective: The course objective was determined according to the following four factors: social demand, the demand from the departments to which the mathematics course is offered, mathematics teachers' opinions, and students' mindset (i.e., their attitudes toward mathematics and mathematical knowledge). To meet social demand, mathematics education should be offered to foster students' ability to use mathematics in real-world situations. Regarding demand from within the departments, students need to acquire sufficient mathematical knowledge for learning about statistics and the mathematical models used in their disciplines. Considering these requirements, the course objective was to develop basic mathematical knowledge and appropriate attitudes for the application of mathematics in the social sciences as well as in real-world situations.

Designing the course: There were two main ideas regarding course design. The first was the order of instruction. In a traditional mathematics class, definitions, formulas, exercises, and applications appear in this order. We changed this order and designed each lesson to begin with an applicational problem in a real-world situation. Mathematics is invisible at the beginning; it only appears after mathematizing the problem. Mathematical exercises come last. Initially, a real-world problem is presented, with the aims of attracting students to the lesson, communicating the importance of mathematics in the real world, and familiarizing students with the mathematical modelling process (Figure 1). The second idea concerned the teacher's language during instruction. Many students in the humanities and social sciences do not have a sufficient understanding of teachers' mathematical explanations in class. Our idea for improving this situation was to explain mathematical concepts and procedures in both mathematical language and everyday language. During the pilot course in the 2011 academic year, we examined the effectiveness of teaching materials and instruction methods and improved them based on our findings.

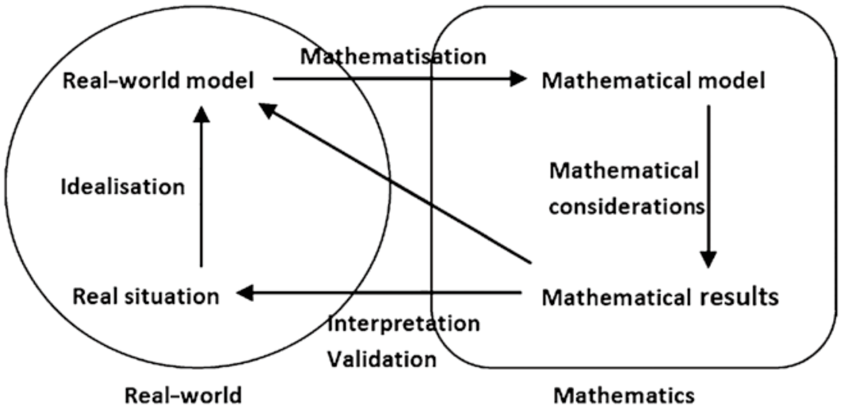


Figure 1: Modelling process from Kaiser-Meßmer (1986) and Blum (1996)

Design principles developed during the preparation phase

The following design principles were developed based on the result of the pilot course (Kawazoe et al., 2013; Kawazoe & Okamoto, 2017):

(1) *Design lessons based on the mathematical modelling framework*: In a lesson designed based on this principle, a real-world problem is presented at the beginning. Then, students engage in mathematization activities and solve the given problem by using mathematics. Following this principle, the course is implemented as a mathematical modelling education.

(2) *Choose topics and contexts with consideration of the mathematical knowledge students encounter in their real life and the situations in which they encounter it*: According to this principle, problems presented during classroom activities should be connected to everyday contexts, so that students can understand when, how, and to what the mathematical knowledge is applicable in real-world situations.

(3) *Connect mathematical language to everyday language*: It is necessary to explain topics in both mathematical and everyday language in order to help students who have difficulty understanding mathematical language comprehend mathematical concepts and methods as well as their value.

(4) *Engage students in group rather than individual activities*: Most students view mathematics learning as individual learning, and they fear it because they are not confident about solving mathematical problems by themselves. This principle aims to reduce students' math anxiety and increase their motivation to learn mathematics.

(5) *Present problems in multiple contexts associated with the same mathematical knowledge*: Mathematics instruction using a real-world situation is sometimes criticized because students' learning is limited to the situation or context used in class. This is the so-called *the situatedness of learning* (cf. Sfard, 2014), which this principle aims to overcome.

(6) *Connect different mathematical knowledge by using different mathematizations of the same problem or by using mutually related contexts*: This principle aims to promote the structurization of mathematical knowledge in students' minds.

(7) *Design paper-based worksheets that follow students' comprehension process and use them as tools for formative assessment*: A course developed with the above principles would be more effective if students'

activities were driven by well-designed worksheets. Worksheets are used to assist students' understanding and also help students assess their learning process by themselves.

As stated in our previous studies (Kawazoe et al., 2013; Kawazoe & Okamoto, 2017), the above principles originated from the four perspectives of learning environments developed in learning science (Bransford, 2000, Chapter 6): student-centered, knowledge-centered, assessment-centered, and community-centered (Figure 2). Principles (1), (2), (5), and (6) originated from discussions about mathematical literacy (cf. Sfard, 2014) and mathematical modelling (cf. Kaiser, 2014). Principles (5) and (6) are aimed at *decontextualizing* and *structuralizing* students' knowledge, respectively, because teaching mathematics in real-world contexts has been found to restrict students' knowledge to the current learning context (cf. Sfard, 2014). Although the above principles do not explicitly state this, the active use of information and communications technology (ICT) tools to perform calculation is also important in our course design.

Figure 2: Four perspectives of learning environments (Bransford et al., 2000, p.134, Figure 6)

Course topics

The topics taught in the spring semester class are as follows: systems of linear equations/inequalities (linear programming), number sequences (savings, loan payment, pharmacokinetics, model of addiction), matrices and vectors (spreadsheets, social network analysis), functions (mental rotation, pharmacokinetics, bacterial growth, pandemic, periodic movement of electric demand), and probability (lottery, disease examination, birthday paradox, Bayesian estimation). The topics taught in the fall semester class are as follows: eigenvalues/vectors (population dynamics, optimal distribution), derivatives (innovation diffusion, population growth, logistic function, marginal profit, optimization), integrals (speed and distance, accumulated radiation level, standard normal distribution), and multivariable functions (loan simulator, formulas for estimating vital capacity).

We provide some examples of the problems presented in the course. The following examples were developed during the pilot course.

Problem 1. You have begun to take hay fever medication. You have to take one tablet every 12 hours. If you continue to take this medication over a long period, what amount of fexofenadine hydrochloride, the medication's main component, will remain in your body? Consider the amount of fexofenadine hydrochloride immediately after taking the drug every 12 hours. Each tablet contains 60 mg fexofenadine hydrochloride, and approximately 40% of fexofenadine hydrochloride remains in the body after 12 hours.

Problem 2. Due to the spread of foot-and-mouth disease in Miyazaki Prefecture in Japan in the spring of 2010, a large number of livestock were killed. The following table summarizes the changes in the cumulative number of livestock suspected to have the disease over 18 consecutive days, starting from April 20, 2010. Analyze the spreading of the disease by using a semi-log plot, and create a mathematical model of it by using an exponential function.

Day	1	2	3	4	5	6	7	8	9
The number of suspected livestock	16	202	266	386	386	1111	1111	1111	2893
Day	10	11	12	13	14	15	16	17	18
The number of suspected livestock	2943	4416	8298	9056	9096	28720	35247	47556	64827

Problem 3. You have tested "positive" for a certain disease. One in 1,000 people is said to have this disease. The test is said to be 99% accurate; that is, 99% of people who have the disease test positive, and 99% of people who do not have it test negative. Estimate your risk of having the disease.

Problem 4. O University has decided to start a bicycle rental service because of its huge campus. Two parking areas are located beside the main gate Nakamozu-mon (Parking Area A) and in front of lecture building B3 (Parking Area B). Students can drop bicycles off in either of the two parking areas. A monitoring investigation showed that bicycles in both parking areas move as shown in the following table every week. The numbers in the table are displayed as percentages. How should bicycles be divided between the two parking areas when the service starts?

	To Parking Area A	To Parking Area B
From Parking Area A	70	30
From Parking Area B	20	80

Problem 1 is presented in a lesson on the application of number sequences. Problem 2 is presented in a lesson on the application of functions (focusing on exponential growth). Problem 3 is presented in a lesson on probabilities. Problem 4 is presented in a lesson on eigenvectors and eigenvalues. It should be noted that Problem 4 is presented to students before they learn eigenvectors and eigenvalues. The lesson using Problem 4 is designed so that students can learn eigenvalues and eigenvectors together with their applications in the classroom activity in order to solve the problem. Similar to Problem 4, some problems are used to teach new mathematical concepts.

Teaching practices for the regular course: Apr. 2012–Feb. 2020

The course started in the 2012 academic year with four classes in each semester. Four mathematics teachers, including the author, taught the classes. The spring semester class is mandatory for all humanities and social sciences students, and approximately 300 students take the class every year. Until the 2017 academic year, the fall semester class was mandatory for students in economic majors. Beginning in the 2018 academic year, only the spring semester class was mandatory for all students. We have been trying to grasp the characteristics of the classes using a questionnaire survey administered at the beginning of the spring semester. For example, the 2018 results showed that about 30% of students who took the course had been in the science-oriented course in high school, while the others had been in the humanities-oriented course. More than half indicated that they were not good at mathematics, and about 40% indicated that they did not like mathematics. The percentages may change from year to year, but this is the average trend.

An Issue encountered at the start of the course and our improvement strategy: When the course began in 2012, we had to consider methods for sharing lesson ideas and tips among four teachers. We developed a textbook (Kawazoe & Okamoto, 2012) and paper-based worksheets for use in all classes, but these did not seem to be sufficient. We decided to develop a document of "lesson plan" for each lesson, which is popularly used in primary and secondary schools in Japan, and to use it as a tool for sharing ideas and tips of a lesson. Using lesson plans, we could control the quality of the classes.

During the eight-year practices in this period, most of the course topics did not change, except for that the Lagrange multiplier method and some other topics were added in the 2015 academic year. On the other hand, students' activities, homework, and assessment tools have changed.

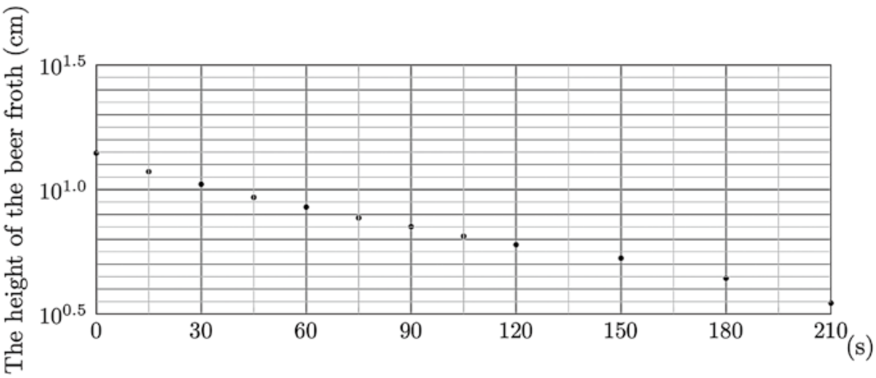
Adding new topics: In 2014, the faculty of economics requested that the course incorporate additional topics. The faculty wanted their students to be exposed to determinants, logic, and the Lagrange multiplier method in their first year. Regarding determinants and the Lagrange multiplier method, they said that it was enough to teach determinants of 2×2 matrices and the Lagrange multiplier method applied to an easy example of functions with two variables. We added exercises on determinants and logic to a worksheet on systems of linear equations and a worksheet on probabilities as optional exercises, respectively, beginning in the spring semester of 2015. Regarding the Lagrange multiplier method, we developed a new 90-minute lesson using the following problem.

Problem 5. Many buildings have water tanks on their roofs, some of which are cylindrical tanks. When looking at a cylindrical tank from the side, it often appears to be a square. In order to consider the reason, solve the following question: "You want to design a 2,000π m^3 cylindrical tank, minimizing the surface area to save materials. Find the height and the radius of the tank."

Problem 5 was created based on an example from a textbook written by Kawanishi (2010). We believe that this easy problem is suitable for introducing the Lagrange multiplier method, although the problem can be solved without use of the method. The lesson was implemented as the final lesson in the fall semester, replacing a 90-min lesson devoted to learning formulas for integration by substitution and by parts. With this change, lessons on integration have put more emphasis on the meaning of integration rather than on calculation. This change has been applied since the 2015 academic year.

Continuous development of new problems: As mentioned in the design principles, we believe that it is important to present problems in multiple contexts. We developed many problems in multiple contexts during the preparation phase, but for some topics, the variety of contexts was restricted. Moreover, problems taken from real-world situations are often associated with a specific time period, and in particular, problems that entail predicting the future lose their connection with reality after the passage of the year indicated in the problem. In addition, we must create new problems with different contexts for the final examination in each semester because students can access the problems given in past examinations. Hence, we have attempted to continuously develop new problems. Here, we present an example. Similar to Problem 2, Problem 6 is on a semi-log graph, but it is in a different context. Problem 6 was used in the final examination in the spring semester of 2019.

Problem 6. It is known that the volume of beer froth decays exponentially with time. Physicist Leike (2002) confirmed this fact by measuring changes in the height of beer froth in a cylindrical glass over time and was awarded the 2002 Ig Nobel Prize[1] in physics. The graph shown below is a semi-log graph representing the results of an experiment using Augustinerbräu München. The horizontal axis represents time (s), and the vertical axis represents the height (cm) of the beer froth in the glass. Answer the following questions: (1) Explain why the height of the beer froth can be considered to decrease exponentially, based on the graph, and (2) create a mathematical model representing the height (cm) of the beer froth at the time t (s) by using an exponential function.



Creating new problems for final examinations is a very hard task for teachers, and it gets harder year by year. Hence, it can be said that creating new problems for final examinations is related to the sustainability of the course.

Changing the balance between group and individual activities: In the early years, we implemented individual and group activities during each lesson. Each teacher has their own method for creating groups. Usually, each group consists of three to four students, but some teachers prefer students to work in pairs. Although we encouraged students to interact even during individual activities, it did not go well because students' mathematical abilities differed. Very capable students could finish the task in a short time, while less capable students could not understand how to do and could not finish the task in time. We then gradually reduced individual activities and increased group activities in each lesson. We also tried to change the way groups were organized so that capable students were distributed across all the groups. In order to accomplish this, groups are organized based on the results of a questionnaire administered at the beginning of the course in which students are asked about their attitudes toward mathematics and their high school learning history. Then, group activities became to be aimed at helping all students understand the meaning of mathematical procedures. Capable students are expected to teach others in a group setting. Individual activities are assigned as homework. We verified the effectiveness of group activities by analyzing the results of a self-report questionnaire administered at the end of the spring semester of 2019. The questionnaire asked the students to write up to three of the most important things that they learned during the semester. In response to this question, 15 of the 70 students in the author's class mentioned group activities or learning with others as one of the three important things that they learned. We also asked them to freely write their thoughts and

[1] The Ig Nobel Prize is a parody of the Nobel Prize. It is awarded annually since 1991. On its official website, the award is said to honor "achievements that make people LAUGH, then THINK."

comments about the class. Sixteen students described group activities as effective in their comments. The following are examples of students' comments pertaining to group activities:

Student 1: By exchanging opinions during group work and so on in order to derive formulas, I could gain a deep understanding of mathematics.

Student 2: By solving in a group, I got explanations from others about things that I could not understand, I felt that there were few cases where I could not understand something during the class, compared to when I was in high school.

Student 3: In mathematics, I found that I get more creative thought and have more fun when we solve in a group than when we do it alone.

(These comments were written in Japanese and translated into English by the author.)

The above comments indicate that increasing the frequency of group activities contributed to reducing the number of students who could not understand the class.

Implementing e-learning homework: In the early years, homework was paper-based, consisting of modelling tasks and basic computational tasks. In the 2018 academic year, we developed e-learning materials consisting of basic computational tasks in order to enrich students' after-class learning and increase its effectiveness. We have been using these materials since then. E-learning materials have been implemented on Moodle, and some have been implemented as STACK question type (The STACK project, 2020). Students can access the materials at any time and receive the immediate feedback. Students can try them multiple times before the deadline, and the highest score is adopted. Regarding e-learning usage, it has been observed that students made repeated attempts until they succeeded, and the percentage of reviews in students' after-class learning has increased. Examples of e-learning materials are shown in Figure 3 and 4.

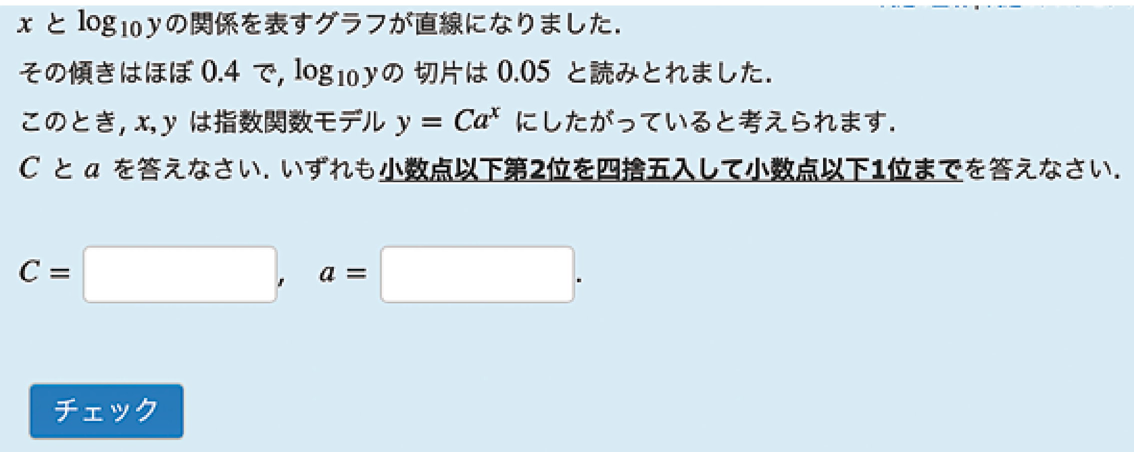


Figure 3: An example of e-learning material: "Assume that a graph representing the relation between *x* and $\log_{10} y$ is a straight line. Its slope is about 0.4 and a $\log_{10} y$-intercept is read as 0.05. Then, the relation between *x* and *y* can be modeled with an exponential function $y = Ca^x$. Find *C* and *a*."

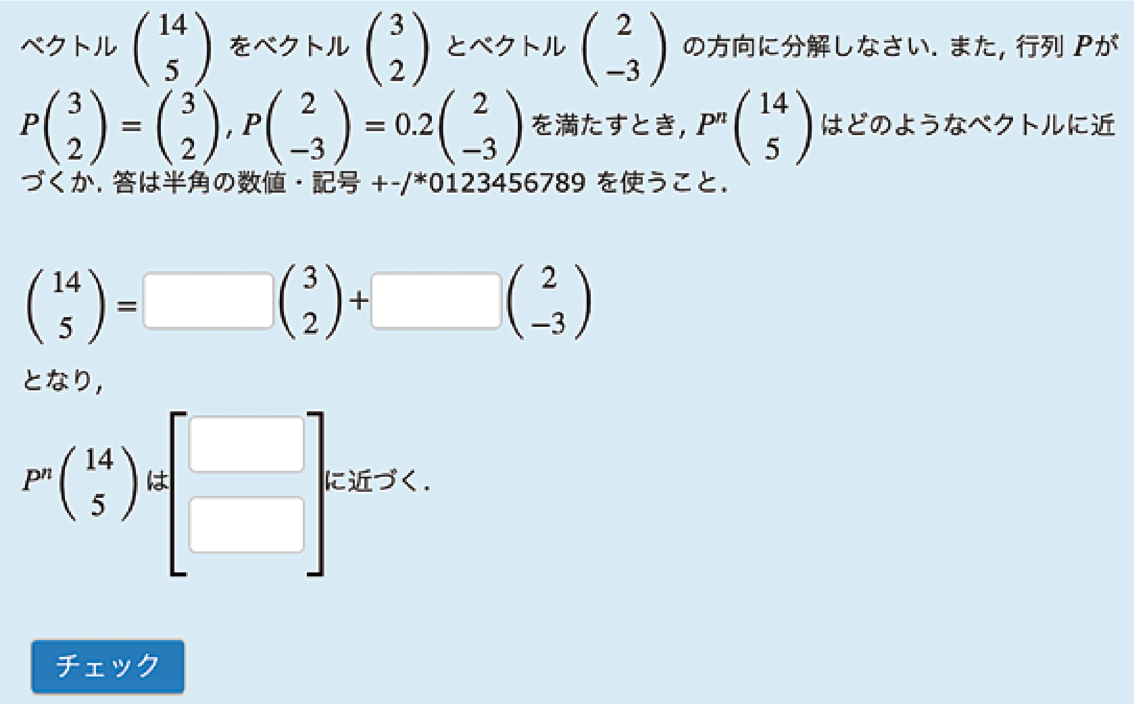


Figure 4: An example of e-learning material: "Decompose a vector $\begin{pmatrix}14\\5\end{pmatrix}$ into two directions given by $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-3\end{pmatrix}$. When a matrix P satisfies $P\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$, $P\begin{pmatrix}2\\-3\end{pmatrix}=0.2\begin{pmatrix}2\\-3\end{pmatrix}$, what vector does $P^n\begin{pmatrix}14\\5\end{pmatrix}$ get close to when n becomes larger?"

Evolution of assessment tools: Students' mathematical knowledge can be assessed using a paper-based test. To evaluate teaching practices, it is important to assess changes in students' attitudes and their mathematical habit of mind. We developed a rubric that we have been using as a self-evaluating tool for students since the 2015 academic year. Development of an effective assessment tool is still in progress (e.g., Kawazoe, 2019) and remains a challenging task.

Another sustainability issue: In 2018, we faced a serious problem due to the retirement of one of the four teachers. We could not find another suitable teacher within our university, so three teachers had to teach the four classes in each semester until a new teacher was hired in April in 2019. This made us aware that teacher development is another important issue for the sustainability of the course.

FINDINGS FROM TEACHING PRACTICES AND CHALLENGES FOR THE FUTURE

As a result of the nine-year practices described in the previous section, we obtained the following findings. First, our design principles and topics were proved to be viable in the long run. As stated in the previous section, the basic framework of the course and the content of each lesson have remained almost the same since 2012, except for the change due to requests from the faculty of economics. Although our lesson design was very different from mathematics classrooms in Japanese high school, students' self-reflections collected via worksheets showed that many were satisfied with exposure to real-world problems to which mathematics can be applied. Many students who were not good at mathematics described a reduction in their negative feelings about mathematics in their self-reflection comments at the end of the semester. Second, it was found that a lesson plan is a useful tool for sharing ideas and tips of lessons among teachers. The main component of a lesson plan is a timetable containing instructions for students' activities, contents on black board, and teacher's questions. Lesson plans have helped three teachers, other than the designer of the course, conduct their own classes smoothly. Additionally, when a new teacher joined as a lecturer of the course in 2019, lesson plans helped him start his first year of teaching. This confirms that lesson plans help a course designer to make other teachers understand ideas regarding lesson aims and strategies for asking questions in a classroom. Third, group activities were found to be more effective than we previously thought. This is evidenced by the growing number of comments from students in their self-reflections about the advantages of group activities since the year when we started to increase group activities. Until the 2019 academic year, groups were fixed during the semester. In the fall semester of 2020, the author reorganized the groups on a trial basis, but due to the spread of the novel coronavirus (COVID-19), the class shifted to an online class, preventing verification of the effect of reorganizing groups on students' learning, which remains a future task. Fourth, the use of e-learning was found to enhance students' after-class learning. When computational tasks are given as paper-based homework, the number of problems is restricted owing to the space constraints of the paper and the time constraints of the teachers who grade them. Additionally, in the case of paper-based homework, students cannot receive immediate feedback because homework is submitted in the next week's class and is not graded and returned until a week later. With e-learning, it is possible to randomly assign problems from a vast problem pool or to randomly generate equations in a problem. Furthermore, students can repeat exercises as many times as needed and receive immediate feedback. From the teachers' perspective, using e-learning materials for computational tasks allows them to focus on grading tasks that assess students' thinking process, such as modelling tasks. These four findings are positive.

However, two issues need to be addressed. One is the need to develop an effective tool for assessing changes in students' attitudes and mathematical habits of mind. The author conducted a pilot study (Kawazoe, 2019) to assess these by analyzing students' self-reflection comments collected via worksheets. The results of the study indicated that students' awareness of the essence of mathematical thinking and knowledge could be found in their comments. However, extraction requires careful perusal of students' weekly comments, which is a hard work for teachers. Further studies are needed to determine whether this method can be established as an evaluation method and whether the evaluative efficiency can be improved. The other issue is sustainability, which contains two sub-issues. One is the need to continue creating new problems, especially for final examinations. This is getting harder each year, and we have not yet been able to find a solution. The other is the need to develop mathematics teachers who can teach the course. It is very difficult to find mathematics teachers who can teach the course presented in this paper. The main reason for this is that almost all mathematics teachers in Japanese universities are pure mathematicians. Establishing strategies for developing mathematics teachers who can teach such courses is an emergent task pertinent to the sustainability of the course.

Finally, we note on our teaching practices in the 2020 academic year amidst the COVID-19 pandemic. Our university requested teachers to give their lectures in the spring semester on an on-demand online basis. The course presented in this paper was heavily affected by it, because group activities cannot be implemented in an on-demand video lecture format. We were apprehensive of students' reactions to online course delivery. Their comments revealed a strong desire to attend classes in person on campus, but on the other hand, students were very satisfied with the online course. Based on students' comments on each lesson, it was observed that they watched the videos repeatedly until they felt that they understood the content and that they engaged themselves in exercises at their own pace. These observations suggest that if we can give students enough time to perform exercises, they will be able to learn without math anxiety, even in an environment where only individual activities are available. This finding may provide an important insight into developing strategies for implementing group activities and individual activities in the classroom.

Acknowledgments

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TEACHING MATERIALS ON CALCULUS AS SEEN FROM THE APPLICATION TO ENGINEERING

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Abstract

The authors have developed teaching materials on calculus including multivariable functions for first-course undergraduate students in the science and engineering fields. The materials differ from commonplace textbooks in that they first introduce the topics encountered in engineering and then explain the mathematical aspects. This style aims to help readers understand mathematical concepts smoothly by identifying their interests and offering topics that appeal to intuition.

Key words: University mathematics education, teaching materials, multivariable functions, application to engineering, intuition

INTRODUCTION

Although many calculus textbooks have been published in Japan, only a few explain the concept of calculus in the context of applications such as engineering. This is especially true for the calculus topic multivariable functions. Generally, textbooks play an important role in learning. However, several researchers have pointed out the necessity of reconsidering the role of textbook usage in university classes. For example, Berry, Cook, Hill, and Stevens (2010) investigated textbook usage in finance classes and pointed out that, although students recognize the importance of reading the textbooks, most still do not spend sufficient time doing so. Randahl (2012) investigated an actual class to determine the characteristics of first-year engineering students' approach to using calculus textbook. The results showed that students preferred to use lecture notes rather than the textbook and that students increasingly tended to ignore the textbook, except in relation to the tasks the textbook offers. Randahl identified students' difficult experience of textbooks' formal treatment of concepts as one of the reasons for this trend.

Teachers might have to take special care of students because first-year engineering students may have serious difficulties due to big differences between learning mathematics in college and school. Robert and Schwarzenberger (1991) considered the differences between elementary mathematics in compulsory education and specialist mathematics at colleges and universities, highlighting considerable changes in the nature of mathematical content being taught. Concepts also differ substantially from what students previously experienced, often involving not only generalisation but also abstraction and formalisation (see also Tall, 1991; Dreyfus, 1991). It is hard to imagine that the engineering students discussed here can understand such

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abstractions and generalisations in the same way as mathematicians do. Robert and Schwarzenberger (1991) wrote that "The change in the ratio of quantity of knowledge to available acquisition time means that it is no longer possible for the student to learn all new concepts in class time alone; significant individual activity outside the mathematics class is now an absolute necessity." (Robert & Schwarzenberger, 1991, p. 128). In such situations, some students may rely on textbooks. However, if the textbooks provide formal definitions of calculus concepts and theorem proofs monotonously, students will have difficulty with concept formation because of their limited knowledge and ability. First-year engineering students relying upon such textbooks need sufficient mathematical and modelling skills to understand and smoothly apply theoretical calculus. However, this is an unreasonable expectation.

It is known that treating formal definitions of the concepts in teaching mathematics is difficult (e.g. Vinner, 1991; Cornu, 1991). Dreyfus (1991) mentioned that it is insufficient to merely define and exemplify an abstract concept. Raman (2002) illustrated the difficulties students encounter when coordinating the formal and informal aspects of mathematics. The gap between the two approaches taken by precalculus and calculus textbooks, namely emphasising the informal aspect at the expense of the formal aspect or the reverse, respectively, complicates coordination. Tall and Vinner (1981) described concept image as the total cognitive structure including all mental pictures and related properties and processes. Additionally, they described concept definition as a form of words used to specify the concept. Textbook content should enrich the concept image so that students understand formal definitions of concepts. Bingolbali and Monaghan (2008) pointed out that environments such as students' departmental affiliations can influence the acquisition of concept images, in the context of research on undergraduate students' understanding of the derivative. Harris et al. (2015) analyzed engineering students' problems with mathematics through interviews. Those scholars challenged both the pedagogical practice of teaching non-contextualized mathematics and the lack of transparency regarding the significance of mathematics in engineering. Moreover, some previous researchers pointed out that practical contexts or situations which could clearly show the necessity of extending the existing knowledge are absent and that strict and pure mathematical contexts cause the situation in which students experience textbooks as difficult to use (e.g. Randahl, 2012; Randahl & Grevholm, 2010).

The authors are currently preparing calculus teaching materials for use in classes. This paper specifies some requirements imposed on mathematics textbooks for engineering students and offers examples of teaching material pertaining to both partial derivatives and area integrals intended to aid students' concept formation of multivariable functions by employing topics in practical contexts. The use of such materials in calculus classes is also discussed.

We assume the following minimum requirements for mathematics textbooks for engineering students:

1. Introduce the concept not only with a formal definition but also by providing practical contexts that represent the characteristics of the concept.
2. Introduce the concept in practical contexts before providing a formal definition.
3. The practical contexts include the reasons for the necessity of extending students' knowledge.

The first and third requirements reflect previous research results that have pointed out the difficulties associated with providing a formal definition without practical contexts or situations. This has demonstrated the necessity of extending students' present knowledge, and it will help calculus teachers who are unfamiliar with the use of calculus in engineering. Preferable mathematics textbooks should include both formal definitions of concepts and practical contexts. If a formal definition of a concept is introduced before the practical context is given, most students will experience difficulty with concept formation, and their comprehension motivation might decline. Teachers should try to facilitate students' understanding of these concepts. Therefore, we adopted the second requirement, which will lead students to abstract and formalize concepts.

On the other hand, from the standpoint of an engineering course that the second author took, calculus is an extremely important topic in university-level mathematics. Derivatives and integrals of multivariable functions are new units for first-year engineering students, and understanding of these concepts is poor for the majority, even though these units are extremely important fundamentals of subsequent subjects in their course of study. Therefore, we prepared calculus teaching materials including derivatives and integrals of multivariable functions, for engineering students.

In this paper, we introduce some of our teaching materials concerning derivatives and integrals of multivariable functions. The materials fully utilize geometry, which is commonly used to explain functions. This is intended to minimize the complexity and, therefore, avoid the confusion associated with introducing dimensions other than length into calculus. For partial derivatives, the geometry of a river-bank, which is familiar to many people, is taken as an example to assist concept acquisition. Subsequently, Taylor series and total derivatives are introduced in relation to the topic of derivatives as employed in fluid mechanics, which students encounter in mechanical, civil, and chemical engineering studies. For integrals of a multivariable function, two examples of area integrals of the first and second moments of the area are introduced. The area integral is the basis for determining the center of an area. We were unable to find such an application in the ordinary mathematics textbooks that we perused. The first and second moments of the area arise as topics in hydraulics and structural engineering, which are subjects that mechanical and civil engineering study.

EXAMPLES OF OUR TEACHING MATERIALS

Partial derivatives

Partial derivatives are defined in the context of multivariable functions as the rate of increase in functions with respect to one selected independent variable with other independent variables fixed. One of the most effective ways to understand the concept of partial derivatives is to consider the geometry of a 3-dimensional body. If we consider a 3-dimensional body located in the x, y and z spaces, we can imagine an edge formed by cutting the body by an arbitrary plane. Any edge formed in this manner can be generally expressed as a curve in a 2-dimensional space. In this case, the partial derivative can be evaluated with respect to the variable whose axis is taken horizontally in the cutting plane. Here, we attempt to explain partial derivatives through the function expressing straight lines formed from a plane in 3-dimensional space as a special case.

Let us consider a river that conveys flood water in a channel (Jain, 2001) and a dike with a flat slope with a tilt angle θ built on the river's horizontal high-water floor, as shown in Figure 1. If the x-axis is taken along the line where the dike slope and high-water floor cross each other, and the y-axis is taken in the perpendicular horizontal direction, the height of the surface of the slope z measured from the high-water floor is expressed by a two-variable function as $z = f(x,y)$.

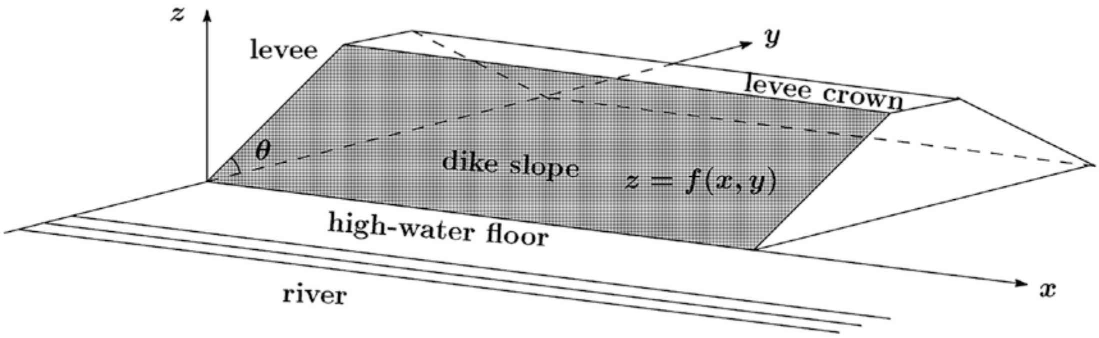
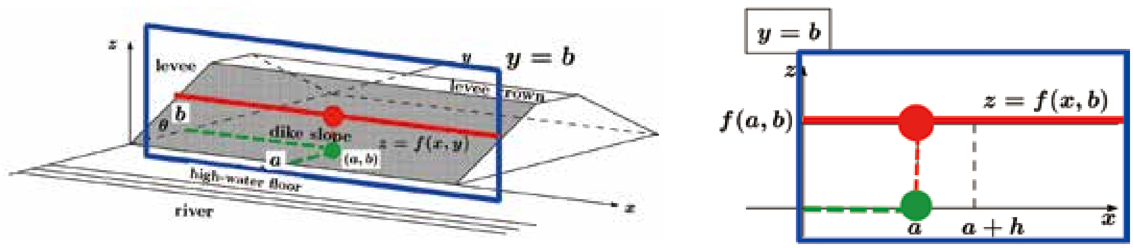


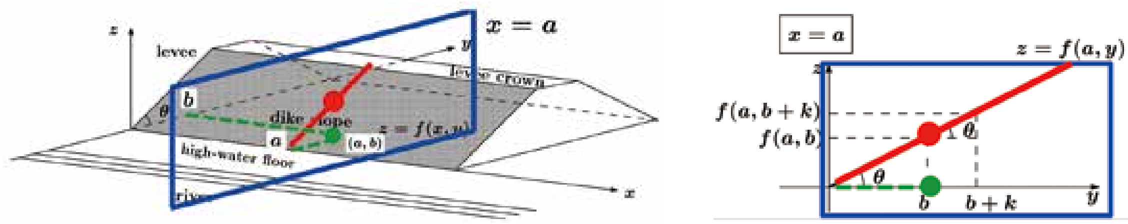
Figure 1: A bird's eye view of a bank with flat dike slope.

In this situation, it is clear that the partial derivative of z with respect to x is expressed as $\partial z/\partial x = 0$ and that the partial derivative of z with respect to y is expressed as $\partial z/\partial y = \tan\theta$ (see Figure 1). We believe that Figures 2 and 3 enhance understanding of these situations.



(a) Bird's eye view showing cross-section $y = b$. (b) Cross-section cut by $y = b$.

Figure 2: Cutting the dike slope by the plane $y = b$.



(a) Bird's eye view showing cross-section $x = a$. (b) Cross-section cut by $x = a$.

Figure 3: Cutting the dike slope by the plane $x = a$.

Next, let us consider the change in the value of the dependent variable z when the independent variables of x and y change, which is expressed by Taylor series. In the case of the situation indicated in Figure 1, the increase in the dependent variable z, dz, owing to infinitesimal increases in the independent variables x and y, dx and dy, is expressed using Taylor series as follows:

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy + O(dx^2 + dy^2)$$

where $O(dx^2 + dy^2)$ is the component of dz due to the derivatives of the second-order and higher, which expresses the effect of surface unevenness and is 0 in this problem. Therefore, the above expression is written as follows for the case shown in Figure 1.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

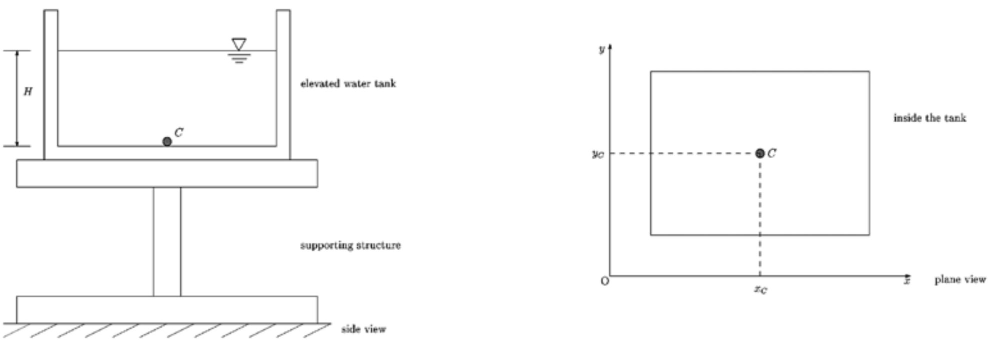
In this case, the above expression shows the height increment when a person walks on the slope via dx in the x-axis direction and via dy in the y-axis direction.

It should be emphasized that Taylor series' connection to total derivatives is an extremely important basis for studying engineering course subjects. In fluid mechanics, physical quantities indicating fluid properties are treated as functions of time t and the position in space, x, y and z. Additionally, the concept of a fluid particle is the minimum unit of the fluid, which is a conceptual matter meaning an infinitesimal element constituting the fluid and moving around in the flow. The time rate of change in physical quantities of a fluid particle, called material derivatives or substantial derivatives, is expressed using total derivatives. This concept is necessary to define the acceleration of flow and is one of the most important basic concepts of fluid mechanics.

Area integrals 1

Let us consider an elevated rectangular parallelepiped water tank with a horizontal bottom supported by a supporting structure, as shown in Figure 4. To maintain a stable condition, it is necessary to support the tank with a structure that can bear the total weight of the water at the position selected so that the central axis of the vertical part of the supporting structure lies below the line of action of the total gravity force exerted on the water, where the weight of the tank is considered to be negligible compared with the weight of water. This problem requires calculating the area integrals. First, the force that the structure should support is determined by integrating the weight of the water on an infinitesimal area of the bottom of the tank over the total area of the bottom. When a force acts normally on an area, the force per unit area is generally called "pressure" and is usually denoted by the character p. If we set the weight of the water per unit volume as w, the water depth in tank as H, and the area of the bottom of the tank as A, the force acting vertically downwards on the infinitesimal area of the bottom of the tank dA due to the pressure p is given by $p\,dA = wH\,dA$. Therefore, the force F the structure should support is given using the area integral, as follows:

$$F = \int_A p\,dA = \int_A wH\,dA = wH\int_A dA = wHA$$



(a) Side view of the entire system. (b) Plan view of the bottom of the tank.

Figure 4: Elevated water tank supported from below.

Second, the position of the center of the supporting structure is determined. Another requirement to ensure the stability of the system is that the axis of the center of the vertical part of the supporting structure should be aligned with the line of action of the total gravity force *F* indicated above. This condition is obtained by evaluating the moments about the *x* and *y* axes, as indicated in Figure 4. Force *F* expressed by the above equation should be considered as an equivalent concentrated force acting on a point, with the restriction that the moment due to force *F* about an axis is identical to the sum of the moments about the same axis due to the pressure acting on the area in a distributed manner. Let us consider the moment of the force about the *x*-axis using Figure 4. If we apply the force due to the pressure *p* on an area element *dA* located at a distance *y* from the *x*-axis as $dF = p\ dA = wH\ dA$, the moment of the force due to this force about the *x*-axis is expressed as $dM_x = dF \cdot y = p\ dA \cdot y = wH\ dA \cdot y$. Therefore, the sum of the moment dM_x for all components of the infinitesimal areas comprising the total area of the tank bottom is as follows:

$$M_x = \int_A py\ dA = \int_A wHy\ dA = wH \int_A y\ dA$$

From the above discussion, the quantity indicated by M_x must be identical to the product of $F = pA = wHA$ and some distance y_c from the *x*-axis (see Figure 4):

$$wH\int_A y\ dA = wHA \cdot y_c$$

The above expression reveals the following relationship:

$$\int_A y\ dA = A \cdot y_c \quad \text{or} \quad y_c = \frac{1}{A}\int_A y\ dA$$

The term indicated by integration in the above equations is called "the first moment of an area *A* with respect to *x*-axis", and y_c indicates the *y*-ordinate of the action point of total force $F = wHA$. The first moment of area *A* with respect to the *x*-axis is defined as the sum of the infinitesimal area of each area component *dA* multiplied by the first power of the distance between the component and *x*-axis. It is interesting to consider that the areas of the individual elements act as weights on the distance, which resembles the calculation of the mean value in statistics. The moment of force about the *y*-axis is considered in the same manner. The expressions corresponding to the forms in the former case are as follows:

$$M_y = \int_A wHx\ dA = wH\int_A x\ dA \quad ; \quad \int_A x\ dA = A \cdot x_c \quad ; \quad x_c = \frac{1}{A}\int_A x\ dA$$

The quantities x_c and y_c derived above indicate the ordinates of the centroid and the geometrical center of an area and are customarily written as x_G and y_G, respectively. That is,

$$x_G = \frac{1}{A}\int_A x\ dA \quad \text{and} \quad y_G = \frac{1}{A}\int_A y\ dA$$

Area integrals 2

Next, we consider the problem of hydrostatic pressure acting on an inclined plane submerged under the water surface, as shown in Figure 5. Similar to the former example, the water pressure at a point *y* vertically below the water surface is given by *wy* using the weight of the water per unit volume *w*. Consider a case where the inclined flat bottom of a body of water has a hole with the shape shaded, as shown in Figure 5, and the hole is covered by a hard plate with its same shape and area to prevent water outflow.

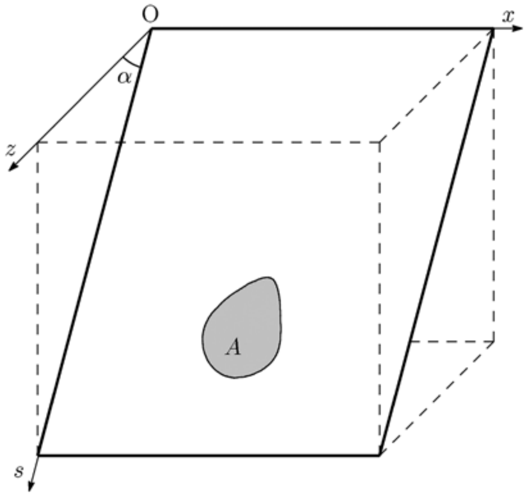


Figure 5: A plate covering a hole in the inclined flat bottom of a body of water.

The model for analysis of this problem is shown in Figure 6, in which the left side is a view alongside the flat bottom, and the right side is a view normal to the flat bottom. The shaded part in Figure 5 is exaggeratedly shown in Figure 6.

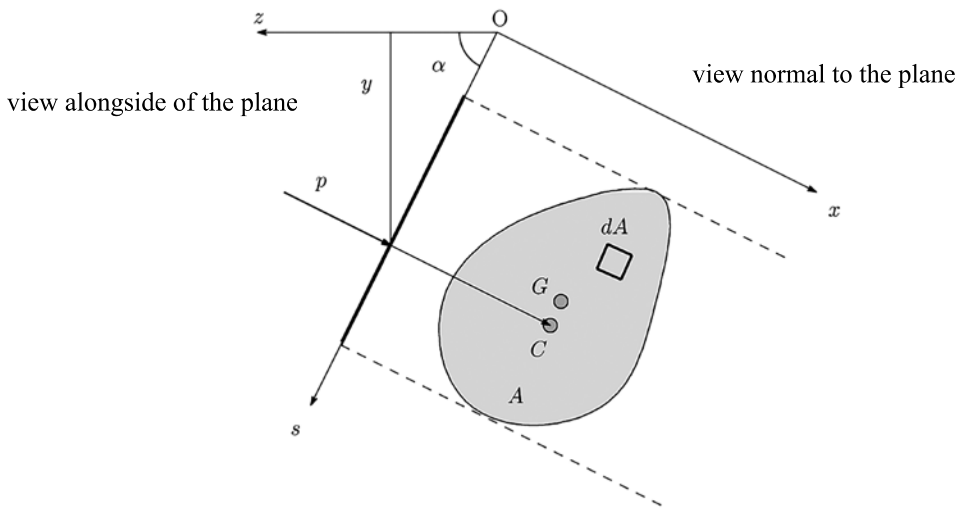


Figure 6: Model for analysis of hydrostatic pressure in Figure 5.

The water pressure at distance s taken downwards along the plane from the water surface is $p = ws \sin \alpha$. Therefore, the force due to the water pressure acting on the plane element of area dA at this position can be written as $p\, dA = ws \sin \alpha\, dA$. The total force F is the sum of $p\, dA = ws \sin \alpha\, dA$ over the entire surface of area A and is given by the following equation:

$$F = \int_A p\, dA = \int_A ws \sin \alpha \cdot dA = w \sin \alpha \int_A s\, dA = w \sin \alpha \cdot s_G A$$

Here, $s_G A$ is the first moment of area A with respect to the x-axis, and s_G is the s coordinate of the centroid of area A.

Next, we consider where to support plate A at the back to obtain a stable force balance. This point is referred to as the point of action of the total pressure and is usually indicated as C, while its coordinate is s_c. Let us consider the finding of s_c. When the moment of the force due to the water pressure acting on the surface element area dA at a distance s from the x-axis $dM = pdA \cdot s$ is summed over area A, the following form is obtained:

$$M = \int_A ps\, dA = \int_A \mathrm{ws}^2 \sin\alpha\, dA = w \sin\alpha \int_A s^2\, dA$$

Here, $\int_A s^2\, dA$ is the moment of inertia of area A in Figure 5 with respect to the x-axis. The area moment of inertia is also called "the second area moment", which is the sum of the infinitesimal area of each area component dA multiplied by the second power of the distance between the element and axis in question. Subsequently, from the relationship $M = Fs_C$, the following is obtained:

$$s_c = \frac{w \sin\alpha \int_A s^2\, dA}{w \sin\alpha \cdot s_G A} = \frac{1}{S_G A}\int_A s^2\, dA$$

If we define a new axis taken from the centroid s_G in the s direction as u the area moment of inertia expressed by the coordinate s is rewritten in the following form:

$$\int_A s^2\, dA = \int_A (s_G + u)^2\, dA = s_G^{\,2} A + \int_A u^2\, dA = s_G^{\,2} A + I_G$$

Here, I_G is the sectional moment of inertia about the axis passing through the centroid s_G of plane A. Therefore, we obtain the position of C as follows:

$$s_c = \frac{1}{S_G A}\int_A s^2\, dA = \frac{s_G^{\,2} A + I_G}{s_G A} = s_G + \frac{I_G}{s_G A}$$

This indicates that action point s_C of the total water pressure is located below the centroid s_G by $\frac{I_G}{s_G A}$.

DISCUSSION

The example teaching materials proposed in this paper provide a clearer image of partial derivatives and area integrals. The sample materials also seem to satisfy the three requirements stated in the Introduction section of this paper. Indeed, these examples introduce concepts in practical contexts before the provision of a formal definition. Moreover, these practical contexts highlight the necessity of extending existing student knowledge. We expect that first-year undergraduate students in engineering courses will use these examples to establish a professional foundation.

Given that the topics the examples cover are all related to civil engineering, students in other fields, such as electrical engineering, may feel that these materials are unfamiliar. Therefore, we believe that materials that would be more widely accepted are required.

On the other hand, all engineering students, not just those in civil engineering courses, seemed to be familiar with the dike slope example given to explain partial derivatives. Indeed, when the first author used this material to deliver a lecture on partial derivatives to economics students, the results of a post-lecture questionnaire showed that most students understood partial derivatives with respect to the individual variables of two-variable functions in relation to the slope in each direction. Furthermore, due to the additional explanation provided, these students understood that partial derivatives play an important role in the extreme value problems of the two-variable functions.

This example, however, has a weakness because the function treated expresses the flat plane whose derivatives of the order higher than one all become 0. Therefore, this example fails to explain the more general aspects of partial derivatives. In this study, the authors attempted to explain partial derivatives through the exclusive use of geometry, that is, without employing any relationship between physical quantities, to avoid confusion arising from complexity. Although this constraint may reduce the potential for explaining concepts by reducing freedom of consideration, the geometry-based explanation proposed here is effective, as evidenced by the first author's experience, which is described in the previous paragraph.

Regarding the materials on area integrals, the authors assume that they are used in classes as an introduction when expanding from the integral of one-variable functions to the integral of multivariable functions. If non-engineering students use these materials, a small supplement to physics may be needed.

The authors believe that using these materials to introduce each math unit has the advantage of allowing students to understand the need for mathematical knowledge and how it can be applied in the real world or in their specialty. As another merit of using this type of material, the authors assert that students in engineering courses can not only understand professional subjects as applications of mathematics but can also deepen their understanding from a mathematical viewpoint. It is difficult for either mathematicians or engineering teachers to create these materials independently. Collaboration between teachers on both sides is necessary to accomplish this goal.

CONCLUSION

We have discussed problems encountered when teaching calculus to students in engineering courses and have tried to provide teaching materials for engineering education. The teaching materials covering the topics treated in this paper employ problems contextualized in daily-life situations and communicated using figures. We believe that the proposed examples will provide a clearer image of partial derivatives and area integrals and that the explanations given in this paper will deepen students' understanding of the technical problems they encounter in engineering courses. We plan to use the examples presented here not only in engineering but also in mathematics classrooms and take into account students' opinions and/or requests in order to refine and further develop the teaching materials. We are also working on creating teaching materials that support various topics of instruction. For example, regarding multivariable functions in calculus as specialized fundamental education in engineering, more exciting exercises are needed so that students in engineering courses can derive ordinary and partial differential equations that arise as the governing equations to evaluate important physical quantities. In engineering, it is necessary to analyze differential equations, particularly partial differential equations, based on key physical laws rather than ideal assumptions. We hope that students will acquire mathematical skills for practical problems, including physical elements, through derivations of the related differential equations.

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