A STRATEGY FOR ENHANCING MATHEMATICAL PROBLEM POSING

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Abstract

In this work, we establish a heuristic strategy, the purpose of which is enhancing the posing of new problems in the school context. The strategy is supported by a cognitive framework consisting of six stages: Selecting, Classifying, Associating, Searching, Verbalizing, and Transforming. The first five actions make up an essentially creative process, while the last stage is present within the nucleus of the previous ones. This provides the process with a high level of complexity. Compactly, we call the strategy SCASV+T. We reflect on the heuristic nature of the strategy, as well as the didactic actions that are required for its implementation. We also describe a didactic situation in elementary geometry, where the posing of new problems based on one already solved is discussed. The analysis is carried out with students who are studying a Bachelor's degree in Mathematics Education, who know the strategy and try to put it into practice collectively. Analysis and discussion are led by a professor, who provides suggestions and demonstrates the importance of each action in the development of heuristic reflection.

Keywords: problem posing, heuristics strategy, elementary geometry, school mathematics, teacher training

INTRODUCTION

Posing new problems is a characteristic of advanced mathematical thinking. By its own nature, this process is basically creative and is closely related to other aspects, such as problem solving skills, imagination, the use of analogies, and the capacity to generalize (Cruz et al., 2016; Silver, 1997; Singer & Voica, 2015; Tuchnin, 1989; Van Harpen & Sriraman, 2013). Numerous researches on problem posing which have been published in recent years highlight the importance and helpfulness of using problem posing in the school context (Baumanns & Rott, 2020, 2021; Cai & Hwang, 2020; Felmer et al., 2016; Gabyshev, 2021; Leikin & Elgrably, 2020). An interesting aspect focuses on the stages that take place during problem posing. Polya (1957) provided a very useful model of the problem solving process. Brown and Walter (2005) recognized the existence of stages within the problem posing process. The fact that problem posing also consists of stages is not surprising since many researchers have remarked that there is a very close relationship between posing and solving problems (e.g., Chang, 2007; English, 2020; Koichu, 2020; Peng et al., 2020; Silver, 2013; Yao et al., 2021).

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Mathematical problem posing is a process of high cognitive complexity that excludes those trivial situations involving the simplified embodiment of questions (Cai & Hwang, 2020). The good question is exactly the final stage, and it is mediated by a highly creative activity that is born from a high level of affect and motivation (Cai & Leikin, 2020; Tuchnin, 1989). Numerous studies have highlighted the need to promote problem posing in the school context. This has not only been consigned in important normative documents, but also in research reports, scientific events and international forums related to mathematics education. In this regard, Kilpatrick (1987) has indicated that: "Problem formulating should be viewed not only as a goal of instruction but also as a means of instruction. The experience of discovering and creating one's own mathematics problems ought to be part of every student's education" (p. 123).

Despite the well-known need to encourage problem posing at school, there are important aspects that have not been sufficiently addressed in the scientific literature. For example, there are not many tests or other types of scientific research instrument, with adequate levels of reliability and validity, that serve to evaluate the levels of development of the process of posing new problems. In the latter case, any advance in the identification of the main stages, actions and operations of thought will be useful from the didactic point of view. As early as the 1950s,, Polya (1957) noted that problem posing and problem solving are closely interrelated processes. He stated that: "To find a new problem which is both interesting and accessible, is not so easy; we need experience, taste, and good luck. Yet we should not fail to look around for more good problems when we have succeeded in solving one" (p. 65). This conception is dialectical and reveals that problem posing and problem solving are difficult to separate from the didactic point of view. Thus, it is necessary to investigate how to educate students in thinking, so that students are ready to generate new problems. This can be approached from the perspective of didactic strategies, but necessarily involves a psychological background that serves as a framework for the mental actions of thought.

In this paper, inspired by the stages established by Brown and Walter (2005), we present a structure that models the process of posing mathematical problems. This cognitive structure works as a strategy on the didactic level, since teachers can adopt the stages of the framework, as a kind of guide in the teaching process. Some new relationships which have not been explored in previous works (cf. Cruz, 2006) are presented. The strategy is illustrated with the help of an elementary geometry problem, which was analyzed jointly with students from the Bachelor of Mathematics Education at the University of Holguín.

THE HEURISTIC STRATEGY AND ITS COMPONENTS

The heuristic strategy is supported by a cognitive framework consisting of six stages: Selecting, Classifying, Associating, Searching, Verbalizing, and Transforming, with the acronym SCASV+T (Figure 1). Although the first five stages express an apparent linear path, the process becomes more complex when transforming is included.

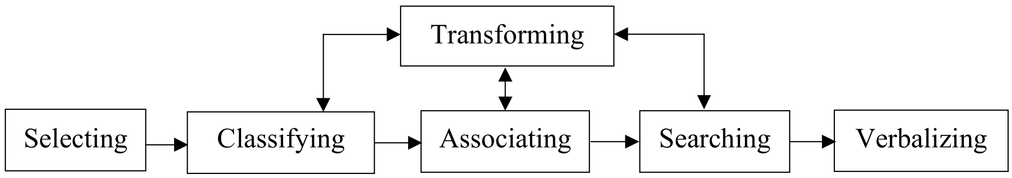


Figure 1. Cognitive framework of SCASV+T heuristic strategy

SELECTING

Structurally, this strategy begins with the selection of a given object or phenomenon which corresponds with "choosing a starting point" described by Brown and Walter (2005) and expresses the intentionality of posing problem as a motivated aware cognitive activity. Silver (1994) states that problem posing involves the generation of new problems about a situation or the reformulation of given ones, so the starting point in the Selecting stage can also be a previously solved problem, or even a problem that is in the process of being solved. Then, the subject breaks up the object or phenomenon (a problem, a situation, a geometric figure, a set of objects, etc.) through an analytic-synthetic process, which is similar to the heuristic strategy "decompose-recompose" described by Polya (1957) in problem solving process.

CLASSIFYING

The second stage is called Classifying, which is a cognitive process that implies listing, comparing and organizing attributes according to certain criteria (Inhelder & Piaget, 1969). Although the possibilities for listing attributes are limitless in mathematics, there are barriers for each person on their individual level. From the personal, institutional and socio-cultural point of view, classification schemes are formed. Under the restrictions of these schemes, each subject or group selects the most familiar attributes within their cognitive patterns. Jacob (2001) pointed out that classification schemes provide a powerful cognitive scaffolding, as this minimizes the perceptive load on the individual by providing tools, selection strategies, as well as criteria for selecting the most likely alternative. However, classification schemes also hinder creative thinking, which requires not only originality, elaboration, and flexibility, but also broad fluency in reasoning (Guilford, 1956).

ASSOCIATION

The next stage comprises the association of related concepts, with elements involved in the classification. For example, if an element resulting from the classification is a segment, then there is a set of related concepts that can be activated with the help of memory processes, such as length, bisector, and midpoint. But instead of looking for concepts of properties, if one prefixes two or more objects, then it is possible to think of relational concepts. By prefixing one segment and a certain angle we can think of the concept of a capable arc, and if we take two lines we can connect our thinking with the concepts of perpendicularity, parallelism, the angle between lines, and the existence or not of an intersection point. The depth and plurality in the associated concepts will be greater, to the extent that the objects extracted from the classification are more complex. If the prefixed object is a triangle, then measurement concepts such as perimeter and area emerge. Similarly, pieces of classification resulting from the lengths of their sides and the amplitudes of their angles come to our mind, as well as proper objects such as heights, medians, inscribed and circumscribed circles, Euler's line, and so on. The fluency and diversity of concepts that emerge are in direct correspondence with our mathematical skills and culture.

SEARCHING

In the fourth stage, the student looks for relationships and dependencies, explores conjectures, establishes analogies concerning already known situations, among other processes of high cognitive complexity. This stage is complex from a psychological point of view since it is directly related to creative processes and divergent reasoning. On many occasions, this stage has been masked with enigmatic terms such as insight. However, the didactic problem consists in modeling what actually happens when the subject intelligently searches for new patterns, relationships and ideas, in order to establish plausible conjectures. Therefore, the teacher's role here is to teach students to think mathematically.

Therefore, the best effort should be focused on investigating how to find connections between concepts and properties that really make sense. The use of mathematics software is a great opportunity, as they help us find and explore promising hypotheses (Abramovich & Cho, 2015). First, exploration can be done using the computer, followed by proving or disproving one's own hypothesis using mathematical tools. For these reasons, the searching stage is very closely related to problem solving activity, because the mere fact of considering the relevance and meaning of a question implies a glimpse of possible ways of solution.

VERBALIZING

Verbalizing appears in the final part of the process. Under Vygotskian epistemology, this stage involves the idea that language is the material wrapping of thought (Vygotsky, 1962). Although this stage may have a communicative purpose, its primary function is to summarize the problem in our own thinking. This is a synthetic process in which what is given or required to prove or find can be specified. This idea is directly connected to the well-known taxonomy of Polya (1957), in which he differentiates problems to find and problems to prove. Once we have specified the problem, then we can try to refine it, and also modify its levels of complexity, establish an inventory of possible solutions, find a real situation that masks it (in order to provoke mathematical modeling), and even find an interesting way to communicate it, and so on. These last actions are eminently didactic and go beyond the cognitive process of posing new problems. By taking them into account, the pedagogical importance of teaching mathematical problem posing in the teacher training curriculum will be realized.

TRANSFORMING

The transforming stage interacts with the previous sequence, Brown and Walter (2005) observed that problem posing process is not linear, since it involves certain cycles where the "What-if-not" strategy emerges. Kilpatrick (1987) pointed out that both this strategy and the "What-if-more" (suggested by Jim Kaput in a personal communication), are typical examples of a more general type of reasoning that he calls "contradiction". Contradiction underlies the epistemological basis of critical mathematical thought. This aspect is intrinsically linked to the epistemic sources of mathematical knowledge, which "…it is necessary, stable, and autonomous but that this coexists with its contingent, fallibilist, and historically shifting character" (Ernest, 1998, p. 259).

From the perspective of social constructivism, mathematics is part of human culture. Ernest (1991) points out that mathematics is not neutral but laden with the values of its makers and their cultural contexts. In particular, Ernest (1991) states that: "Mathematics consists primarily of human mathematical problem posing and solving, an activity which is accessible to all. Consequently, school mathematics for all should be centrally concerned with human mathematical problem posing and solving, and should reflect its fallibility" (p. 265).

Contradiction is a situation that activates thought and motivates transformation, encouraged by a creative need to search for new ideas. However, there are other reasons that lead subjects to carry out transformations during this process. On the one hand, there is the case in which one encounters a problem, but senses or realizes that it is excessively complicated. Then one can try to transform it into a simpler problem, setting the value of certain parameters or abandoning one idea to undertake another. On the other hand, the poser may want to make more complicated things and finds that his/her finding is too trivial, or maybe uninteresting. Then he or she also has the opportunity to change things through transformation. The context in which a reasoning by contradiction takes place constitutes an expression of the sociocultural environment. This justifies the fact that one person can provide a kind of personal stamp to the problem. Although the reflexively critical and fallibilist spirit constitutes a catalyst for reasoning by contradiction, this is regulated by own barriers of each individual cognitive development.

If during the classification process one does not find any interesting aspect or some suggestive idea, there is the option of transforming the mathematical object. Then one can associate properties, or reclassify them in search of new components. The same is true in both Associating and Searching stages in which regressive subprocesses are admissible. From our point of view, the transforming phase is intrinsic in the three intermediate stages. If one sees the transformations in the object, problem or phenomenon to be selected, at some point one will decide to choose something to start with. So, this would be Selecting stage itself, and one must avoid a vicious circle in modeling this process. On the other hand, transforming the results of verbalization leads to previous stages. In this closing moment, the individual has conceived a question with mathematical meaning. An eventual reformulation of an algebraic problem in another geometric one would imply that the previous stages happen again. Likewise, aspects such as the refinement of the conceived question, the clarity in the approach and the aesthetic retouch, are less cognitive and more didactic.

AN EXAMPLE OF ELEMENTARY GEOMETRY

As Silver (1994) points out, the discovery of new problems can occur before, during, or at the end of the resolution of a problem. This idea is well connected with observations by Sharygin (1991a, 1991b), related to the invention of problems for mathematics Olympics (cf. Kontorovich, 2020; Poulos, 2017). Sharygin suggests the importance of looking for reformulations for a problem that has already been solved, as if one idea were encapsulated within another ("matryoshka" problems). For example, looking for a geometric interpretation of an algebraic result leads to interesting problems, which not only help the development of reflective thinking but also form a more interconnected conception of mathematical knowledge.

Below we present an example, which was discussed collectively with students of the Bachelor of Mathematics Education at the University of Holguín. During the data acquisition process, the professor was the third author of this work. The group was made up of 12 students gathered in a problem solving session, which lasted two class hours. The students were previously familiar with the heuristic strategy, both regarding structure and interrelationships. The analysis takes place in a professional practice session, where students can combine their mathematical and didactic knowledge.

SELECTING

This stage consists of choosing the situation or mathematical object that serves as a starting point. Specifically, we start from the following problem already solved by the students.

Suppose that the side *DB* of a square *BEFD* is the diagonal of a second square *ABCD*. Calculate the ratio of the area of the first square to that of the second square.

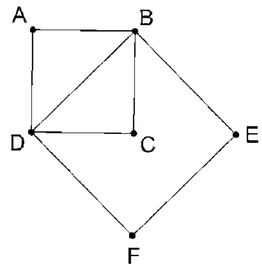


Figure 2.Problem selected as starting point

CLASSIFYING AND ASSOCIATING

The determination of attributes and components already has an advance, since the original problem directly refers to two squares and a diagonal. The concepts of area and ratio are also associated, and the calculation of the quotient between two areas is demanded. For the solution, it is assumed that the small square is of unit length, hence the side of the large square measures $\sqrt{2}$, and finally it can be concluded that the ratio between the areas is 2. The following are the different attributes that can be determined:

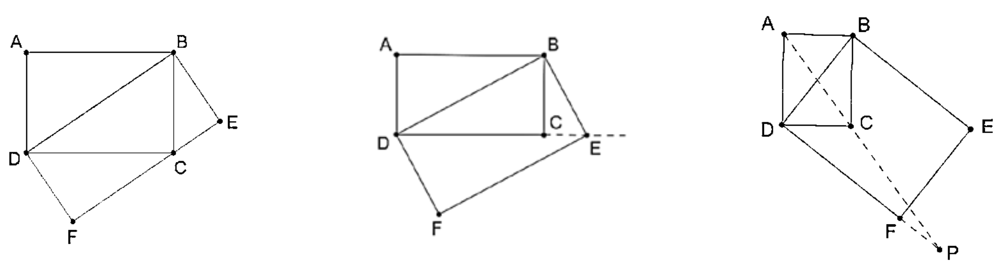
– In the initial problem, two plane geometric figures appear.
– The two geometric figures belong to the same class of quadrilaterals.
– The two geometric figures are squares.
– One side of a figure is a diagonal from the other.
– There is a relationship between the areas of geometric figures.

In the discussion with the students, it is highlighted that the best benefits are obtained when many elements of the mathematical object are imagined. Heuristic thinking works best when a plurality of components not drawn in the original figure is perceived, so that related concepts can be established. For example, that the point *C* is the center of the square *BEFD* may motivate one to imagine the center of the other square, at the midpoint of the diagonal *DB*. Before looking for relationships, it is fruitful to envision a variety of possibilities, which can potentially raise interesting questions.

SEARCHING, VERBALIZING AND TRANSFORMING

The possibility of establishing transformation is inherent in the whole process of Classifying-Associating-Searching. However, it is especially effective when the search does not produce interesting results, or when the subject does not find appropriate questions. Transforming is more feasible with the help of the "What-if-not" strategy, developed by Brown and Walter (2005). A consistent way to implement this strategy consists of the generalization-specialization technique, in Polya's sense. An immediate example is to replace the concept of square with the concept of a rectangle, which is more general. Now, based on this general case, it is possible to examine particular special cases. In fact, the original problem is the result of imagining the special case where both rectangles are squares. The students proposed numerous variants to analyze, of which three were primarily interesting. Figure 3 illustrates these three variants, worth exploring.

I	II	III



What if vertex *C* lies on side *FE*?	What if the vertex *E* lies on the ray *DC*?	What if the rays *AC* and *DF* are concurrent at a point *P*?

Figure 3. Three special cases after generalization

In the first variant (Figure 4), the students noticed that after drawing the segment CG, perpendicular to the diagonal BD, both rectangles have the same area. Indeed, this follows directly from the equalities $\Delta CBE = \Delta BCG$, $\Delta DCF = \Delta CDG$ and $\Delta BCD = \Delta DAB$. In fact, Figure 4 constitutes a kind of "proof without words" of this assertion. Therefore, verbalization focuses on expressing a question whose solution the student already knows. This consists of verifying that under the given conditions both rectangles have equal area. However, one student suggested verbalizing like this: Which of the two rectangles has a greater area? Give reasons for your answer.

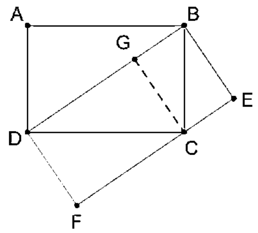


Figure 4. Equality of rectangles in the first variant

Regarding this last reformulation, some students raised objections. If the context of the problem responds to an affective and motivated environment with open reflection, then the question about which of the two rectangles occupies the largest surface is intended to show that the rectangles have the same area, contrary to what is expressed in the text of the problem. This requires the solver to act confidently and answer that neither of the two rectangles covers a larger surface, since they both have the same area. If the question is asked in a tense environment, this way of presenting the problem can lead to confusion, and even fear of refuting the demand expressed in the question.

In the second graph II of Figure 3, it can be seen that according to the position of point E, the segments AB and DE are parallel, so the right triangles DAB and DBE are similar. Therefore, $\frac{DB}{DE} = \frac{AB}{DB}$ and $DB^2 = AB \cdot DE$. A student observed that this property is present in the original problem, where obviously points D, C, and E are aligned. Therefore, the student discovered that another interesting question is to show, from the two squares in the original figure, that DB is the geometric mean of AB and DE. This fact suggests that the return to the starting point was not static but dialectical, since a new problem was perceived in the same object.

Next, the professor proposed that the students reflect on what would happen if the figures were not squares or rectangles simultaneously. For example, the equality $DB^2 = AB \cdot DE$ connects the lengths of both bases of the right trapezoid $ABED$ with one side of the rectangle $BEFD$. If the lengths of AB and DE are given, then it is possible to calculate the length of DB and then BE, from Pythagorean relations and also from the equality $CE = DE - DC = DE - AB$. Finally, if the lengths of the bases of the right trapezoid are known, then all the areas of the triangles and rectangles represented in the second variant can be calculated. The previous observations were summarized in the following problem. In Figure 5, $ABED$ is a right trapezoid at A and D. $BEFD$ is a rectangle, DE = 13.0 cm and AB = 4.0 cm. Find the area of triangle ABD and also of rectangle $BEFD$.

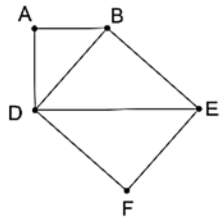


Figure 5. Geometric object resulting from the analysis in the second variant

There are two important aspects, related to the way ideas are verbalized. In the problem comfortable numbers are used, which facilitate the calculation since their Pythagorean nature provides the problem with greater elegance. For example, the results of the calculation of both areas constitute whole numbers. The complexities are not centered on numerical calculation, but on geometric reasoning. At this time, it is useful to discuss with the students about the veracity or impossibility of the figure. The default numerical values are not always appropriate, as the geometric figure described could be impossible to construct. On the other hand, Figure 5 hides segment *BC*, and by not drawing said element, the person who solves the problem is expected to draw it, as part of a heuristic reflection.

Regarding the variant represented in III of Figure 3, a student noticed that the triangles *CDA* and *FDB* are similar. The flexibility that transformation provides allows us to return again to the search and association process. In this case, it was feasible to imagine what would happen if the sides of these two triangles were extended. One possibility is to analyze relationships between lines *AC* and *FB*, which contain a diagonal in the corresponding rectangle. However, exploration was more successful using GeoGebra. Indeed, collective discussion led to exploring the relationships between the rays *AC* and *DF*, since both are parallel in the original problem. This is an example of the importance of considering special cases.

At this time, a high level of motivation was perceived in the students. The time consumed for the activity did not allow to continue exploring, so the professor advised to continue investigating this situation at home. In a subsequent session, the students presented various ideas, which were appropriately discussed. The most ingenious proposal corresponded to a student involved in the Mathematics Olympics. She noted that under condition $0° < \measuredangle CBF < 90°$,, which implies that $AB < BC$, the rays *AC* and *DF* intersect at a point *P*, as illustrated in Figure 6. In the original problem, $AB < BC$ and $\measuredangle CBF = 0°$, so *P* is the point at infinity, corresponding to the direction of the parallel lines *AC* and *DF* (see a special case in Figure 2).

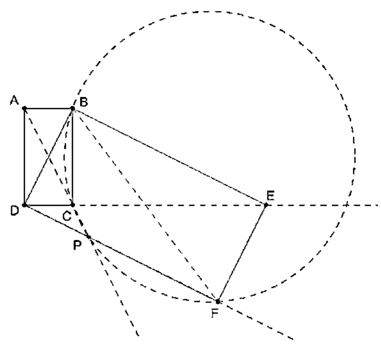


Figure 6. Exploring the third variant with GeoGebra

The most interesting observation was that the points P, C, B, and F are concyclic. The student explained that she had reached this conclusion by noting that $\measuredangle PCB$ and $\measuredangle BFP$ turn out to be supplementary when she enlarged the length of the segment AD in GeoGebra. This fact reflects the great heuristic value that underlies the ability to move and compare items. The dynamic geometry software, in this case, became a kind of catalyst, due to its wide possibilities in this sense and with an economy of time. After adding the circumference and testing her hypothesis experimentally in GeoGebra, the student also presented her proof of the property. Indeed, since the triangles FDB and ABC are similar, it turns out that $\measuredangle BFD = \measuredangle ACB$. Therefore, we have the following: $\measuredangle PCB + \measuredangle BFP = \measuredangle PCB + \measuredangle BFD = \measuredangle PCB + \measuredangle ACB = 180°$. Finally, a new problem consists of proving that under the conditions of the third variant, the points P, C, B, and F are concyclic (see Figure 3 and a complimentary animated GIF in Cruz, 2021).

Again, the analysis of the hypothesis and its verification, constituted an opportune space for debate and discussion in class. Another student observed that if instead of increasing the length of segment AD, this length decreases approaching the length of segment AB, then the quadrilateral $PCBF$ is no longer convex. This is precisely the case that appears in part III of Figure 3. Although this fact does not influence the proof, the idea served to establish a new open question: What happens if in the third variant III of Figure 3 the point P and the vertex F are coincident? Again, the professor pointed out the importance of specializing, which does not mean, in general, the tacit identification of a particular case. Although specialization is a particularization, its primary purpose is to select relevant cases. In other words, particular aspects that are notable and significant. This is the sense in which Polya (1957) describes generalization and specialization. Right at this moment, the professor suggested continue exploring further variants, even starting with other problems already solved.

DIDACTICS

In our strategy, it is difficult to separate the cognitive framework from the heuristic reasoning. The cognitive framework is structural and it underlies on the abstraction that we make about the thinking process itself. On the other hand, the heuristic strategy constitutes an expression of the cognitive framework at the didactic level. It is well known that Polya's scheme (1957, pp. xvi-xvii) shows us an ideal model, is made up of stages that demarcate the problem solving process. With the help of this scheme, the teacher, then, shows a general path to follow. Similarly, the structural component of our framework speculates about what the process should ideally look like. However, the functional component reveals the challenge of how to teach mathematical problem posing.

If we draw on components and relationships of the model, then we can provide a way to organize thought. One can start by selecting a mathematical object, and then suggest enumerating several of its components to establish relationships. It helps a lot to ask questions such as: What visible elements appear in the figure? What properties can we associate with these elements? What non-visible elements could we draw? What relationships could exist among certain elements? Let's try the computer, see what happens! Does this question make sense? These types of questions are heuristic in nature so they do not guarantee anything, instead they favor the search for original problems. The new questions are important, especially when these make mathematical sense and when these are the result of a process of conscious reflection. However, the most important aspect is the imprint that this heuristic reasoning leaves on thought.

During the discussion of the strategy, the students recognized that the framework provides them with a way of guiding their thought with a creative sense. They recognized that these ideas are useful in their training as mathematics teachers. However, this framework could hardly be explained in a school context. It is necessary to follow a more expeditious path, where guidance is specified in a synthetic and enjoyable way. As Newton asserted: “Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things” (translation from Manuel, 1974, p. 120). An example is the apparent simplicity of Polya’s scheme (1957), set out in four phases. *How to Solve It* is a book designed for students, and for this reason the author presents his profound ideas in a comfortable way. The “looking back” itself provides the scheme with a fertile conception. In addition, the phases are accompanied by suggestions, heuristic questions, argumentation and exemplification. This is the didactic mark that Polya leaves in the mathematics classroom.

Similarly, it is convenient to present the SCASV+T heuristic strategy in a simpler way. Table 1 contains six stages of heuristic reasoning, which can be developed in math class. The elements and relationships of SCASV+T heuristic strategy emphasize the cognitive level, while succeeding stages constitute a didactic expression. Figure 1 reflects one way of thinking, while Table 1 summarizes one way of doing it in class. Both aspects are useful for the teacher, since their professional training requires cognitive and didactic knowledge.

Table 1

A didactic expression of the heuristic strategy

Stages	Heuristic suggestions
Choosing a starting point	– Select one or more familiar math object – Consider a real and interesting phenomenon, that can be mathematically modeled – Start from a problem already solved – Finding problems within other problems
List explicit and non-explicit components	– Consider the essential elements of the situation – List elements of math object
Establish for concepts associated with each component	– Determine what concepts can be associated with each listed item – Think of other similar or analogous concepts – In addition to concepts inherent to an object, also consider relationship concepts between two or more objects
Search relationships and dependencies	– Remember problems in analogous situations – Transform elements of the situation or object – Consider special cases – Explore more general variants
Ask questions	– Distinguish what is most interesting – Present ideas clearly and rigorously – Assess whether the question could be interesting to other people
Analyzing the problem	– Use computational resources to establish relationships and dependencies – Analyze if the question makes sense. Data and figure are possible? – Find other more appropriate or attractive ways to approach the question. Can elements of the figure be hidden? – Assess the real possibilities to establish a solution path. Try to solve the problem

The first and last stages in Table 1 respectively correspond to levels “0” and “IV” of the strategy developed by Brown and Walter (2005, p. 64). The intermediate stages are based on SCASV+T heuristic strategy framework. “What-if-not” question is not part of the sequence of stages, since it is possible to think about other problems even without changing or varying the initial attributes. However, the possibilities of transformation are always present, which favors the emergence of cycles in reasoning. This freedom to transform affords the process with flexibility, which is very closely related to creative thinking (Silver, 1997). In this regard, Brown and Walter (2005) point out: “The process of varying one attribute followed by varying another suggests a systematic technique we could employ for brainstorming new problems. We call this technique cycling. Here we have a systematic way of generating new forms by combining the preceding two What-If-Nots” (p. 60). As can be seen, Figure 1 illustrates several cycles that are established with respect to the possibility of transforming the attributes of the problem. The identification and analysis of these cycles form an interesting aspect of experimental studies, where Schoenfeld’s episodes (2016) can be useful as they have already been in problem solving studies (Cruz, 2006).

On the other hand, it is necessary to pay attention to the regulatory processes that occur during the implementation of the strategy. In a classical model of cognitive monitoring, Flavell (1979) refers to a wide variety of cognitive activities that occurs through the actions of and interactions among metacognitive knowledge, metacognitive experiences, goals, and strategies. For example, metacognitive knowledge is related to the self-perception of strengths and weaknesses to perform a task or to fulfill an objective. In the case of problem posing, this aspect involves planning, monitoring, evaluation and self-regulation during the creative process as a relevant dimension (Baumanns & Rott, 2021). In particular, beliefs and affects are established as relationships between individuals and mathematical knowledge (Schoenfeld, 2016), so it can be expected that these processes also occur during problem posing, such as the belief that in geometric objects one finds greater diversity of problems (Cruz, 2006). However, some typical beliefs about the resolution of problems can influence and affect the formulation of problems, as in the case of “problems are designed to test procedural knowledge and be solved quickly” and also “the solitary source of mathematical problems is textbooks” (McDonald, 2017). Identification and enquiry of beliefs and conceptions that affect the problem posing process constitutes a complex challenge that goes beyond a typical teaching/learning error, as it also brings up certain training deficiencies in the field of the philosophy of mathematics itself.

It is also important to reflect on the epistemological premises that support the strategy at the didactic level. In a setting where mathematics is presented rigidly and dogmatically, it is difficult to promote the creative posing of new problems, since critical reflection and fallibilist conception of mathematical knowledge are blockaded (Ernest, 1991; Lerman, 1990). It is necessary to accept the possibility of mistakes, an aspect that not infrequently alarms the teacher and confuses the student. In the training of mathematics teachers this is especially important, since the student is also learning to teach. Transforming the attitudes of the prospective teacher requires not only the learning of didactic resources, but also the apprehension of adequate epistemological bases. One consistent path is to promote inquiry-based learning, where problem posing is central.

CONCLUSION

SCASV+T heuristic strategy provides a theoretical framework, which suggests how to organize the reasoning to pose new mathematical problems. The structural and functional components of the strategy reflect complexity, cyclicality, and flexibility. However, the framework expresses what happens on the cognitive level, and that is why it requires didactic recommendations. The latter are presented in the form of stages complemented with heuristic suggestions that can be enriched.

The content of Table 1 is a didactic expression of the heuristic strategy modeled in Figure 1. This means that the stages can constitute a teaching content in teacher training. However, the framework depicted in Figure 1 is too complex to be used in elementary and secondary school teaching. The didactic actions are reformulated to facilitate teaching. This is what the stages in Table 1 consist of, which are complemented by heuristic suggestions. Therefore, teachers in training can prepare themselves to teach mathematical problem posing, following the stages of Table 1 (explicit aspect oriented towards didactic action), but aware of the foundation provided by the model in Figure 1 (implicit aspect oriented towards didactic foundation)".

The example developed reflects the wide possibilities of imagining new problems, where verisimilitude and mathematical sense are essential aspects. These ideas may be useful in other contexts of mathematics teacher training. The process of posing problems not only helps to achieve a better mathematical education, but also to the equipping of professional tools. Indeed, knowing how to ask interesting questions, establishing a variety of solutions, promoting reformulation and problem posing in students, and favoring the development of self-regulatory mechanisms, are important components of the teacher's professional competence.

The cognitive component of the strategy reflects a certain link between the activity of elaborating a problem for students and the process of conceiving a problem for oneself. There are differences regarding the purpose of formulating and the way of presenting a problem. Making a new problem for students, although it may be open to creative imagination, has limits related to the teaching objectives. This problem constitutes an open field for research, which was pointed out by Silver (2013). The example that we have shown is not an experimental but an experiential result, however it makes us think that the common aspects are in the cognitive component, whereas differences are manifested in the didactic and professional components.

The education of creative reasoning encounters numerous obstacles in mathematical problem posing, which constitute challenges for teaching. A motivating environment is required, where the fear of making mistakes is minimized. Collaboration and collective work are very helpful, as well as the use of computational tools that facilitate exploratory work. On the other hand, the deployment of this heuristic strategy also requires a high mathematical and didactic preparation of the teacher. The teacher must be aware that the variants are infinite, which underlies the very nature of mathematics: contingent, fallible and historically changing. However, the challenge is to foresee the main opportunities for the conduct of reasoning and thus leave a favorable mark on the mathematical thinking of students. As Halmos (1980) pointed out, problems are the heart of mathematics, so the art of solving them must be combined with the art of posing them.

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