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AIMS AND SCOPE

Hiroshima Journal of Mathematics Education (HJME) is the official English-language journal of the Japan Academic Society of Mathematics Education (JASME), and is devoted to research that seeks to improve the mathematics education. It is an annual peer-reviewed and open-access international journal that publishes original papers written in English. HJME was founded by the Department of Mathematics Education at Hiroshima University. Articles have been published by JASME since Volume 12 was issued in 2019.

The journal is dedicated to the dissemination of research findings on mathematics education at all levels (e.g., from preschool to university, professional development, lifelong education) on a variety of issues related to the teaching and learning of mathematics (e.g., students' understanding, classroom teaching, curricula, policy, teacher education). It is open to any type of research (e.g., theoretical, empirical, methodological, qualitative/quantitative). Papers dealing with Japanese mathematics education issues or issues of special interest to the Japanese mathematics education community are particularly welcome.

Paper submissions are welcome from contributors from different sectors (e.g., researchers, educators, and practitioners) who are interested in contributing to the development of mathematics education and its research. HJME is expected to promote reflection on mathematics education and communication among international researchers, educators, and practitioners in mathematics education. Papers are published in either of two HJME sections: original paper and lecture notes. The original paper reflects the main topics of the journal. All papers submitted to this section are peer-reviewed, with an emphasis on papers of an excellent level. Lecture note includes records or materials of scientific events (e.g., invited talks, symposia) organized by JASME

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Editorial for Volume 15

Tatsuya Mizoguchi
Editor-in-Chief of the *Hiroshima Journal of Mathematics Education*,
Japan Academic Society of Mathematics Education

Throughout the ongoing coronavirus disease (COVID-19) pandemic, our research practice, especially empirical research, has been subject to significant methodological constraints. This is also true, but to a lesser extent, in theoretical study. We have been forced to communicate online, having lost an “on-the-spot atmosphere,” so to speak, and the opportunity for informal conversation with coffee and beer.

In 2022, several international conferences have been held face-to-face (e.g., The 45th Conference of the International Group for the Psychology of Mathematics Education [PME 45], Alicante). Researchers have also begun to interact with each other as they had before the pandemic. We are increasingly aware of the importance of discussion in facilitating research activities. At least in Japan, the primary elements of conferences and seminars are presentations and lectures; I am under the impression that there is less dialogue on shared topics in Japan than in other countries. This difference cannot simply be attributed to national variances. While social and cultural influences on research content may be valid, the research activities themselves must be shared internationally.

Discussion generates mutual understanding. Results and conclusions of research are undeniably important to the construction of our knowledge. It is even more important to comprehend the theoretical position/framework and the methodology that led to the findings and conclusions. The various articles published in journals are the outcomes of such discussions. The articles go on to generate new discussions. Different theories and different methodologies focused on the same topic will produce different conclusions. It is through mutual understanding achieved through discussions that we can advance research in mathematics education as a whole.

In current mathematics education research, there is a diverse range of international journals. Some journals specialize in a particular area of interest. Other journals cover a wide range of genres comprehensively; the *Hiroshima Journal of Mathematics Education* is one such journal. Volume 15 of the *Hiroshima Journal of Mathematics Education* contains two Special Issues that present the work of the two Topic Study Groups (TSG) of the 14th International Congress on Mathematical Education (ICME-14) held online/hybrid in 2021. The volume presents four papers focused on the issues of “Problem posing and solving in mathematics education” from TSG17, and seven papers focused on the issues of “Mathematics for non-specialists/mathematics as a service subject at the tertiary level” from TSG45. The Special Issues were organized by Guest Editors. The detailed structure of each Special Issue and the introduction of the included papers were left to the respective Guest Editors. We would like to express our sincere thanks to the authors of each article and, in particular, to the Guest Editors of both Special Issues for their contributions.

Finally, please note that the *Hiroshima Journal of Mathematics Education* is published once a year. Accepted papers are published online first, before the publication of the next volume. Submissions are accepted at any time. We look forward to receiving your contributions.

HIROSHIMA JOURNAL OF MATHEMATICS EDUCATION

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Special Issue (1):

Mathematical Problem Solving and Problem Posing,

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Guest editorial

EDITORIAL FOREWORD FOR THE SPECIAL ISSUE OF HIROSHIMA JOURNAL OF MATHEMATICS EDUCATION ON MATHEMATICAL PROBLEM SOLVING AND PROBLEM POSING

Tin Lam Toh, Manuel Santos-Trigo,
Puay Huat Chua, Nor Azura Abdullah and Dan Zhang

The theme addressed in this special issue, *Mathematical Problem Solving and Problem Posing*, arose from the contributions that were presented and discussed during the sessions of the Topical Study Group (TSG) on *Mathematical Problem Solving and Problem Posing* at the 14th International Congress of Mathematics Education (ICME) held in Shanghai in 2021. The papers presented in this issue continue the discussion we had during the meetings of the TSG. This special issue affords some form of platform for more interested researchers to extend and refine the discussion and engagement on this important research and practicing area in mathematical problem-solving themes.

Although problem solving and problem posing are closely related, research on problem solving has gone through many decades of significant development while problem posing remains a relatively new field. This collection contains three papers on problem solving and four on problem posing. The problems on problem solving, although mainly built on the early works of Polya and Schoenfeld, have expanded beyond their seminal work on this field.

BOS and BOGAART (2022) advocate the use of heuristic trees, a form of digital tools as a scaffold to foster students' independence and to achieve problem-solving competencies. In order to scale up the use of heuristic trees, teachers must be able to design a heuristic tree, which in itself, is a challenging task. In analyzing the difficulties that teacher participants in a professional development course encountered in designing heuristic trees, Bos and Bogaart collated a list of mistakes, and provided the design principles of heuristic trees.

ABDULLAH and ABBAS (2022) conducted a study on teachers' exploration of using Graphic Organizers (GO) for teaching problem solving in the primary mathematics classroom. A group of teachers participating in the professional development workshop was introduced to the use of GO to help primary school students in solving word problems. They implemented the GO and assess their students' learning. The teachers' reflections on the affordances and challenges of the GO were also discussed.

Some of the models of the problem-solving processes are linear as proposed by Pólya's model and others are cyclical such as the Schoenfeld's model. However, the large majority of descriptions of the problem-solving processes do not fall neatly into these two models. FAVIER (2022)'s work on a characterization of the processes shown by the students in problem solving within the context of a classroom setting, provides one example of this alternative model for teaching problem solving. The analysis of the empirical data collected in this study led to the enrichment of the Rott et al. (2021)'s descriptive model of problem-solving processes , with an additional dimension involving the interactions between students and teacher.

Ramírez et al. (2022) proposed a heuristic strategy for enhancing problem posing in the mathematics classroom. The strategy is supported by a cognitive framework consisting of six stages (in short, SCASV+T framework). The six stages are Selecting, Classifying, Associating, Searching, Verbalizing, and Transforming. The authors presented the didactic considerations in implementing the heuristic strategy. In particular, the authors illustrated with a problem of elementary geometry how new problems can be posed based on an existing problem.

WANG and WANG (2022) carried out a textbook analysis of the mathematical problems and problem-solving tasks found in six series of mathematical textbooks in China. They focused on the distribution of the number, types of problems from the perspective of historical comparison, the types of problem-posing tasks and the distribution of these problem-posing tasks across various mathematical content strands. It was found that since the 1990s, a large number of mathematical problems has been included and evenly distributed throughout the textbooks. With the recent emphasis in problem posing, there had been an increase in the number of problem-posing tasks. But these tasks were not evenly distributed throughout different topics. Moreover, the total of problem-posing tasks was found to be still very low.

LUO et al. (2022) conducted an international comparative study between students from two regions in China and the United States, investigating the rural elementary students' thinking about division by analyzing the problems that they posed about division. Bruner's paradigmatic and narrative modes of thought was used as a framework for this study. The problems posed by the students about division were only about partitive and equal groups division, with none from either country posing problems on array or area. Most of the context of the problems posed were on food, though with cultural variants across the two countries.

CHUA and TOH (2022)'s exploratory study on secondary school students' problem posing, used two types of problem-posing tasks, one free-posing and one semi-structured task. The authors found a variety of problem-posing strategies used by the students, with the free-posing task elicited a wider variety of problem-posing strategies from the students. The number of problems-posed by the students was also not dependent on the students' achievement type. The authors further discussed the implications of the findings for teaching problem-posing in the mathematics classroom.

We hope that this collection of papers could contribute to the contemporary knowledge of problem solving and problem posing. The Editorial Team also like to raise the readers' attention that while this issue is being prepared, another set of authors from the same TSG of ICME14 have been invited to contribute to chapters of another book on problem posing and problem solving, which is expected to appear sometime in 2023.

From the Editorial Team

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TEACHERS' DESIGN OF HEURISTIC TREES

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Abstract

In scaling up the use of heuristic trees to facilitate students' mathematical problem solving, we developed a design course and design protocol for heuristic trees. However, designing heuristic trees is a challenging task. The study reported in this paper aims to collate an inventory of teachers' difficulties in designing heuristic trees. We analyzed a sample of heuristic trees teachers designed after participating in the course. Through open coding we arrived at a list of mistakes in the designs, and showed how these relate to the design principles for heuristic trees. We see support for the conclusion that most of the design principles are not straightforward to implement. In particular, teachers need to address the general techniques and concepts in the problem, and provide support for students in a way that needs no further intervention by the teachers.

Keywords: mathematics education, teacher education, problem solving, heuristic trees.

INTRODUCTION

The importance of both teaching problem solving and teaching through problem solving is generally acknowledged by researchers (e.g., Lester & Cai, 2016). Problem solving is considered an important 21st century skill that should be central in the modern curriculum. Additionally, teaching *through* problem solving invites students to raise their ability to apply knowledge to more challenging situations. The goal is to promote a reorganization of the students' knowledge of concepts, theory, and techniques in such a way that not only reproduction in familiar situations is achievable, but also a creative approach in new problem situations.

However, it is a challenge for teachers to implement problem solving (Lester, 2013). Students get stuck in a problem and need help, but a teacher cannot always be there in time to support the students, either because of the large amount of students that need individual help, or because the teacher is not around (e.g., in homework situations). This causes frustration for students, who have the tendency to consult the "model answers" instead of thinking about the problem themselves. To support problem solving in the absence of a teacher, the authors proposed a digital tool called heuristic tree (Bos, 2017; Bos & van den Bogaart, 2021, for an example of a heuristic tree, see https://edspace.nl/htree/heuristiekboom.php?boom_id=146). The tool provides students with just in time support through hint cards ordered in a tree structure. These hints are

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specific for the problem at hand, so each problem needs its own, tailor-made heuristic tree. For designers of heuristic trees we provide detailed design principles on how to formulate and structure the help within the tree. The design principles of heuristic trees, reflect the underlying theoretical ideas on the phases of problem solving (Pólya, 1945), and on compression of mathematical knowledge (Tall, 2013; Thurston, 1990). Studies have shown that heuristic trees allow students to work in absence of a teacher and to engage strongly with problems, maintaining ownership of the solution methods (Bos, 2017; Bos & van den Bogaart, 2021; Lemmink, 2019).

In order to scale up the use of heuristic trees, teachers need to become designers, creating heuristic trees that match the requirements of their own students. To facilitate this, we designed an author environment on our website https://edspace.nl/htree/. All tools on this website are open source. However, designing heuristic trees following the design principles is not a straightforward task. Hence, our aim in this study is to investigate how teachers design heuristic trees, and what support teachers need in this process. To this purpose, we developed a heuristic tree design protocol and a short course. We analyzed the heuristic trees designed by the in-service teachers that attended the course. We expected that following the design protocol and implementing the design principles would be a challenge for the participating teachers. The way teachers did this informed us to what extent they were able to identify and bring to the fore in their help the compressed, general concepts, techniques, and theory that underly the problems they set their students. The designed heuristic trees also reflected their personal ideas on how to support students' problem solving. In addition, this study also contributes to our understanding of how teachers prepare and structure help for students' problem solving tasks.

THEORETICAL FRAMEWORK

Compression and decompression. As one learns, knowledge is organized and reorganized. A central way of reorganizing one's mathematical knowledge is compression. Compression can be applied to concepts, techniques, and statements (propositions) and it can concern cases or steps. Our notion of "technique" ranges from general heuristics, like "drawing a helpline" (Pólya, 1945), to more specific algorithms, such as long division. *Compression on cases* is characterized by a shift of attention from a multitude of phenomena (cases) to the common properties of those phenomena (Bos & van den Bogaart, 2021; Tall, 2013; Thurston, 1990). *Compression on cases for a concept* refers to the conception of a new category for a multitude of cases, see Table 1 for an illustration. *Compression on cases for techniques or statements* means recognizing that the technique or statement does not only apply to a single case but to an overarching category of cases. *Compression on details of a concept* means that the definition or important properties of the concept are stored in long-term memory or reference books. *Compression on details of a technique or statement* means that relevant steps and conditions are stored in long-term memory or reference books. Both forms of compression entail a form of hierarchical reorganization in which a multitude is represented by a singular, allowing thought with less working memory involved. Such thought in bigger chunks is essential in overcoming the multiple steps that might constitute a mathematical problem.

Table 1. Examples of six types of compression

	Cases	Details
Concepts	Cases of similar triangles are considered as a new category or concept, e.g., right triangles.	The defining property of a right triangle is that it has one right angle.
Techniques	Considering the Pythagorean Theorem as a technique for cases where a length needs to be computed	For the Pythagorean Theorem as a technique storing steps like identifying a right triangle, labeling the sides, filling in the unknowns, substituting in $a^2 + b^2 = c^2$, etc., and replacing them by a more general description, like "computing one side, knowing the others".
Statements	Recognizing the Pythagorean Theorem (as a statement) applying to all cases of right triangles	For the Pythagorean Theorem (as a statement) storing details of a line of reasoning that supports the statement (a proof).

Compression is key to problem solving. It is essential, for example, to learn to recognize and apply techniques to a wider class of situations than where they are taught. Moreover, solutions to problems are discovered by thinking in terms of general techniques and reasoning with general statements, rather than in terms of the multiple steps and the details that compose them. This relates precisely to some central challenges that students face while problem solving: Recognizing which general techniques and reasonings to apply in concrete cases, and supplying and applying the adequate details and steps for the concrete problem situation. Decompression refers to the inverse cognitive process – applying concepts, techniques and statements to concrete cases and supplying adequate steps and details. Decompression is a central feature of heuristic trees.

Heuristic trees: design principles. A heuristic tree consists of help cards ordered in the shape of a tree, see Figure 1. The content of a card can only be accessed by clicking on it, and the content can only be revealed if the card closer to the trunk has already been accessed. There are several branches for each phase of problem solving: orientation, making and executing plans, completion. The design of a heuristic tree is shaped by a list of six design principles (Bos, 2017).

P1. *Compression-decompression ordering:* The order along a branch should be from general to more concrete hints, thereby decompressing the initial concepts, techniques and statements.

P2. *Logical ordering:* The structure of the tree (both along and across branches) should represent a logical order of reasoning within a solution model. It should highlight the main structure of the argument, and separate main and side issues.

P3. *Problem solving phases ordering:* The branches should be ordered following the phases of problem solving: orientation, making and executing plans, completion.

P4. *Independence:* The help offered in different branches should be independent stepping stones, in the sense that for the help offered in one branch no information in any of the other branches should be needed.

P5. *Rationing:* Each click should not give more help than needed.

P6. *Revelation:* The questions that are shown on the unopened cards should give an indication of what help can be obtained along that branch, but not give away any content. In the same way, a hint on a card

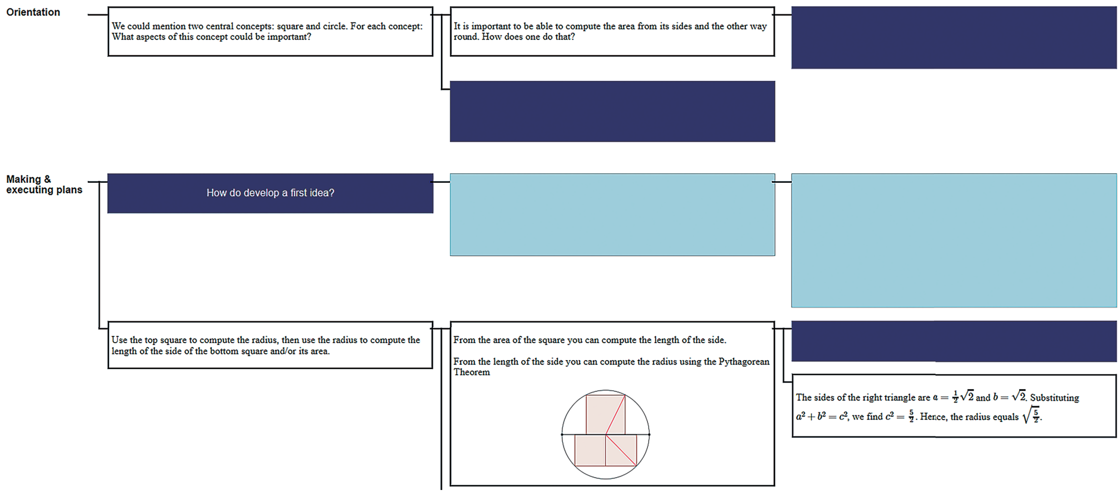

Figure 1. Top half of a heuristic tree(see https://edspace.nl/htree/heuristiekboom.php?boom_id=146# for the full heuristic tress. Note also that blue cards are closed, where dark blue cards are in line to be opened next).

should indicate what can be expected in the next hint, but not give away the content.

Principle 1 (P1) emphasizes the importance of inviting students to compress their mathematical knowledge on concepts, techniques and theory with respect to the problem. Each time students need help, they will first be exposed to a compressed version. Only next, students will find what the details are and how they apply to the problem at hand. Principles P2, P3, and P6 when implemented, help to navigate the tree and find the help suitable for the phase at which they are stuck. Implementing principles P4, P5, and P6 supports the students to maintain maximal ownership of the solution by providing no more help than is needed. Even though opening all cards would reduce the problem to following a set of steps, the goal of the tree is for students to first encounter help on the general concepts, theory and techniques, thereby inviting students to develop conceptual and heuristic thoughts. The goal is for students to develop so that they no longer need to proceed to the end of the branch before they know how to continue their problem-solving approach.

Consider the problem: You have a string of 1 meter in length. Place this string against a wall in such a way that the string forms three sides of a rectangle and the wall the fourth side. How should you lay the string so that the area of the rectangle is maximal?

Figure 2 shows the orientation phase branches for the supporting heuristic tree. The questions for the closed cards are placed above the card content for the left-most cards. Note how these questions and the following information is carefully chosen to suit the orientation phase, following P3. Note that the cards address activities that prepare for embarking on a plan, but do not suggest one. The branches in the orientation phase might lead to an idea, but do not suggest any concrete plans of attack. Also note that the questions always indicate but not reveal the content of the next card following P6. The figure shows precisely which type of decompression is used in each step to the left, making distinctions between decompression on cases and on details.

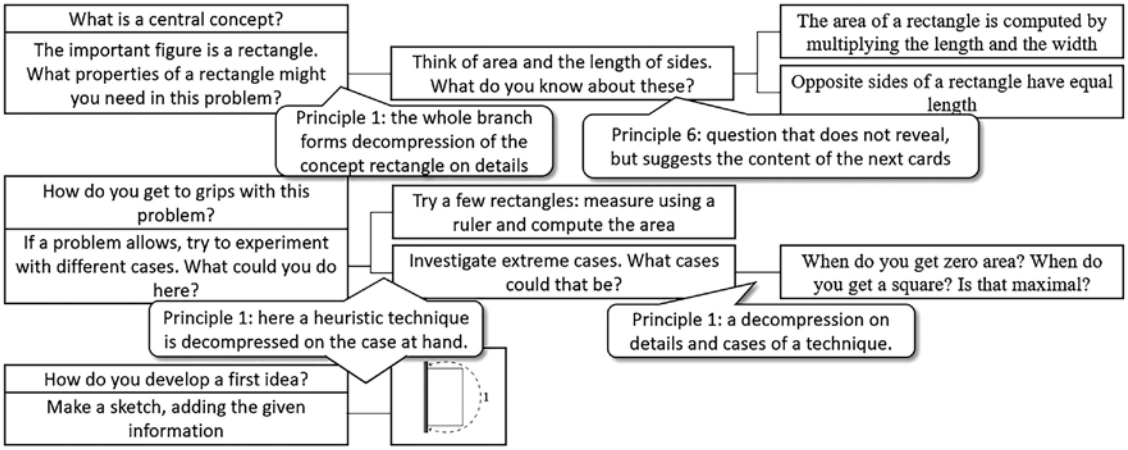

Figure 2. Annotated branches of the orientation phase

Figure 3 shows the branches of the making and executing plans phase. Note how the first branch supports the development of an overarching plan. The second and third branch represent the logical ordering of the problem approach (P2). Note how the first card of the third branch deals with independence (P4), since inevitably the formula for *A* is pre-knowledge for this branch. Branch three demonstrates the rationing (P5) by each time trying to leave as much of the work and discovery for the student to do. Figure 4 shows a single branch of the completion phase.

Note that decompression on statements does not play a role here. Compression on statement plays a more important role if the solution of the problem needs more theoretical development, such as proof-situations. Bos and van den Bogaart (2022) provides more information on how heuristic tree design and implementation relates to other developments in the research field of problem-solving in mathematics education.

Research questions. As stated before, designing a heuristic tree for students involves several challenges for the teacher: Foremostly adhering to the design principles, but also finding the right level for a diverse set of students, choosing what solutions to include, etcetera. With an interest to scaling up the use of heuristic trees and educating teachers in the design process, we want to investigate what difficulties teachers have with designing heuristic trees. In particular, in this paper we aim to answer the following research questions: (1) To what extent do teachers apply the design principles, in particular the principle of compression and decompression? (2) What difficulties do teachers encounter while designing heuristic trees?

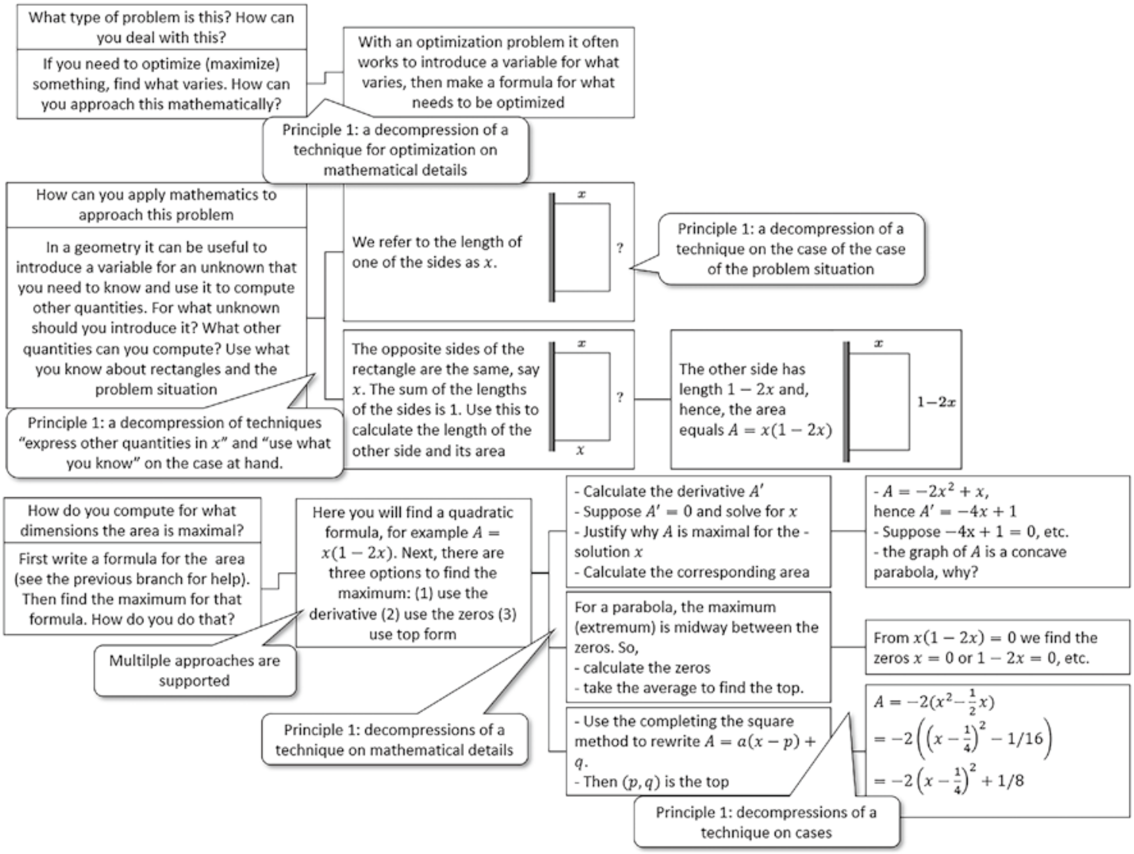

Figure 3. Annotated branches of the making and executing a plan phase

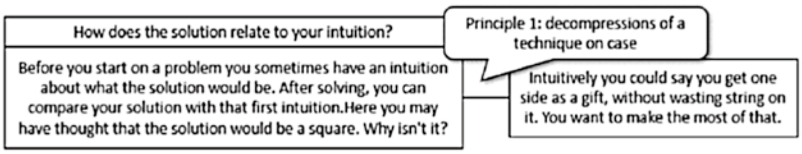

Figure 4. Branch of the completion phase

METHOD

The intervention consisted of a workshop on heuristic tree design, which was made up of two meetings of approximately 120 minutes. This workshop was pretested in a workshop at the ICTMT15 conference in Copenhagen before it was used for this research. The main components of this workshop were for the first meeting (1) a problem solving session supported by existing heuristic trees (online), (2) some theoretical background on heuristic trees, (3) a paper-and-pen heuristic tree design session; and for the second meeting (1) an online implementation session, (2) a try-out session of the participant-developed materials, (3) a critical discussion and feedback session.

The workshop participants received the heuristic tree design protocol developed by us, as a guideline for designing heuristic trees. The protocol consists of the eight steps listed in Table 1.

Table 1. Steps of a design protocol for heuristic trees

Design stage	Guiding steps
Preparation	1. List relevant concepts 2. List the useful general techniques and theory 3. List key insights 4. List where you expect difficulties for the students
Implementation	5. Make horizontal branches exhibiting decompression of a concept, technique or statement, on cases and techniques. 6. For each branch, formulate a closed card question. Use general phrasing, e.g., – What are central concepts for this problem? – How do you start? – What is the key step? – How can you generalize this problem? 7. Make the tree, putting the branches in a logical sequence rooted in three phases: – Orientation – Making & Executing Plans – Completion
Validation	8. Check your design against the design principles

The workshop was held in October and December at HU University for Applied Sciences in the Netherlands. The workshop was attended by participants of a master course for in-service mathematics teachers. The participants of the workshop were invited to share the heuristic trees they had designed during the workshop on the website www.edspace.nl/htree. These heuristic trees were analyzed using an open coding approach (Robson, 2011), with some predefined categories of codes based on the design principles. The final set of codes is presented in Table 2. The two authors coded the heuristic trees independently. The differences in coding between the two authors were then discussed, after which a final coding was settled. Each author also selected representative examples of each registered code, from which the authors jointly decided on a selection of the most informative for the paper.

RESULTS

We analyzed seven heuristic trees sent in by participants of the course. These trees featured a total of 153 cards in 48 branches. Table 2 shows the codes that were the result of the open coding process.

Table 2. The set of codes and the counted frequencies of occurrences in the trees in the data set

Design princple	Codes	Frequency
1. (De)compression order	i. An opportunity for decompression is **missed**	
	a. Concept on cases	1
	b. Concept on details	2
	c. Technique/statement on cases	14
	d. Technique/statement on details	7
	ii. An opportunity for decompression is **taken**	
	a. Concept on cases	1
	b. Concept on details	5
	c. Technique/statement on cases	17
	d. Technique/statement on details	15
	iii. The order along the branch is sequential	13
	iv. A concept, technique or statement could be formulated in a more general way as an advice	12
2. Logical order	Jumps in the reasoning going to the right or down along the tree	3
3. Phases order	Placing support in the wrong phase	9
4. Independence	A hint is dependent on information in a different branch.	11
5. Rationing	A hint is suddenly given, without opportunity for students to come up with this idea themselves.	13
6. Revelation	i. The content of the hint is neither announced on the closed card nor on the previous card	9
	ii. A question on a closed card reveals part of the solution	7
7. General	i. Questions and hints are not well formulated	21
	ii. Irrelevant information is presented	5
	iii. Help on important parts is missing	2
	iv. Information is unnecessarily repeated	1
	v. Low quality support in the completion phase	3

In some cases, decompression on cases and on details was done in one step, as in our example in Figure 2. In this case both codes 1.ii.c and d were scored.

In order to get an idea of how much a tree branches out, we computed the average number of cards that directly follow a card that is not at the end of a branch. For sample trees 1 to 7 this equals respectively: 1, 1, 1.2, 1.3, 1.6, 1, 1. Note that an average of 1 means that each card has at most one following card.

Example 1 (codes 1.i.d, 1.ii.c, 1.iii, 6.i, 6.ii). The problem concerns an isosceles triangle ABC, with $AB = 10$ and $AC = 15$. In the situation of Figure 5 on the right, compute DE. In Figure 5 on the left the two last branches of the orientation phase of the first sample tree are displayed.

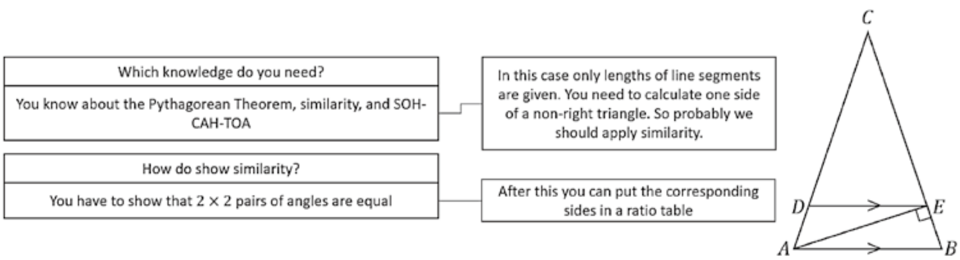

Figure 5. Two branches of the orientation phase of sample tree number 1 (left) and a picture to support the problem.

The second card in the first branch explains why the technique of applying similarity should apply to this case. We interpret this as a decompression of a technique on cases (code 1.ii.c). However, the content of the second card is not announced in the first (code 6.i). In the next branch, the visible question reveals a central technique, thus not allowing students to come up with that themselves (code 6.ii). Again, the content of the second card is not announced in the first card; the student would probably expect more information on what pairs of angles are equal (code 6.i). Instead, the cards in this branch are ordered as two consecutive steps (code 1.iii). Since this branch is part of the orientation phase the idea is to give more information about the technique of using similarity in general, for example providing more details without applying it to the specific case. This is what the question suggests that might be the content, hence we code this as a missed opportunity to decompress a technique on details (code 1.i.d).

Example 2 (codes 1.i.d,1.ii,1.iii,1.iv,2,3,4,6.i,7.i,7.iii). The problem concerns the question what the width is of a rectangular raft with length 56 that would fit in a circular moat with inner radius 37 that is twice as wide as the raft (see the pictures in Figure 6). Figure 6 shows the last three branches from the making and executing plans phase of the second sample tree.

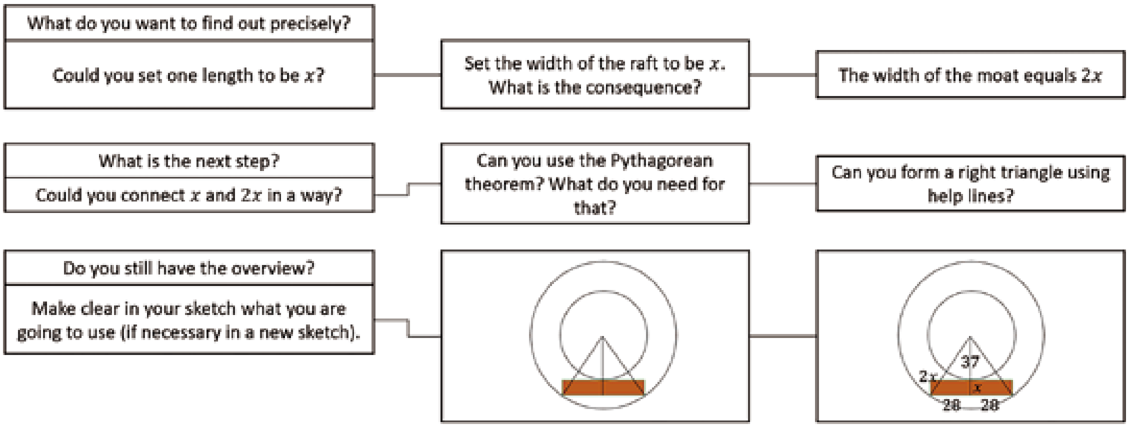

Figure 6. Three branches of the making and executing plans phase of sample tree number 2.

The top branch suggests a technique, introducing x for a length, which is then applied to the case of the raft in the moat. This is coded as decompression of a technique on a case (code 1.ii.c). This technique and also the technique of expressing the other lengths in x could be phrased more generally though (code 1.iv): "For a geometry problem try introducing a variable for a central unknown length, express the other relevant lengths in this variable, and express a relation as an equation in this variable". There is a mismatch between the question on the first card and the content. The question belongs to the orientation phase (code 3) and does not announce the content of the card (code 6.i). The question on the second card "What is the consequence?" does not have a single clear interpretation (code 7.i). The middle branch begins with the question "what is a next step". For a student that has not opened cards of the first branch this does not help navigating to the right help. That is: it is not independent (code 4). The more generally phrased technique above could have also been the beginning of this branch. Moreover, the order of suggestions is now sequential, in the sense that each next card gives a next step in a line of reasoning, instead of a decompression (code 1.iii). Moreover, both suggested and related techniques, using the Pythagorean Theorem and drawing suitable help lines, should be the starting point of a decompression on details (code 1.i.d). Generally, in this fragment the tree does not follow the logical line of reasoning, which would begin with making a sketch and possibly drawing some helplines, then introducing the algebra, and finally forming the equation (code 2). The question on the card of the third branch does not make clear that the support is about drawing a sketch and suitable helplines (code 6.i). The technique of drawing a sketch is clearly applied to the case at hand: a case of decompression on cases (code 1.ii.c). The last card suddenly shows extra information that was not announced in the previous card (code 6.i). Finally, this was the last branch of the "making and executing plans"-phase and a branch of help on the algebraic part of the solution is missing (code 7.iii).

Example 3 (codes 1.i.d, 1.iv ,1.ii.c, 1.iii, 4, 5, 6.ii, 7.i, 7.v). In the problem students are asked to give equations for all lines in the plane that have a given distance to two given points. Figure 7 shows the two branches from the making and executing plans phase, followed by the two branches from the completion phase.

The first two branches exhibit clear examples of decompression of a technique on cases (code 1.ii.c): in the top branch the distance formula is applied to the specific problem, while the second provides help on solving an equation that is specific for the problem case. An opportunity is missed to give decompression on details (code 1.i.d), especially in the second branch which fails to emphasize a general technique for solving a system of two equations. The second branch also contains a sequential progression through the algorithm to solve a system of equations (code 1.iii). In the first branch, the question on the closed card is not clear (code 7i) and the suggestion to use the distance formula comes too soon (code 5); it would be better if the card started with a text such as "How can you make an equation out of the given requirements?". Lastly, the

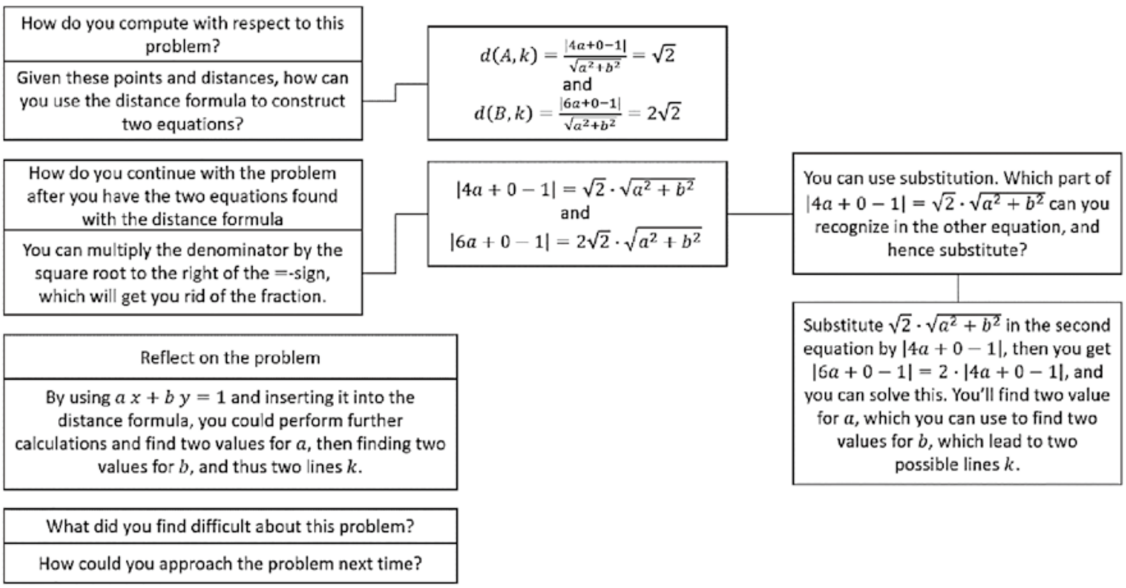

Figure 7. Two branches from the phase Making and executing plans followed by two branches from the completion phase, both from sample tree number 6.

first branch involves information about the distance formula that was on a card in the orientation phase, so the branch is not independent (code 4). Also notice that the question on the closed card in the second branch refers to the distance formula, so the closed card already gives away part of the solution (code 6.ii). The third and fourth branch are short ones; this turns out to be a typical characteristic of branches in the completion phase. But the third one is actually too short: an opportunity is missed to emphasize the general heuristic (code 1.iv) "Summarize the techniques you have used in the problem's solution". Instead, a summary of the solution is given away in the first card (code 5). The last hint is an example of low quality support in de completion phase (code 7.v). It is unclear how to make sense of the question "How could you approach the problem next time", if the problem has just been solved a certain way, and moreover, it is no support in trying to answer the question of what was difficult.

DISCUSSION AND CONCLUSION

In our discussion we follow the order of the design principles P1 to P6, and address to what extent teachers managed to follow them, and implement the advice given in the design protocol.

Table 2 shows that in total 38 opportunities for decompression were taken in the seven sample trees. Most of those opportunities concerned decompression on techniques. This might be a reflection of the approach of teachers and school book being more procedural than conceptual. In many cases, decompression on cases and details was performed at once, as in our example in Figure 2. Instead, it would be better to first present details of the techniques, allowing the students to apply the details to the case. Support on this could in turn be the content of the next card. In this way, maximal ownership of the solution process is allowed, which is a central aim underlying the heuristic trees' design principles.

We observed 24 instances where an opportunity to apply decompression along the branches was missed, and in 12 cases an opportunity to phrase a technique more generally was missed (code 1.iv). This casts doubt on whether the protocol steps 1 and 2 were performed in those cases, and whether the idea of decompression as a central structuring element was fully understood. This leads us to reconsider our approach of the design course. The notion of compression may be to technical; the participants may be better supported by a more practical approach. For example, a more concrete advice to promote a compression structure – without using the term – could be: "start a branch by formulating advice in such a way that it could apply to a similar problem. Try to introduce concepts, technique, and statements in general, before providing details or applying them to the context of the problem".

Teachers seem to have little difficulty following the logical order of a solution method (see Table 2, code 2). However, support might end up in the wrong phase. In particular, we observed too much detail on solution plans already being stated in the orientation phase, or the other way around, suggestions that would suit in the orientation phase – like making a sketch – are presented in the second phase. The support in the orientation phase might at best lead to an idea of an approach, but should not address any concrete plan.

Teachers have a tendency to order information sequentially along a branch (13 occurrences, see Table 2, code 1.iii). The disadvantage of this is that students need to view information on steps that they have already solved to arrive at suitable help, and that opportunities for decompression are missed. The course needs to draw attention to this issue. If, for example, the solution plan consists of three major heuristic techniques, then these should be announced in a first card, followed by a branching into three outgoing lines to cards that decompress on details of those techniques. We observed that four out of seven trees have an average of one outgoing card, that is, no use of branching is made. Even in the other three trees, that have higher averages, we do hardly observe use of this method. We conclude that this should be addressed explicitly in the design protocol.

Let us address design principles P4, P5, and P6. Maintaining independence between information in different branches is a challenge for teachers. There are 11 cases in our sample where this goes wrong (Table 2, code 4). This is difficult indeed, and the only way out is either to strategically repeat information when necessary, or to refer explicitly to other branches for more help. See for example how this is solved in the bottom branch of our example in Figure 3. There are 13 cases where teachers present a hint that gives away information of the solution, without announcing it properly so that students have a chance to find this part of the solution themselves. This happens in particular when revealing a useful technique in the question on a closed card, as in example 3 and Table 2, code 6.ii. It is also related to a lack of decompression, when a technique is immediately presented in detail and applied to the case in the first card of a branch.

Finally, we observed 21 cases where there was an issue with the formulation of the support on the cards. The text is not clear enough in suggesting what to do or think of next. This is a consequence of teachers applying a conversational tone on the cards, as they might use in classroom while supporting a student. The problem is that, whereas in classroom teachers can always clarify themselves, in a heuristic tree the text should be completely self-explanatory. The text should be precise and reveal a well-considered amount of new information. Even though it is good practice to support students by posing questions, revealing a general technique can take a different shape. A good card often contains some new information in the form of a statement, then followed by some questions announcing the following cards.

For the validity of this study, it is important that the participants were given a fair chance to acquaint themselves with the protocol and the principles, and we believe they did through the course. We conclude that, even though the teachers designed heuristic trees that satisfied many of the principles we set out and communicated through the protocol, many issues need to be addressed more effectively in the course, most notably decompression and formulation. It is not clear whether missed opportunities to use decompression come from an inability of the teachers to formulate concepts and techniques in more general terms, or by a lack of attention for this aspect. This could be an issue for further research in the form of interviews with the involved teachers. As discussed above, we believe that many issues could be addressed by improving the course and the protocol. This could also give rise to a follow-up study.

The sample size in this study is small. We stopped including trees after saturation occurred: new trees no longer gave reasons for new codes. Our aim was to give an inventory of difficulties in heuristic tree design. We do not claim the frequencies in Table 2 to be telling more than a rough indication of what are common and what are less common mistakes. The (relative) frequencies will in the end strongly depend on the prior training of the designers. However, there are no reasons why a different group of teachers would make different sorts of mistakes after a similar course.

Our findings also shed light on the way teachers support students generally, i.e., in situations where heuristic trees are not used. Compared to a classroom situation, the teachers have a lot of time to think and discuss about their choices in the way they shape the support for students. In that sense, their heuristic trees present a version of how they support students that should be at least as good as what they would do in classroom content wise—we do not refer to affect here.

Designing heuristic trees turns out to be a demanding and time-consuming enterprise. Nevertheless, we believe this worth the teacher's investment. Firstly, heuristic trees, once designed, can be used over and over again; and, if necessary, be improved. Secondly, the process of design allows teachers to develop their problem-solving guiding skills. Thirdly, joint design invites teachers to discuss their ideas on both the problem content and on their ideas on how to guide problem-solving. And finally, the main goal, a well-designed tree allows successful problem-solving in absence of the teacher.

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TEACHERS' EXPLORATION USING GRAPHIC ORGANIZER FOR PROBLEM SOLVING IN PRIMARY MATHEMATICS

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Abstract

This study is an exploratory study of primary school teachers conducting in utilizing Graphic Organizer as a tool to teach problem solving in a professional development setting. The importance of problem solving among children has been highlighted in the Brunei educational reform, particularly in mathematics. One of the inhibiting factors in teaching problem solving is the low level of comprehension and transformation skills needed to solve mathematics word problems. Graphic Organizer is an instructional strategy used to help students to compartmentalize the necessary information to solve word problems. A group of mathematics teachers was introduced to embedding Graphic Organizer as a tool to address issues in problem solving in a professional development workshop. They implemented the strategy in their respective classes and found ways to apply and assess students' problem-solving strategies. Their reflections on the challenges and affordances Graphic Organizer posed in teaching problem solving are also discussed.

Keywords: Graphic Organizer, problems solving, Mathematics, professional development

INTRODUCTION

Brunei's education system, known as SPN 21(Ministry of Education (MoE), 2009), has put a greater emphasis on teaching and learning that covers the 21st century skills encompassing communication skills, collaborating skills, critical thinking skills and problem-solving skills as stated in the National numeracy and literacy standards. Problem solving has been identified as one of the basic skills in the 21st century (e.g., Trilling & Fadel, 2012). It should be incorporated in the mathematics instruction in teaching and learning as it is regarded as the cornerstone of mathematical learning (National Council of Teachers of Mathematics, 2000; Common core state standards, 2010). In general, Problem solving is a difficult skill to master but must be taught in Mathematics (Myers et al, 2022). Problem solving however has long been in existence in our life. It permeates in every aspect of our life in various forms including those that are of mathematical based problem. Exposing students early to this skill is essential to prepare them to be future-ready. Through early exposure to problem-solving skills, students will experience learning the processes of when and how as well as justifying their actions while applying the skill (Root, Browder, Saunders, & Lo, 2017).

Despite the introduction of various innovations for assisting children to see the relationship between texts and mathematical concepts, word problems remain difficult for children (Simamora et al., 2017) and

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studies have shown that students generally possess low problem-solving ability (NAEP, 2008; Wulandari et al., 2015). Over the years teachers have continuously witnessed children struggling to solve word problems. In this study, a schematic diagram tool known as Graphic Organizer, originally used to improve reading comprehension, was introduced to mathematics teachers. Replacing mathematics symbols, expressions and equations into text makes Graphic Organizer applicable in mathematics education. Graphic organizer enables students to understand concepts and see relationships encompassing mathematical symbols, expressions, and equations, and express them in a graphical manner (Ives & Hoy, 2003).

PROBLEM SOLVING IN MATHEMATICS EDUCATION.

Problem solving does not only require students to perform mathematical computation, but it also requires them to understand the underlying reason for applying the mathematical skills whenever necessary (Browder et al., 2018). This is because problem solving often comes in the form of text embedded in mathematical concepts and problems to be solved. The issue of students struggling to perform well in solving mathematical problems is not new. It is challenging for students to solve mathematics problems (Browder et al., 2018), and in particular, word problems. Many aspects contribute to the difficulty include the absence or lack of familiarity with the context of the problem and the lack of ability to comprehend what is being read. Problem solving often requires the comprehension of textual, graphical numerical and symbols, and this adds to the complexity of the problem (Braselton & Decker, 1994). This situation could be more problematic for children whose language of instruction is not their mother tongue.

Since gaining independence in 1984, Brunei has adopted a bilingual education policy whereby the medium of instruction in schools is in English and in Bahasa Melayu (*Malay language*). All subjects except Malay were taught in English starting from Year 4 primary schools. Since 2008, with the education reform of *Sistem Pendidikan Negara ke-21 (SPN21)* translated as the National Education System for the 21st century, the medium of instruction for Mathematics subjects at lower primary starting Year 1 to Year 3 has changed to English (MoE, 2009). There are challenges and difficulties for teachers to teach mathematics in a language that the students have yet to master. Kirkpatrick (2012) recommended to delay the use of English as medium of instruction until later years of primary schooling or at secondary level. This is because there are no advantages for younger children in Southeast Asia specifically to learn academic subjects in a language other than their first language. Abedi and Lord (2001) supported this claim, where children who are not proficient in English have a disadvantage in mathematical performance, especially word problems. They found that English language learners scored significantly lower in mathematics tests than their peers who have good command in English. In addition, they found in their study that the students with low problem-solving skills would have difficulties with text heavy problems.

In Brunei primary classroom, Mahdan (2020) found that primary school children who are exposed to keyword approaches to solve word problems are unable to correctly comprehend mathematics word problems. Mahdan taught her students; one group using English only and the other group using English and Malay to solve word problems. There were no significant differences other than both groups of students have errors in their understanding of the problems. This is similar to study done by Yusof and Malone (2003), who observed upper primary students can read mathematical word problems, but error increased in terms of comprehending, transforming and processing the word problem. Thus, word problems and problem-solving questions that are heavy in text are among the most problematic topics to teach and learn. Rasidah (1997) found that students performed poorly in solving word problems irrespective of the language used, English or Malay in this case. She encouraged educators to develop students' language skills especially comprehension skills and their problem-solving skills. Thus, students' difficulty with word problems could be due to the complexity of the text in word problem as found in Abedi and Lord (2012).

In the Brunei primary mathematics curriculum, the mathematics skills are interweaved with the mathematics processes and values as depicted in Figure 1. Since the reform in 2009, the mathematics syllabus has placed emphasis on students' mathematical processes such as mathematics computational skills, mathematical thinking, estimation and mental computation, communication, and attitudes and values. Thus, this has gradually changed the mathematics instructions to a more learner centered developing these skills while maintaining the development of the mathematics contents.

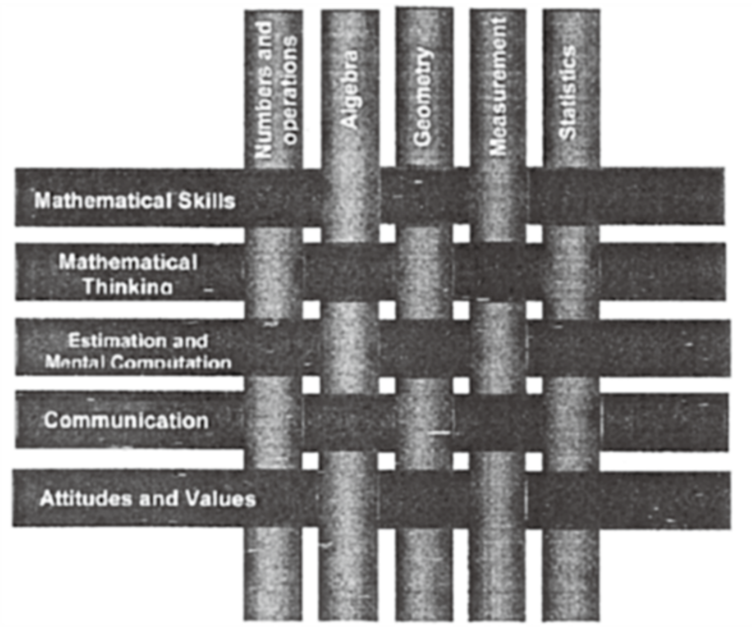


Figure 1. Brunei Mathematics Curriculum Conceptual Framework

Madihah (2006) reported that teachers have difficulties in implementing problem solving and honing mathematical thinking skills to students due to exam-oriented culture that exhorted teachers to take more traditional approaches in teaching to prepare their students for written examination skills. She reported that teachers are also committed to complete the content-heavy syllabus to make sure students are equipped with the knowledge to be tested at the end of the academic year. However, a slow shift in emphasis has emerged; for example, a case study of primary mathematics teachers (Haji Abdullah, 2021) was reported to plan their learner centred lessons focusing on communication and mathematical thinking. However, during the implementation, the instructions shifted to a more teacher-centred approach as the students and the classroom learning environment proved to be an obstacle and hindered the intended planned lesson to be carried out.

Numerous efforts have been initiated to assist students in coping with their difficulty with word problems. Among the initiatives done by scholars and educators includes the use of Graphic Organizers (GO) as a tool to support reading of mathematics problems (Barton & Heidema; 2002) and mathematical problem solving (Khoo et al., 2016; Zollman, 2009). In terms of reading mathematics text, Barton and Heidema (2002) suggested that GO is a useful tool to break down the information presented in the text into smaller parts where students can make connections of these information with their prior knowledge. Furthermore, students can organize and comprehend these pieces of information which is meaningful to them. With improvement in their reading and comprehension skills, students are able to make connections the mathematical concepts in order to solve word problems. This can be found in the study done by Zollman (2009) where middle school children improved in their problem-solving skills specifically where GO aided the students to "coordinate their mathematical ideas, methods, thinking and writing" (p. 8). Similar results were found in local studies done by Khoo et al. (2016) and Sai et al. (2018), whose students showed improvement in their problem-solving skills and, furthermore, with the aid of GO, students' attitudes, and confidence towards problem-solving also improved. GOs do not only help students' language skills but are also used to get a clearer picture for learners to communicate their mathematics thinking in solving non-routine word problems. Khoo et al. and Sai et al. agree that with the use of a graphic organizer, students can communicate their mathematical thinking during problem solving.

GRAPHIC ORGANIZER (GO)

The use of GO originated from an initiative to improve reading comprehension in language classes. Later, mathematics educators 'borrowed' this instructional tool to assist students in understanding the concepts and facilitate their attempt in connecting the relationships between the mathematical symbols, expressions and equations. All these can be expressed graphically in the GO worksheet (Ives & Hoy, 2003). In other words, GO provides scaffolding strategies for learners to unfold the questions asked into digestible information.

Despite having an array of GO formats to choose from, the common feature among the different types of GO is that all these formats aim to depict the thinking process by reorganizing the information into a graphical or pictorial format (Ellis, 2004). The GO is designed visually in such a way that the students or solvers need to write down the known information based on the texts and only then find or propose the solution. Essentially, GO is a visual representation showing the relationship among the key concepts in a topic with the purpose to improve students' learning outcomes by going through the process of comprehending and arranging information before trying to solve the questions. In short, it is a tool that represents the text concepts. In the authentic mathematics classroom, GO serves as an instant tool to check students' learning progress. Teachers can quickly identify from the GO the student's progress or confusion in solving the problem given to them.

In this study, we used Zollman's (2009) adaptation of GO that originated from literacy teaching tools. The adapted GO is a tool that helps students to solve word problems by organizing the processes, information, and possible solutions in no exact order. The *four corners and a diamond* graphic organizer consist of five areas that asked students to 1) think what they needed to find, 2) list what they already know, 3) plan the possible solutions, 4) try out their solutions and 5) explain what they have learned in solving the problem.

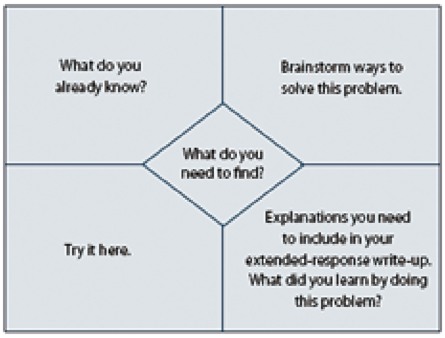


Figure 2. Zollman's (2009) four corner and a diamond graphic organizer

CONCEPTUAL FRAMEWORK

The objective of Zollman's GO is similar to Polya's (1945) approach of solving a mathematics problem. Polya's approach consists of four steps in sequence i) understand the problem ii) devise a plan iii) carry out the plan and iv) check and extend. It is in a linear order where students are required to execute the problem in ordered steps. Although the GO has the same purpose as Polya's (1945) strategies in solving mathematics problems, the solving process is differnt (Zollman, 2009). The GO's spatial model helps students in terms of visualising their strategies and in a non-linear schema diagram approach. The similarity of the sequential steps between Polya's approach and Zollman's GO model is depicted in Figure 3 for clarification.

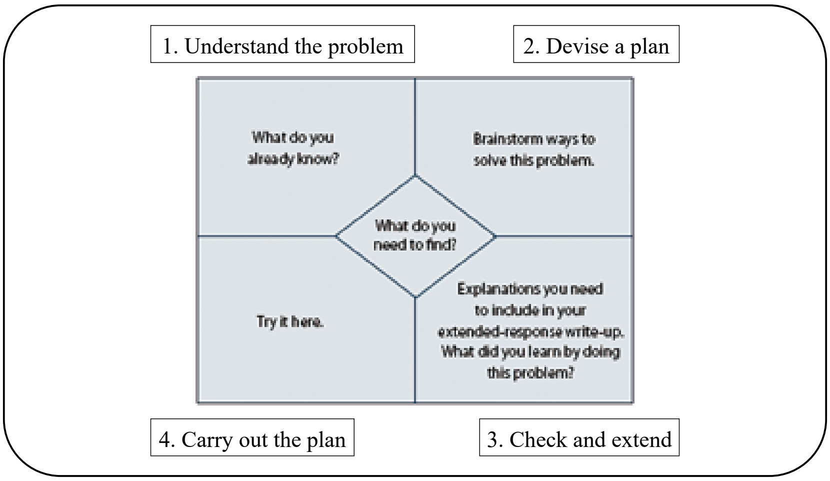


Figure 3. The combination of Polya's (1945) steps of solving mathematical problems onto Zollman's (2009) graphic organizer.

The GO as shown in Figure 3 does not have sequential order to process a word problem. Students can read a mathematics word problem and write the information at no specified order. For instance, upon reading a problem, if students identify a mathematical operation as possible solution, they can fill in the information on the top right corner of the GO. The information then can be stored on the GO while students process other information to solve the problem.

RESEARCH DESIGN

The purpose of our study is to investigate the use of GO in the teaching and learning of problem-solving skills in mathematics lessons in a bilingual education setting. The research aim of the study presented here was to explore the use of GO in primary mathematics classrooms from teachers' perspectives in order to answer the research question "How do teachers reflect on the use of Graphic Organizer for problem solving in primary mathematics lessons?" We explored the experience of teachers in using GO as a tool to aid students in solving non-routine problems based on their written reflections.

The research design of this study is based on a qualitative inquiry to evaluate the use of GO as an aid in problem solving. Patton (2002) termed this as evaluation research in qualitative inquiry where the "qualitative findings in evaluation" is used "to deepen understanding" (p. 10). The grounded theory is used as the foundation of this research design as the methodological approach that is best suited for this study (Lambert, 2019). Inductive analysis was done on the documents consisting of teachers' written reflections.

A total of 30 certified primary mathematics teachers from various government schools enrolled in a professional development workshop on problem solving. They were selected by the Ministry of Education from various schools across the four districts in Brunei Darussalam and were of various academic backgrounds.

METHOD AND PROCEDURES OF THE STUDY

In the workshop, the teachers were introduced to the use of GO as an innovative tool to cater to students' difficulties in comprehending and transforming mathematics word problems. The version of GO introduced by Zollman (2009) was adapted to suit mathematics word problems. The GO consists of four corners space and a diamond space as depicted in Figure 1. There are five questions asked in the graphic organizer, that is, on what to find, on what is known, brainstorming of how to solve the problem, space to try out and space to provide explanation and reflection.

GO was introduced and the teachers had the opportunity to experience how to use the GO in groups during the workshop. They were given time to solve a set of problems. The whole group sharing their application of GO was discussed. In this study, the teachers learnt from the learners' perspective in solving problems with the help of GO by trying out a variety of problem-solving questions and write their answers and thoughts on GO like shown in Figure 1.

In the workshop, the teachers familiarized themselves with eight strategies of problem solving (Hatfield, Edwards, Bitter and Morrow, 2008). The eight strategies that the teachers were exposed to were 1) estimation and check, 2) developing formulas and writing equations, 3) drawing pictures, graphs, and tables, 4) modeling, 5) working backwards, 6) flowcharting 7) acting out the problem and 8) looking for patterns. To communicate their mathematical processes, GO was used to show the strategies used.

In the workshop, teachers developed a set of rubrics they agreed on in assessing students' problem-solving skills. They agreed to look at four categories of students' learning process in solving non-routine word problems. The four categories were 1) understanding, 2) knowledge and process, 3) strategy and 4) explanation. Each learning process had five levels of descriptors with scores ranging from 0 to 4, with 0 indicating no planning shown in the 'strategy' category to 4 indicating students got everything planned with relevant strategy and using clear representations. The detailed rubric is shown in Table 1. In addition, teachers learned and adapted the use of a student scoring guide to solving a problem based on Hatfield et al.'s (2008) assessment on mathematical proficiency.

Table 1

Problem solving rubric

Learning process	Descriptors				
	4	3	2	1	0
Understanding	They understand the whole concepts	They understand almost everything	They understand some of it	They understand a little bit	They did not understand the problems
Knowledge and process	They gave the correct answer and shown clear correct workings	They gave the correct answer and provided appropriate workings	They provided answer with some correct workings shown	Little working was shown, or they provided answer only or they have a little misconception	They did not try at all, or No working was shown at all
Strategy	They got everything planned with relevant, strategy and using clear representations	They got almost everything planned and/or using other appropriate representations	They got some of it planned and/or using diagrams	They got a little of it planned or planned with some misconceptions	They did not try to plan at all
Explanation	They explained why they did everything using mathematical terms and steps	They explained most of why they did things and/or using keywords and steps	They explained some of why they did things and/or with some keywords	They explained a little of why they did things and/or showed some steps	They did not try to explain at all

Groundwork was conducted before the teacher implemented their action research in their own classrooms. Upon discussion, upper and lower primary teachers confirmed one similar written problem and another problem of their choice. In total, there were two problems to be solved by their students. The teachers encouraged their students to utilize GO in solving the problems. They were given more than 4 weeks to conduct the action research before writing a report to the facilitators. It was agreed to have the minimum of three lessons: introduction of GO, students applying the GO in solving at least 2 problems; each in a lesson.

Research evidence is based on the reports done by the teachers at the end of the workshop series in which they were required to share their findings with all the workshop participants via group presentation. This explorative study involves collecting multiple forms of evidence encompassing students' work using GO, students' rubric, data of students' achievements and teachers' reflections. These data were then analysed qualitatively. First, the authors read the written reports multiple times especially the reflections part of the reports and took notes by highlighting information of teachers' experience in implementing GO in their lessons. In an open coding phase, this highlighted information was categorised into two categories namely challenges and opportunities. In these categories, several subcategories were found to support the categories. The students' works were collected as evidence to corroborate the claims of the teachers. The authors compared notes, analysed and synthesised the findings separately and then collaboratively to validate our analysis through researcher's lens (Creswell & Poth, 2018).

However, it is a limitation of the study where participants' validation was not available. This is largely due to logistical constraints. With participants' lens on the findings, it could provide further credibility to the findings of this study (Creswell & Poth, 2018). Furthermore, this is a study involving a small sample of teachers in Brunei and may not be generalised to the population.

RESULTS AND DISCUSSION

The authors analyzed all the reports and evidence provided by the teachers. The first section describes the implementation of GO in solving non-routine problems and the reflections from the teachers on their experiences implementing GO in their lessons. The second section is to extend from the teachers' experiences on the possibilities of using GO as an aid in solving non-routine problems, as an aid in problem-solving skills, as a mediator for communication skills and as an assessment tool in problem-solving.

Teachers' experiences and reflections

All the participating teachers implemented the non-routine problems in the lessons and used GO as an instructional and learning tool. They reported their students on mixed-ability students. In the teachers' reflections report, teachers shared and described their experiences of using GO for non-routine problem solving. These reflections are further categorised into opportunities and challenges as shown in Table 2 and elaborated in next sections.

Table 2

Main coded themes of teachers' reflections on using GO

Teacher Group	Opportunities	Challenges
TG1	Expressed ideas in own language It encourages the pupils to express their ideas Helps pupils to plan how to solve the problem Help pupils to get marks even if they don't answer correctly Helps pupils to think and communicate their process	Language barrier Problem in mathematical terms
TG2	Excellent tool for visual learners who struggle with information that is presented in written form Gives every student starting point for problem-solving process	Limited time frame High achiever dominant Difficulty in whole sentences explanation (language barrier)
TG3	Graphic organizer helps express students' thinking Group work is better than individual task	Difficulty in brainstorming session and explanation corners Allocate more time to explain and familiarise students on how to use GO
TG4	Graphic Organizer a tool to organising information and relationship As a strategy that may assist pupils (who are) identified as having difficulties with problem-solving. Increase pupil's willingness to attempt the problem. Mitigating hesitancy and resistance. Allow pupils a strategy to organise the information before answering, hence may increase the confidence to answer.	Need extra support for pupils with learning difficulties Continuously use graphic organizer for familiarisation
TG5		There are too many ways to answer the question. The writing of the question should be very clear and unambiguous.
TG6	Clearly shows their planning and derived their answers By using the graphic organizer the process of solving a problem is taken into account and pupils may score marks in the process regardless of final answer Low achievers at least are able to score marks Rubric help to identify the strength of each pupil	Took a lot of time to familiarize with GO but once students figured it out less help from the teacher

OPPORTUNITIES

The teachers reported the opportunities that the use of GO had presented to them in their teaching. They reported that GO is a good tool for representing students' knowledge, help to express students' thinking, and enable students to express their ideas in their own language styles. These affordances supported the benefits of GO as recommended by Zollman (2009). A sample of the student's work using a graphic organizer is shown in Figure 3.

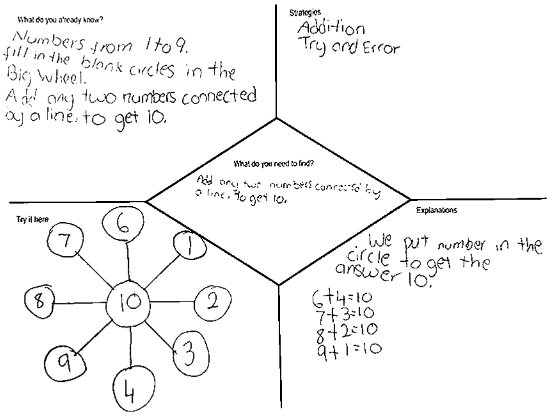


Figure 3. Sample of student's work using graphic organizer

The teachers reported that, through the use of GO, their students were able to explicitly express their knowledge and thought processes on the spaces provided in GO. With the students' written work on the GO, the teachers were able to collect evidence and discern the type of strategies used by their students. The schematic diagram of the GO breaks down the steps into non-linear sections that enable students to write down their thinking and their solving strategies to find the solution. For students to be able to articulate their thinking processes. As with any higher order thinking, for students to be able to articulate their thinking processes, it is not an easy feat as reported by the teachers, who asserted that it is an added value but disrupts their normal routine. To change a habit, a new technology as a disruptor may be useful. Maybe graphic organizers are a possible disruptor in this context.

Based on the results we have learned from the teachers' actual use of GO in their mathematics lessons involving problem solving, we would like to recommend a few strategies that might help in the learning for problem solving.

GO AS AN AID IN PROBLEM SOLVING SKILLS

Despite language barriers, the students were able to hone their problem-solving skills specifically in the mathematical learning process of understanding, knowledge and process, strategy, and explanation. This is in agreement with Khoo et al. 's (2016) study of using GO with secondary students in solving word problems. The student participants in their study showed an increase in mathematics performance particularly Mathematical knowledge, Mathematical strategy, and Mathematical explanation.

GO AS A MEDIATOR FOR COMMUNICATION

The GO can not only help students as a reading strategy, as GO was used in learning English language, but it has also further aided the students to communicate their mathematical knowledge and mathematical thinking skills. This could be seen from the following statement made by teacher TG1A (pseudonym).

TG1A: The pupils that used Malay languages in their writing were able to tell what they were asked to do. In other words, they know what to do; it's just that they have a language barrier.

Despite teacher TG1A's students using Malay language to explain their solutions, they had understood the question. This is different from studies done by Yusof and Malone (2003) and Mahdan (2020) where their students were unable to understand problem solving involving word problems, irrespective of the medium of language used, which is Malay and English. This interesting incident might be due to the use of GO that compartmentalizes on finding out the pre-concept ideas that the students have on the problem. This statement supports findings by Sai et al. (2018) where their student participants were able to make progress in word problems by breaking down the information using GO into smaller sections. These findings correspond to the advantage of using GO as Zollman (2009) recommended "helps students reduce and organise information, concepts, and relationship" (p. 5).

TG1A: It can reduce information processing demands as it doesn't need to process as much semantic information to understand the question given.

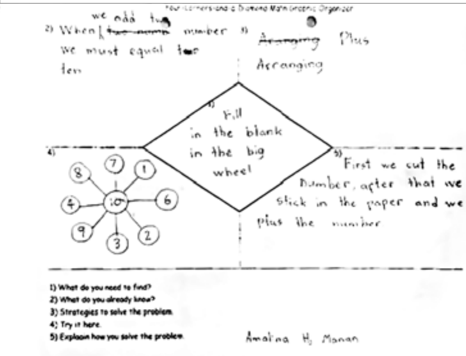
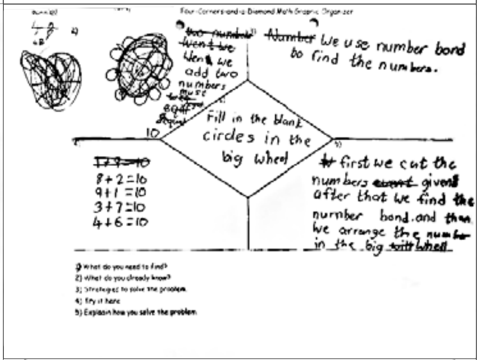
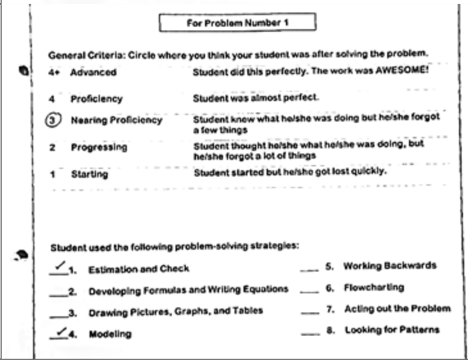
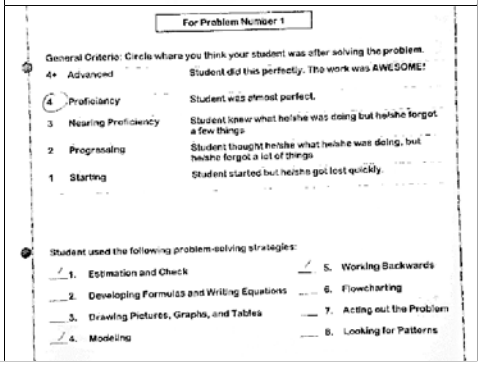
GO AS ASSESSMENT TOOL FOR PROBLEM SOLVING

Madihah (2006) reported that teachers may not be equipped to teach mathematical thinking then. In the action research performed by the teachers, none of them showed their unreadiness in teaching problem solving skills. In the implementations, none of the teachers reported on the difficulties of using the rubrics to assess students' problem-solving strategies (Table 2). This could mean that with the use of rubrics, it helps teachers to assess students' mathematical processes more efficiently and systematically.

The teachers also reported that they were able to identify the problem-solving strategies their students had used on the GO as well as their proficiency in solving problems. Sample of students' work and assessment from teacher TG1B are shown in Table 3 below. The pictures show how the teachers focused on the proficiency of students' problem-solving skills as well as the problem-solving strategies based on the information written on their GO.

Table 3.

Students' GO and Problem-solving rubric

Student	TG1BSA	TG1BSD
Graphic Organizer		
Problem solving proficiency and strategies	For Problem Number 1 General Criteria: Circle where you think your student was after solving the problem. 4+ Advanced — Student did this perfectly. The work was AWESOME! 4 Proficiency — Student was almost perfect. 3 Nearing Proficiency — Student knew what he/she was doing but he/she forgot a few things 2 Progressing — Student thought he/she what he/she was doing, but he/she forgot a lot of things 1 Starting — Student started but he/she got lost quickly. Student used the following problem-solving strategies: ✓ 1. Estimation and Check 2. Developing Formulas and Writing Equations 3. Drawing Pictures, Graphs, and Tables ✓ 4. Modeling 5. Working Backwards 6. Flowcharting 7. Acting out the Problem 8. Looking for Patterns	For Problem Number 1 General Criteria: Circle where you think your student was after solving the problem. 4+ Advanced — Student did this perfectly. The work was AWESOME! 4 Proficiency — Student was almost perfect. 3 Nearing Proficiency — Student knew what he/she was doing but he/she forgot a few things 2 Progressing — Student thought he/she what he/she was doing, but he/she forgot a lot of things 1 Starting — Student started but he/she got lost quickly. Student used the following problem-solving strategies: ✓ 1. Estimation and Check 2. Developing Formulas and Writing Equations 3. Drawing Pictures, Graphs, and Tables ✓ 4. Modeling ✓ 5. Working Backwards 6. Flowcharting 7. Acting out the Problem 8. Looking for Patterns

With these practices as evident in Table 3, the assessments of mathematics problem solving were seen to be *shifting* towards a more process-oriented assessment. This could be the first step to mitigate the norm in Brunei mathematics teachers where Abdullah and Leung (2019) study, the Brunei mathematics teachers normally focused on product-oriented lessons. Although the 2019 study was not specifically on problem solving, mathematics teachers in this study shared their struggle to implement different approaches in the classroom when the norms or *cultural scripts* of mathematics lessons are strong. These cultural scripts include Brunei teachers to rely on product-oriented lessons preparing for examinations. Due to this norm, low-ability students were mostly left behind. However, with GO, these low ability students were found to be motivated in attempting solving the problems and they were still able to score marks based on their processes rather than the final answer. This sentiment was highlighted by the teacher participant:

TG1B: The pupils get some marks even if they don't get the answers correct.

CHALLENGES

The challenges reported included time constraints in using a GO because students would take a long time to be familiarised with it, too much class discussion surrounding the use of GO, and assessing the GO was time-consuming. Although the teachers felt that the use of GO was time consuming, they also saw the benefits and suggested for GO to be used in the early years of primary schooling, more frequently and beyond problem-solving lessons.

TG1C: The use of graphic organizers in solving any problems for the primary pupils should be implemented as early as in Lower Primary Year 1 since it can improve the pupils' thinking skills to solve mathematics problem questions.

TG1A: It is hoped that the graphic organizer can be used in the classroom more often especially when it comes to problem solving.

TG3: We should use the graphic organizers in the routines Mathematics questions to make them understand well on the purpose of this method.

TG3: We also found out that pupils also work better than doing it individually.

However, we believe as suggested by some of the teachers, that when students have become accustomed to using a graphic organizer, time limitation will no longer be an issue. To address this, the authors recommended that graphic organizers be used in other subjects as well. Originally, GO are used in English language teaching, thus adhering to Brunei's Whole-of-Nation approach, we can also use graphic organizers in other subjects and let students become accustomed with the tool in their learning.

CONCLUSION

As with any other novel tools employed in the classroom, challenges are inevitable, and opportunities are forthcoming. In this study, graphic organizers do help with the communication skills of the students in terms of expressing their mathematical thinking. Furthermore, it could be a useful tool to assess students' mathematical performance in the process stage and less focus on the final product of problem solving. The authors believe that with effective instructions, graphic organizers can be a powerful tool that can help develop students' problem-solving skills as well as promote high quality communication of their mathematical thinking as aspired by the Brunei education reform of SPN21.

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A CHARACTERIZATION OF THE PROBLEM-SOLVING PROCESSES USED BY STUDENTS IN CLASSROOM: PROPOSITION OF A DESCRIPTIVE MODEL

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Abstract

My Ph.D.'s research (Favier, 2022) aims at characterizing the processes used by students when they solve problems in the ordinary context of the classroom, i.e., when the problem-solving session is led by the teacher. The chosen problems may require students to make trials and errors. I consider two levels of characterization: the outer structure (Lehmann et al., 2015) of problem solving processes in terms of timing and organizing of processes and the inner structure (Ibid) considering heuristics. In this paper, I focus on the outer structure of the processes. Embedded cameras installed on the students' heads were used to collect audio-visual data as close as possible to the students' work. The recorded work of 33 groups of two or three students (20 groups at the primary school and 13 at the secondary school level) for a total of 79 students are coded independently by a research assistant and by us using the framework for the analysis of videotaped problem-solving sessions by Schoenfeld (1985). It consists of cutting the students' work into macroscopic chunks called episodes. The analysis of these empirical data leads us to discuss and enrich the descriptive model of problem-solving processes proposed by Rott et al (2021) with an additional dimension that allows us to take into account the interactions between students and teacher. The use of this enriched model allows us to identify three problem solvers' profiles in terms of the process implemented. Some of these processes are linear like Pólya (1957) proposed in his model. Some others are cyclics like in Schoenfeld's model (1985). The large majority are not supported by these two very well-known models which shows the contingency of the student's work and, therefore, the limit of these two models (in particular Pólya's model) used to teach problem solving.

Keywords: Mathematical problem solving, Descriptive process model, trials and adjustments

INTRODUCTION

Problem solving in mathematics as a method for developing students' learning is put forward by different actors in the educational system, whether they are researchers or institutional leaders, in many countries and at different school levels (Dorier & Garcia, 2013). Houdement (2009) distinguishes two functions among the learning objectives of mathematics problems. On the one hand, there are problems that contribute to the construction of mathematical concepts. These problems best cater to learning *through* problem-solving. On the other hand, there are problems that form a part of the mathematician's activity such

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as searching or validating, which can be used for *learning about problem-solving*. In this paper, the second category is the focus of my study. More specifically, my research aims*[1] at characterizing the approaches used by students when they solve mathematical problems in classroom. In particular, we have chosen problems that may require students to make trials and adjustments (often called trial and error in the literature). The complexity of the problem-solving processes is taken into account at different levels of analysis. Lehman et al. (2015) discuss the inner structure and the external structure. In this paper, I focus on the external structure, i.e., the temporal organization of the process.

THEORETICAL BACKGROUND

In this section, the main models of problem solving found in the literature are discussed, followed by a discussion of the similarity and differences across these models. The first model to be discussed is Pólya (1957)'s model, which is seminal in the field of mathematical problem solving. Pólya models the problem solving process through the sequence of four successive phases as shown in Figure 1.

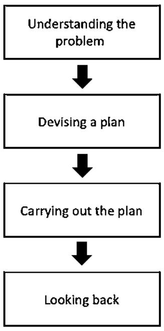

Figure 1. Pólya's model (1990, p. xxxvi)

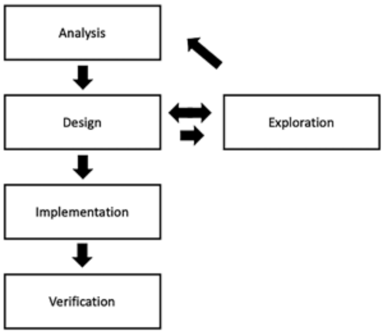

Figure 2. Schoenfeld's model (1985, p. 110)

To develop his model, Santos-Trigo (2014) explained that "Polya (1945) reflected on his own experience as a mathematician to write about the process involved and ways to be successful in problem-solving activities" (p. 498). This model has a prescriptive function since its intention was to teach students to solve mathematical problems. Moreover, this model has been used in many mathematics textbooks (Santos-Trigo, 2014; Wilson et al., 1993) and is still used as it is in French-speaking Switzerland (CIIP, 2018; CIIP, 2019).

The main criticism of this model is its linear nature. Nevertheless, it has served as a basis for several researchers to develop their own approaches, in particular Schoenfeld. Indeed, Schoenfeld (1985) has enriched Pólya's model by adding a phase that he calls *exploration* (Figure 2) to account for the part of the research that tends to move away from the understanding of the statement as such, which is therefore no

*[1] My doctoral research is part of a more global project, financed by the Swiss National Science Foundation, whose title is : « La résolution de problèmes comme objet ou moyen d'enseignement au cœur des apprentissages dans la classe de mathématiques : un point de vue fédérateur à partir d'études dans différents contextes ». Co-requestors Jean-Luc Dorier and Sylvie Coppé, Subside n°100019_173105 /1.

longer part of the appropriation phase, but which does not (yet) constitute a plan that needs to be implemented. By adding this phase, Schoenfeld differs from Pólya in that he affirms that the process is not necessarily linear, but can even be cyclical, that is, when there is a sequence between the three phases of analysis - planning/implementation - exploration in this order or if there is a back-and-forth between the planning/ implementation and exploration phases. Mason et al (1982) also criticizes this linear aspect and propose a model which joins that of Schoenfeld on the cyclical character. Schoenfeld's model can also be described as normative, since it is supposed to represent "what may be called the *ideal* problem solver, or the most systematic behavior of good problem solvers" (Schoenfeld, 1985, p. 107). Wilson et al (1993) take the development of the model a step further by proposing a dynamic and cyclical interpretation of the different stages of Pólya.

Figure 3. Wilson et al 's model (1993)

Wilson et al's idea also breaks away from the idea of linearity, which is not very consistent with the genuine problem-solving activity (Ibid, p. 5). This model has also been used in a teaching perspective. Fernandez et al. (1994, p.196) noted that the model is being used as the framework in a mathematics problem-solving course at the University of Georgia to aid the discussion of issues involved with teaching mathematics problem-solving in elementary and secondary schools.

One of the common points of these three models is that they are built from the phases of the Pólya's model. This makes it possible to highlight their major differences, which concern the sequence possible of these phases. Another common point between the Pólya's and Schoenfeld's models is that they are built from the study of expert resolution. These processes are taken as a model in the sense of an example to follow. This is not the case with Wilson et al's model as these researchers attempt to take into consideration the fact that students may go back and forth between the different phases during their search which results in a more complex model. Despite this important difference, all three of these models are primarily used for educational purposes (Rott et al., 2021, p. 3).

Rott (2012) puts these different models to the test by analyzing the processes actually mobilized by 10-12 year old students to solve problems under laboratory conditions (students participated voluntarily and were taken out of school time). His analyses show that the processes mobilized by the students are not all linear (30 out of a total of 98 observations). This number, far from being negligible, highlights the fact that a strictly linear model such as Pólya's is not adapted to realistically describe students' work. Moreover, among these 30 resolutions, Rott counts 12 for which the process can be considered as cyclic and thus 18 of these resolutions present non-linear processes. Also, non-linear analysis models must allow for junctions between different phases of the resolution process. The cyclic Schoenfeld's model is a sub-case of all the cases that can be considered by a non-linear model. Furthermore, Rott observes that students do not necessarily make their solution plan explicit before implementing it, which leads him to group the planning and implementation phases together. Finally, students do not always move directly from ownership to planning and implementation, but often go through an exploration phase. This confirms, according to him, the importance of considering this exploration phase to account for the unstructured dimension of some of the processes.

Rott identifies some properties that a model should possess to describe problem-solving processes for 10-12 year-old students:

- There should be a distinction between structured and unstructured behavior (Planning and Exploration) as in Schoenfeld's model.
- It should be possible to intertwine Planning and Implementation.
- It should be able to display both linear and cyclic processes – with the majority of those processes being linear (Ibid, p. 105-106).

These different ideas are represented by the following diagram (Figure 4):

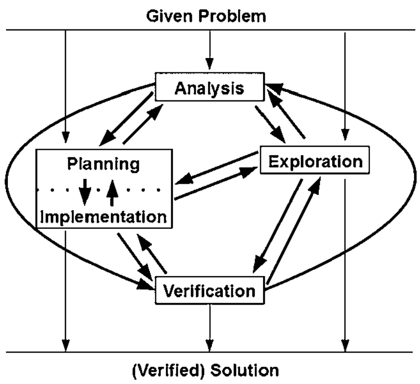

Figure 4. Rott's descriptive model of problem-solving processes (2012, 2021)

Thus, according to Rott, this model allows for a description of the problem-solving processes. A limitation of these different models is that they focus either on experts or on students who come to solve problems in the laboratory. None of these works studies the work of students in an ordinary classroom situation, with the possible interventions of a teacher. However, what interests us is precisely to characterize the approaches of students who solve problems during sessions lead by teachers. This leads us to formulate the following research questions:

- How can we characterize the problem-solving process of students when they solve problems in class?
- What characterizations emerge from this analysis?

METHODOLOGY

Research Context

My research was carried out in the canton of Geneva (Switzerland) with the intention of covering the three cycles of obligatory education. To do so, I focused on three different levels of education, from primary to secondary levels:

- Primary 4 (4P), which is the last year of cycle 1 and corresponds to students aged 7-8 years;
- 8th grade (8P), which is the last year of cycle 2 and corresponds to 11-12 years;
- 10th grade (10^{th}), which is the middle grade of the orientation cycle and corresponds to students aged 13-14 years.

Each degree is represented by two different classes.

The Problems

Problems that were used for this study can all be solved by trials and adjustments. The problems for the three different levels are presented below.

In 4P, Game Of Cards:

> Each card in my deck represents either a triangle or a square. I pull 15 cards at random. I count all the sides of the figures drawn on the cards I pulled and find 49. How many triangles and squares do you think I pulled?

In 8P, Dragons and company:

> On a computer screen parrots, crocodiles and dragons are seen. In total, I counted 20 heads, 72 legs and 30 wings. How many parrots, crocodiles and dragons are there?

In 10^{th}, Test ball:

> In order to renew its sports equipment, a school makes a first command of 2 rugby balls, 4 basketballs and 4 soccer balls for a total amount of 72 CHF. Then it makes a second order of 2 rugby balls and 2 basketballs and paid 30 CHF. We know that a rugby ball, a soccer ball and a basketball together cost 20 CHF.
> What is the price of each ball?

The 10th graders had learnt algebraic equations but had not learned to solve systems of equations. By making certain deductions, 8P and 10th graders could quickly find one of the answers. For example, in the Dragons and Company problem, the students could have used the data 30 wings to deduce that there are 15 animals that have wings (because dragons and parrots each have two wings). They then could deduce that there are 5 crocodiles. At this point, they could solve a problem of the same difficulty as the one proposed to the 4P students.

The Experimental Conditions

The research condition was not restrictive as it was intended to observe the students' approaches to solving problems as closely as possible to their usual working conditions. Thus, the arrangement that was proposed to the different teachers were the following:

- Introduction of the statement, answers to possible questions related to comprehension:
- 1st time of individual problem solving
- 2nd time of group problem solving

The teachers were asked to allow the students do as much problem solving as possible within the stipulated time limit of 45 minutes.

Data Collection

In order to attempt to document student's work, experimental data as close as possible to the students' work was collected. This being the case, the fact that my research was carried out in the usual classroom context, and that in order to guarantee the most ordinary and stable conditions for conducting the experiment, I faced certain constraints. In particular, the classic means of data collection (such as "thinking out loud" techniques, recordings with a camera on a stand, explanatory interviews (Vermersch, 1994)) were not feasible. Technical innovations, in particular the on-board camera, were exploited. According to Morieux (2016), the on-board camera allows "original shots and access to images of great didactic, pedagogical interest [...]" (p. 68). If using a camera external to the actor (classic data collection conditions) allows for a "third person" perception (Andrieu & Burel, 2014), filming with an on-board camera on the actor's body offers a "first person" perception (Ibid). This provides access to the students' workspace from their own perspective. In this way, the following became visible to us:

- anything the students pointed to or designated;
- items in students' field of vision;
- all the gestures that the students made in their field of vision;
- all manipulations performed with materials; and
- everything the students wrote, including those that were erased.

We also recorded everything the students uttered and probably (almost) everything they heard. These different elements are valuable for documenting the approaches used by the students during problem solving.

This first-person recording gave us the complete and real chronology of the different steps that constitute the students' problem solving. We should thus have access to the students' "private work" (Coppé, 1993), including all the trial-and-errors produced by the student and not only to the "public traces" (Ibid.) of their work that they want to show.

We therefore equipped one student per group with an on-board camera attached to the head. For each class, we collected the audio-visual recording of the work of each group filmed by the equipped student. It should be noted that we excluded from this corpus the few groups for which the audio or visuals were not usable.

Here is a summary of the number of groups filmed and the number of groups making up the corpus of data to be analyzed:

Table 1. Distribution of groups filmed and to be analyzed by class

Classroom	4PC	4PS	8PS	8PV	10CT	10LS	Total
Number of groups filmed	7	6	6	6	5	8	38
Number of groups to be analyzed	6	5	4	5	5	8	33

Each group is referenced by a code composed of the grade, the initial of the teacher (or the course of study for the 10th graders), the reference to the problem and the camera number. For example, the file 8PVDr7 corresponds to group 7 of Valerie's 8th grade class working on the problem Dragons and Company.

Framework For The Analysis Of The Empirical Data

To analyze the audio-visual data collected in each group, Schoenfeld's (1985) method of video analysis was used. It consists of parsing the students' problem-solving work into macroscopic chunks called episodes: "an episode is a period of time during which an individual or a problem-solving group is engaged in one large task or a closely related body of tasks in the service of the same goal." (Ibid, p. 292). Schoenfeld characterizes five types of episodes labeled by action verbs: read, analyze, explore, plan/implement, and verify. Here is a more precise description of how each of these episodes is characterized:

- *Reading*: this episode, as its name indicates, corresponds to the student's reading of the statement and also includes the time spent appropriating the various elements of the statement, whether in the form of silences, verbalizations, or silent re-reading.
- *Appropriation*: this episode corresponds to the attempts made to better understand the problem, to adopt a point of view and reformulate the problem in its terms, and to examine any principle that might be appropriate.
- *Exploration*: this episode differs from the previous one in its structure (it is much less structured) but also in its content, which moves away from the initial problem, in search of relevant information.
- *Planning/implementation*: in this episode, the emphasis is on the control dimension, which is why elements relating to the formation of the plan are not mentioned. Thus, it is more crucial whether the plan is structured or not, whether the implementation of the plan is methodical, whether the student monitors the process with feedback on the planning or on the evaluation at local or global levels.
- *Verification*: Schoenfeld considers the nature of this episode to be obvious.

He completes these five episodes with a so-called *transition* episode:

- *Transition*: this episode encodes the junctions between the other episodes, which are nodes during most of which decisions are made that can impact the solution in one way or another.

Rott (2011, 2012) adapted this method by introducing two additional episodes: *digression* and *writing*.

- *Digression*: this episode takes into account different student behaviors not related to the problem-solving task, such as when students talk about cartoon characters or TV series, etc., instead of working on the mathematical content.
- *Writing*: this episode allows for coding time spent writing (such as copying an answer) without getting new information or making real progress in problem solving.

Since this study took place in the classroom, with students working in groups, with the teacher ensuring the conditions of the study to be as close as possible to ordinary sessions. Therefore, the implementation of this method of analysis in the context of our research requires an additional adaptation. An additional category of episode, *regulation*, was introduced in order to take into account the exchanges that can take place between the students and the teacher.

Moreover, only one protocol per group was coded. In the event that two students produced different actions at the same time, the most informative interpretation from the point of view of the problem-solving was selected.

Finally, the audio-visual data were coded independently by a research assistant and by the researcher. We compared our coding results and when they did not coincide, we reached a consensus by recoding together. In terms of inter-coder agreement, we calculated a percent agreement*2 (Jacobs et al., 2003, p. 100) of 0.76 for the nature of the episodes and 0.82 for the time codes.

RESULTS

The results of the parsing into episodes are presented in Table 2.

Table 2. Two examples of segmentation in episodes

	4PCJc7	
01:00	02:16	Reading
02:16	05:03	Implementation
05:03	06:28	Transition
06:28	08:46	Implementation
08:46	10:12	Regulation
10:12	11:53	Exploration
11:53	12:59	Transition
12:59	14:05	Implementation
14:05	14:15	Verification
14:15	15:15	Planification
15:15	16:23	Transition
16:23	16:55	Implementation
16:55	18:12	Digression
18:12	19:05	Implementation
19:05	19:22	Verification
19:22	20:15	Transition
20:15	23:39	Implementation
23:39	24:47	Regulation
24:47	25:32	Implementation
25:32	25:48	Verification

	4PCJc14	
00:20	00:30	Reading
00:30	01:42	Implementation
01:42	03:10	Reading
03:10	03:40	Implementation
03:40	05:07	Transition
05:07	06:40	Implementation
06:40	07:09	Regulation
07:09	10:40	Implementation
10:40	11:50	Transition
11:50	12:30	Implementation
12:30	13:03	Regulation
13:03	16:53	Planning - Implementation
16:53	17:58	Digression
17:58	20:09	Implementation
20:09	20:42	Planning - Implementation
20:42	22:54	Digression
22:54	25:45	Planning - Implementation
25:45	30:33	Digression
30:33	37:20	Regulation
37:20	43:20	Planning - Implementation
43:20	57:17	Digression
57:17	58:10	Implementation
58:10	01:00:08	Digression
01:00:08	01:01:40	Implementation
01:01:40	01:01:50	Verification
01:01:50	01:03:13	Transition
01:03:13	01:05:05	Regulation
01:05:05	01:06:14	Implementation
01:06:14	01:06:55	Transition
01:06:55	01:08:45	Regulation
01:08:45	01:10:20	Planning - Implementation
01:10:20	01:10:30	Verification

The first column corresponds to the start time of an episode while the second column marks the end time of this episode. The name of the coded episode appears in the third column. The tables obtained from the episode coding show a relatively high number of episodes and, moreover, a strong dispersion as shown

*2 Calculations of the percentage of agreement could be calculated for the 8PV class, and the two 10th grade classes. For the classes of 4P and 8PS, we recorded the result of the confrontation on the same document as our initial coding. We thus lost the initial coding depriving us of the possibility to calculate the inter-coder agreement for these three groups.

by the box plot (Figure 5).

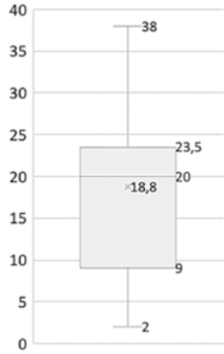

Figure 5. Box plot representing the distribution of the total number of episodes for each group

The students' work was divided into a minimum of 2 episodes and a maximum of 38 episodes. In addition, 75% of the segmentations have more than 9 episodes and 50% have more than 20. As it stands, this division and coding into episodes made comparisons difficult, since a great deal of richness was expressed through the chronology of the different episodes, their duration and their nature. In order to interpret these results, the complexity of these divisions was reduced by operationalizing the Rott's descriptive model (2012, 2021). Then, after having pointed out an important limitation highlighted by the experimental data, an enrichment of this model in order to allow the continuation of the analyses was proposed.

Using The Descriptive Model Of Problem Solving Processes

Following these authors, I call *phase* the different parts of this model; the term *episode* is reserved for the coding of audio-visual data. This model is completed by the number of passages from one *phase* to another which is inscribed in a circle next to the corresponding arrow. As far as our experimental data are concerned, this model seems to be operational for describing the work of groups of students for whom there was no teacher intervention, i.e., who did not present any regulation episodes. This concerns seven groups. The work of these seven groups using this model are presented below. Next to each number of passages (circled number), we have added another number, written with a grey italic font, which specifies the order of the passages in the model. Each of the following figures groups the results by proximity to known normative models.

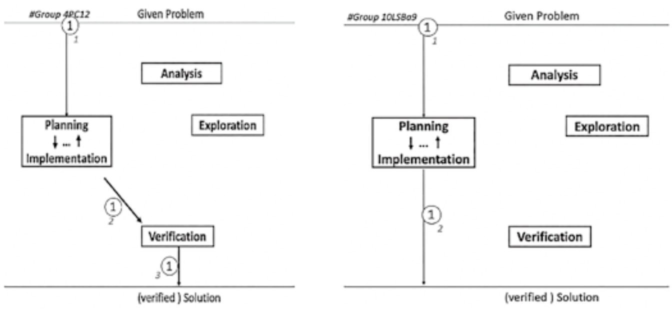

Figure 6. Groups whose resolution process is linear (in the sense of Pólya)

These first two groups show a linear "top-down" pathway of the model. This linear characterization corresponds to Pólya's model (1957). However, we can already see a small difference in that these two groups do not go through an analysis phase (i.e. *understanding the problem*) as envisaged by Pólya.

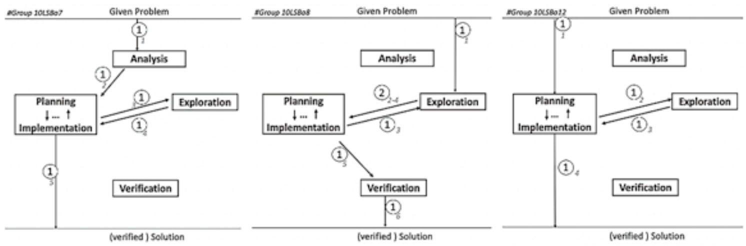

Figure 7. Groups whose resolution process is cyclic (in the sense of Schoenfeld)

These three groups are comparable in terms of the back and forth between the planning-implementation and exploration phases. This cyclical aspect of part of the process can be related to Schoenfeld's model (1985). As we noted a few lines above, these three groups illustrate three different ways of starting the resolution. Only group 10LSBa7 (on the left in Figure 7) is faithful to Schoenfeld's model since the process starts with an analysis phase. The other two groups highlight the need to enrich Schoenfeld's model by planning to enter directly into a planning or exploration phase as allowed by the model of Rott et al.

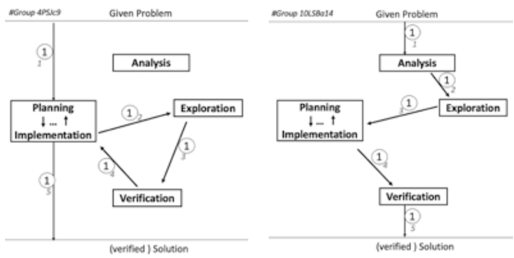

Figure 8. Groups whose process is not related to a normative model

Finally, these last two groups mark a break with the previous five groups. The one on the left (4PSJc9) shows an "ascent" in the model after the verification phase. The one on the right (10LSBa14) shows an appropriation phase followed by an exploration phase which corresponds to a sequence not supported by Schoenfeld's model. The processes of these two groups are not similar to the two models mentioned above. It allows us to understand and confirm (Rott et al., 2021) that the Pólya and Schoenfeld's models are not sufficient to account for the contingency observed in the classroom.

All of these seven groups highlight the operationality of Rott et al's model since it allows for the description of processes that are both linear, cyclical but also those that may escape them due to the more dynamic and non-linear aspect of the course. It also offers the possibility of describing processes that begin with a planning-implementation or exploration phase and not just an appropriation phase. This being said, the confrontation of this model with the other experimental data at our disposal, points to an important limitation which has led us to enrich it.

Proposal Of A Descriptive Model Of Problem-Solving Processes In The Case Of Teacher Interventions

To make this limitation explicit, we note that the problem solving by the subjects in the Rott et al. (2021) study was accompanied by as little help as possible from a tutor. It appears that this help was reduced enough to be neglected in the coding of the data. In this study, the teachers' management of the session and the interventions they might have had with certain groups of students were far from negligible. Indeed, to characterize the problem-solving process of the students in class requires the integration of the participation of the teacher, which necessarily brings a new complexity in the description of the phenomena. These intervention by the teachers were not taken into account by Rott et al. Moreover, the moment when the teachers intervened cannot be considered as being of the same nature as the other phases of the model. We therefore propose to enrich this model by adding a dimension to account for *regulation*. Thus, this additional dimension is on a different plane than resolution and we choose to materialize it in a third dimension (in the geometrical sense), in the enriched model. By testing this proposal for an enriched model on my experimental data, it appears that this *regulation* phase can be connected with all the other phases of the basic model. Double arrows are therefore necessary to represent these different possible connections. Moreover, we could observe some groups for which *regulation* occurred at the very beginning of the research, which explains the arrow that starts from the problem and points to regulation. In the same way, it happened that the solution was found or checked during a regulation: either because it was the teacher who did the work, or because he guided it. It is therefore necessary to add an arrow that starts from the *regulation*'s phase and points to the solution. Therefore, the model I use to describe the problem-solving processes in cases where the teacher intervenes is the following:

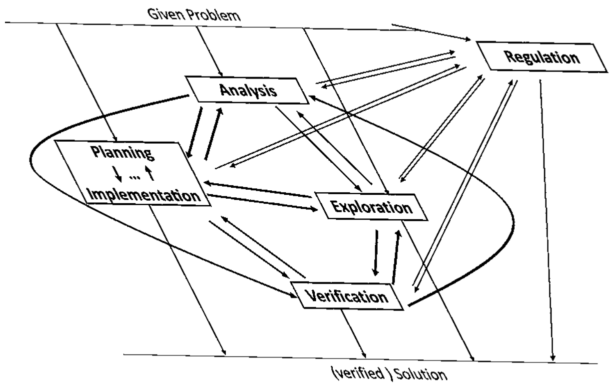

Figure 9. Proposed model to describe students' resolution processes in the case of teacher interventions

The following figures (Figure 10) show two examples that visualize the descriptions I obtain using this model:

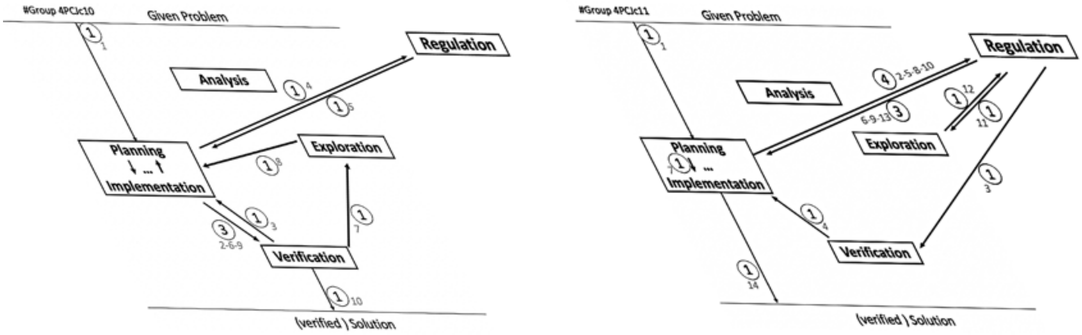

Figure 10. Two examples of students' resolution processes

The comparison of the different processes also reveals three general tendencies: linear processes (in the sense of Pólya), cyclical processes (in the sense of Schoenfeld) and those, very dynamic, which are not taken into account by these two models.

Linear processes

The three coded groups 8PSDr3, 8PVDr2, and 10CTBa12 show a linear process that is characterized by a solely top-down model path. The regulations do not deviate from the linear path of the model run.

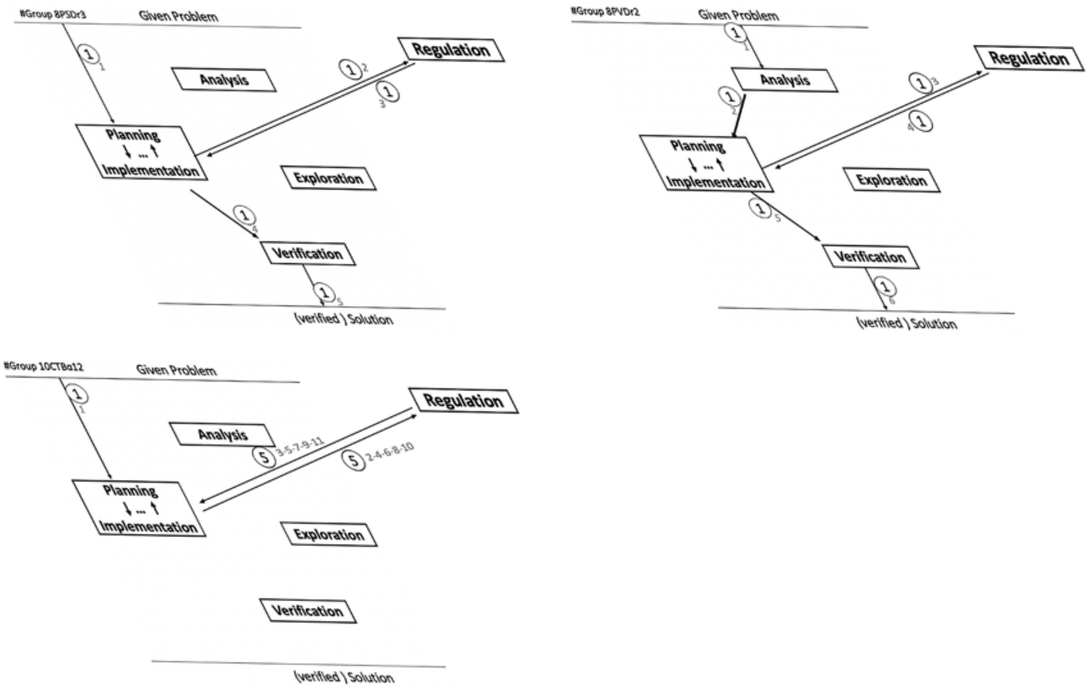

Figure 11. Groups whose process has a linear trend (in the sense of Pólya)

These graphs describe very different realities that are highlighted by the number of regulations (number circled next to the arrows that start or point to the regulation). Indeed, the group (10CTBa12) shows many regulations while the other two groups (8PSDr3 and 8PVDr2) have only one regulation each. I come back to this idea in a following section.

Cyclic processes (in the sense of schoenfeld)

A process to be cyclical (in Schoenfeld's sense) if there is a sequence between the three phases of analysis - planning/implementation - exploration in this order or if there is a back-and-forth between the planning/implementation and exploration phases. Thus, among the set of processes analyzed, the following three correspond to this cyclical characterization in Schoenfeld's sense:

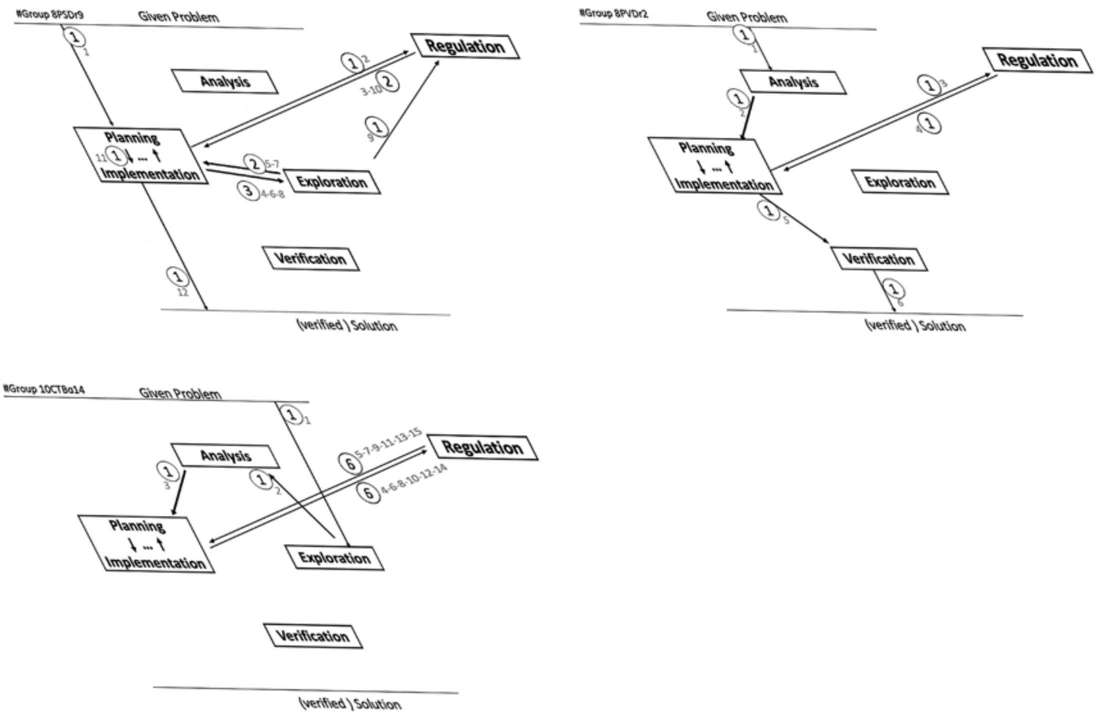

Figure 12. Groups whose process has a cyclical tendency (in the sense of Schoenfeld)

The first two groups show direct back and forth between the planning/implementation and exploration phases, corresponding to steps 4-5-6-7-8 for the 8PSDr9 group and steps 6-7-8 for the 10CTBa13 group. There is also some back-and-forth regulation: steps 9-10 for the 8PSDr9 group and steps 11-12 for the 10CTBa13 group.

Some groups can be associated with the Schoenfeld's model insofar as they are the regulations that allow for movements described by this model. Three groups are representative of this aspect as shown in Figure 13.

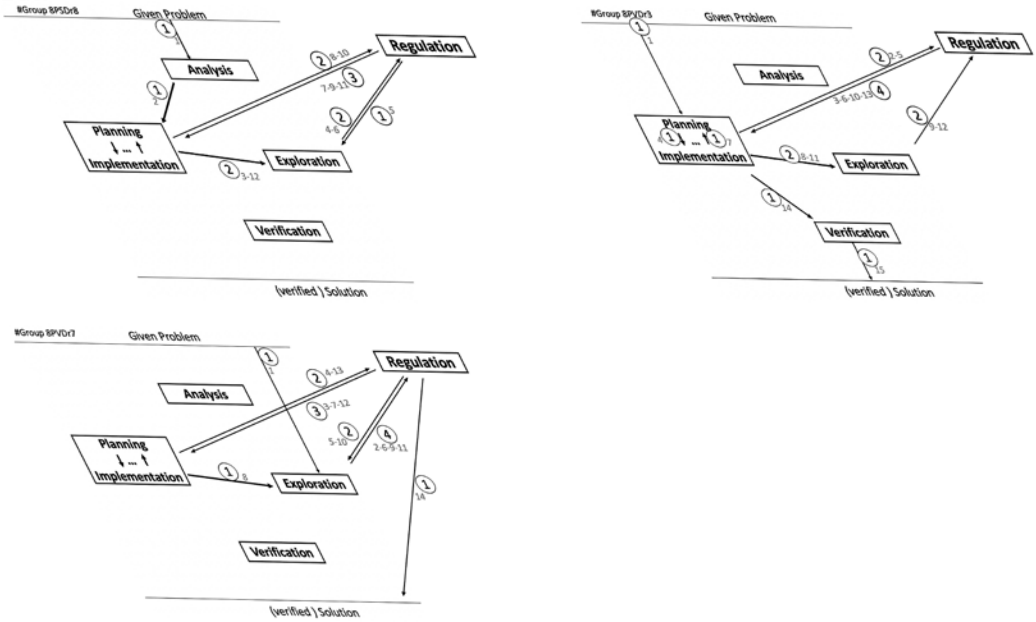

Figure 13. Groups whose process has a cyclical tendency due to regulations

Consider the group 8PSDr8. It is the sequences 6-7 that allow the return from the exploration phase to the planning-implementation phase. For group 8PVDr7, different round trips are materialized by steps 2-3; 4-5-6-7; 11-12. For these three groups, the regulations make it possible to move from an exploration phase to a planning-implementation phase rather than the reverse.

Processes not related to a normative model

All the other groups present processes whose tendency is not similar to the normative models mentioned above, that is to say that the course of the model is made according to a sequence of phases which exceeds the scope of those allowed by the two preceding models. For certain groups, these sequences are directly ensured by the students, i.e., without going through regulations.

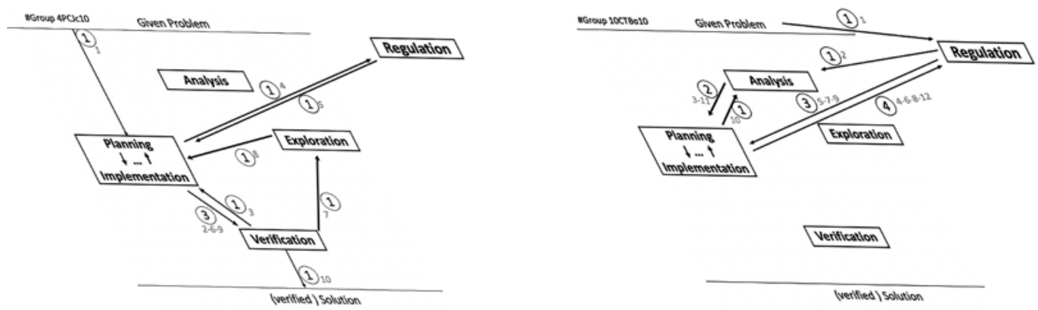

Figure 14. Groups whose process has a tendency not related to a normative model

This is the case, for example, for the sequences numbered 3 and 7 in group 4PCJc10, which highlight that students move from a verification phase to a planning-implementation phase (3) or from a verification phase to an exploration phase (7). For the 10CTBa10 group, it is step 10 that reveals the sequencing of a planning-implementation and analysis phase. For other groups, these sequences are made possible by the regulations. Thus, steps 9-10 of groups 4PCJc14 and 10CTBa7 (Figure 15) show that the students' work has moved from a verification phase to a planning-implementation phase via regulation.

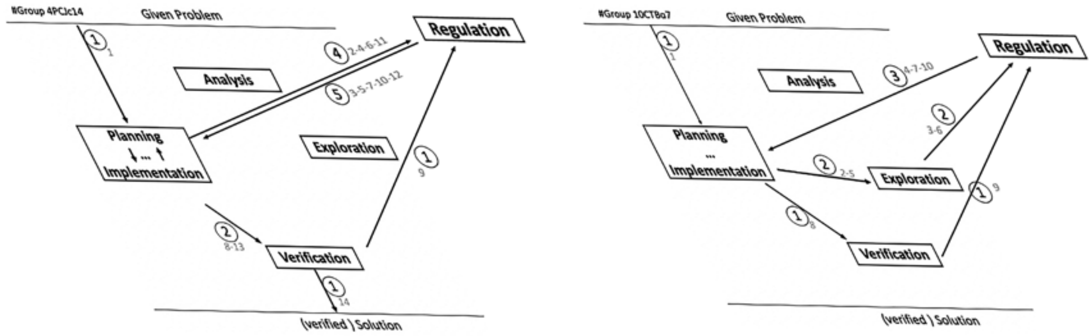

Figure 15. Groups whose process has a tendency not related to a normative model due to regulation

In conclusion, the enriched model I propose is operational to describe students' resolution processes in the case where the teacher intervenes. It allows us to identify the same trends as the model of Rott et al. (2021) when there is no teacher intervention. We thus counted a total of 3 groups whose processes have a linear tendency, 6 groups whose processes have a cyclical tendency in the strict sense of Schoenfeld, and 17 groups escaping these two tendencies. These 17 groups still represent two thirds of the corpus for which teacher interventions are recorded, which is very substantial. This reveals the limitations of normative models, particularly when we seek to describe the actual work of students in the classroom. However, this model has allowed us to highlight all the diversity and contingency that students' solving processes in class can present.

Moreover, I note that one type of process identified by Schoenfeld (1985) does not seem to appear in my analyses. This will be explained in the next section.

A Type Of Problem Solving Process Absent

Schoenfeld (1985) identified a type of problem-solving process that he called "wild goose chase" and described it as follows "The students [...] embarked on a series of computations without considering their utility and failed to curtail those explorations when (to the outside observer) it became clear they were on a wild goose chase." (p. 316) An operationalization of this type of process is proposed by Rott et al. (2021):

> A process is considered by us to be a "wild goose chase", if it consists of only *Exploration* or *Analysis* & *Exploration* episodes, whereas processes that are not of this type contain *planning* and/or *verifying* activities (only considering content-related episode types). (Ibid, p. 14)

None of the groups in my corpus can be classified a priori in this type. At first look, this result may seem surprising since my corpus, composed of 33 groups divided into six classes, represents a sample that seems sufficient to observe the diversity of processes. Also, it is unlikely that there are no such groups in my data but rather that the characterization proposed by Rott et al is not operational to identify them among my data. Indeed, the fact that my experiments take place in classrooms with interventions by a teacher who manages the session leads me to believe that this type of student manifests itself through other behaviors. We hypothesize that the teachers in our experimental set-up have identified these students or groups of students who might have tended to waste their time on calculations without considering their usefulness and to get lost in false leads and that they have intervened with these groups. Thus, it would be the teachers' interventions that impact the students' behavior and prevent me from identifying the processes that might characterize this "wild goose chase" profile. That said, if I cannot identify these groups of students, I can characterize another type of process. Indeed, five groups, 4PCJc14, 8PVDr8, 10CTBa10, 10CTBa12 and 10CTBa14, present processes that are quite remarkable from the point of view of the significant number of connections, almost always consecutive, between a planning-implementation phase and the regulation provided by the teacher. For example, for the 10CTBa14 group, the description allowed by our model shows the sequence of 6 back-and-forth movements between a planning-implementation phase and the regulation phase. These round trips are consecutive since they correspond to the sequences numbered from 4 to 15. We record 5 round-trips of the same type for the 10CTBa12 group (sequences numbered from 2 to 11) and 4 for the other three groups.

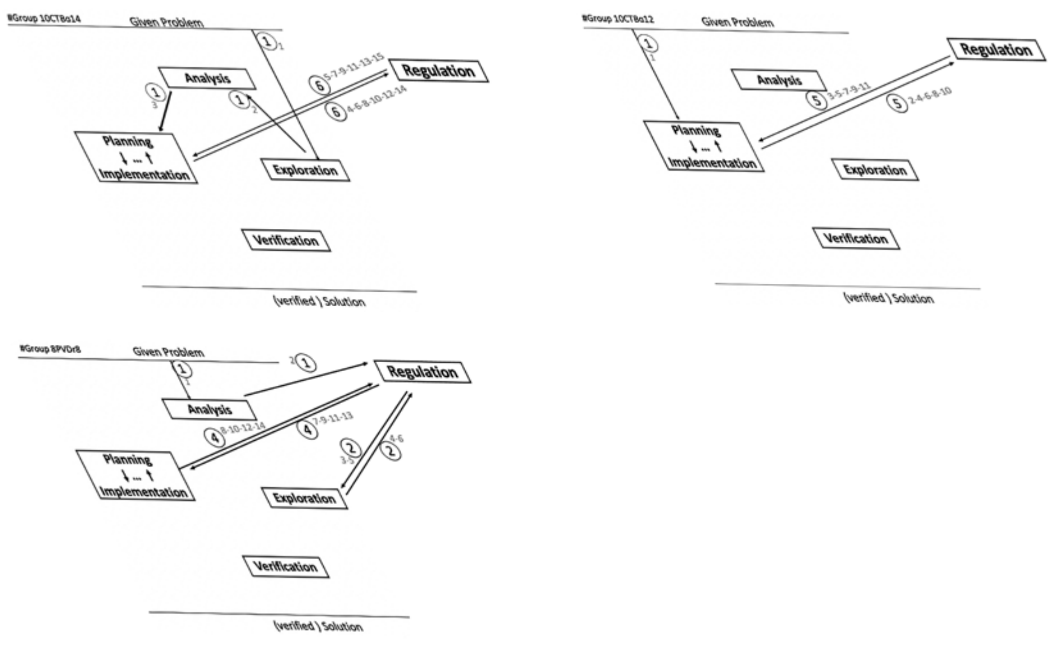

Figure 16. Examples of groups with high teacher involvement

Thus, the common point of these five groups is the important presence of the teacher through a large number of regulations. It seems that these groups need to be supported in their work, which would explain why their process is punctuated by the teacher's numerous interventions. However, the fact that regulations are linked to a planning-implementation phase indicates that after each regulation, the students are able to implement a track that has emerged from the regulation, even if it is not known, a priori, whether the track is proposed by the teacher, by the students or if it is co-constructed. This finding is all the more significant since these five groups belong to three different classes and therefore three different teachers.

Relationship Between Problem Success And Model Type

In this section, the distribution of the different groups according to the type of model and to the success of the problem is examined.

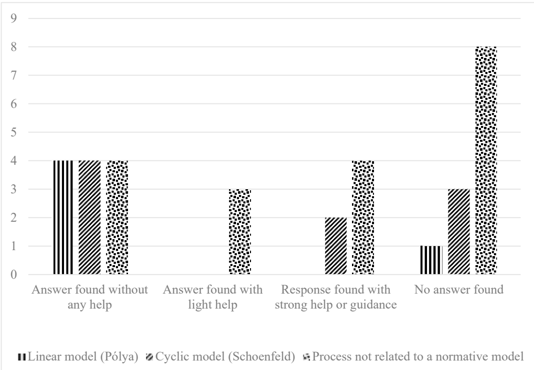

Table 3. Distribution of the groups according to the type of model to which it refers and according to the success of the problem.

This distribution indicates that four of the five groups whose process is similar to Pólya's model solved the problem and, what is more, without help. Moreover, these four groups were among those who found the answer in the shortest time (1-15 minutes) compared to the other groups who found it in 20-30 minutes. These indicators suggest that the problem may have seemed simpler to them. This confirms that Pólya's model is better suited to account for the work of groups for whom the problem did not offer much resistance.

Among the groups that did not manage to solve the problem, i.e., those for which the resistance was sufficiently strong, I note that processes not related to a normative model largely dominate (12 groups out of 18).

CONCLUSION

In my research, I aim to characterize the processes used by students when they solve mathematical problems in class. To do so, I first used Schoenfeld's video analysis framework to divide the students' work into episodes. An adaptation of this framework was necessary with the introduction of a regulation episode in order to take into account the moments during which the teacher intervenes with the students. This first analysis shows a very large disparity, particularly in terms of the number of episodes. In order to be able to interpret this disparity, I chose to reduce the complexity by focusing on the episodes relating to the students' work and by analysing their sequence. I thus operationalized the descriptive model of resolution processes developed by Rott et al (2021). This operationalization allowed us to identify three main trends with respect to the 7 groups that do not present a regulation episode: linear processes (in the sense of Pólya), cyclic processes (in the sense of Schoenfeld) and non-linear processes. This being said, a limitation of this model, inherent to my experimental data, is the complexity brought by the teacher's intervention in the students' work. To take this into account, we have enriched the model of Rott et al. by inserting a phase that is likely to be connected with all the other phases of the model. When I operationalized this enriched model, I was able to characterize the resolution processes according to the same three tendencies.

Finally, I was able to highlight groups of students whose resolution process is marked by numerous back and forth movements with a regulation phase. Most of the time, this back and forth is done with a planning-implementation phase. This leads me to wonder whether the research is left to the groups of students or whether they apply ideas proposed by the teacher.

I could realize that focusing on the external dimension of the problem-solving process is interesting but obviously not sufficient. This is why I complete this work by analyzing the internal dimension and more precisely the heuristics and knowledges that students invest in problem solving. The exploration phase is, according to Schoenfeld (1985), the heuristic core of problem solving. This invites me to investigate whether there are links between the different phases of the descriptive model and the heuristics used.

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A STRATEGY FOR ENHANCING MATHEMATICAL PROBLEM POSING

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Abstract

In this work, we establish a heuristic strategy, the purpose of which is enhancing the posing of new problems in the school context. The strategy is supported by a cognitive framework consisting of six stages: Selecting, Classifying, Associating, Searching, Verbalizing, and Transforming. The first five actions make up an essentially creative process, while the last stage is present within the nucleus of the previous ones. This provides the process with a high level of complexity. Compactly, we call the strategy SCASV+T. We reflect on the heuristic nature of the strategy, as well as the didactic actions that are required for its implementation. We also describe a didactic situation in elementary geometry, where the posing of new problems based on one already solved is discussed. The analysis is carried out with students who are studying a Bachelor's degree in Mathematics Education, who know the strategy and try to put it into practice collectively. Analysis and discussion are led by a professor, who provides suggestions and demonstrates the importance of each action in the development of heuristic reflection.

Keywords: problem posing, heuristics strategy, elementary geometry, school mathematics, teacher training

INTRODUCTION

Posing new problems is a characteristic of advanced mathematical thinking. By its own nature, this process is basically creative and is closely related to other aspects, such as problem solving skills, imagination, the use of analogies, and the capacity to generalize (Cruz et al., 2016; Silver, 1997; Singer & Voica, 2015; Tuchnin, 1989; Van Harpen & Sriraman, 2013). Numerous researches on problem posing which have been published in recent years highlight the importance and helpfulness of using problem posing in the school context (Baumanns & Rott, 2020, 2021; Cai & Hwang, 2020; Felmer et al., 2016; Gabyshev, 2021; Leikin & Elgrably, 2020). An interesting aspect focuses on the stages that take place during problem posing. Polya (1957) provided a very useful model of the problem solving process. Brown and Walter (2005) recognized the existence of stages within the problem posing process. The fact that problem posing also consists of stages is not surprising since many researchers have remarked that there is a very close relationship between posing and solving problems (e.g., Chang, 2007; English, 2020; Koichu, 2020; Peng et al., 2020; Silver, 2013; Yao et al., 2021).

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Mathematical problem posing is a process of high cognitive complexity that excludes those trivial situations involving the simplified embodiment of questions (Cai & Hwang, 2020). The good question is exactly the final stage, and it is mediated by a highly creative activity that is born from a high level of affect and motivation (Cai & Leikin, 2020; Tuchnin, 1989). Numerous studies have highlighted the need to promote problem posing in the school context. This has not only been consigned in important normative documents, but also in research reports, scientific events and international forums related to mathematics education. In this regard, Kilpatrick (1987) has indicated that: “Problem formulating should be viewed not only as a goal of instruction but also as a means of instruction. The experience of discovering and creating one’s own mathematics problems ought to be part of every student’s education” (p. 123).

Despite the well-known need to encourage problem posing at school, there are important aspects that have not been sufficiently addressed in the scientific literature. For example, there are not many tests or other types of scientific research instrument, with adequate levels of reliability and validity, that serve to evaluate the levels of development of the process of posing new problems. In the latter case, any advance in the identification of the main stages, actions and operations of thought will be useful from the didactic point of view. As early as the 1950s,, Polya (1957) noted that problem posing and problem solving are closely interrelated processes. He stated that: “To find a new problem which is both interesting and accessible, is not so easy; we need experience, taste, and good luck. Yet we should not fail to look around for more good problems when we have succeeded in solving one” (p. 65). This conception is dialectical and reveals that problem posing and problem solving are difficult to separate from the didactic point of view. Thus, it is necessary to investigate how to educate students in thinking, so that students are ready to generate new problems. This can be approached from the perspective of didactic strategies, but necessarily involves a psychological background that serves as a framework for the mental actions of thought.

In this paper, inspired by the stages established by Brown and Walter (2005), we present a structure that models the process of posing mathematical problems. This cognitive structure works as a strategy on the didactic level, since teachers can adopt the stages of the framework, as a kind of guide in the teaching process. Some new relationships which have not been explored in previous works (cf. Cruz, 2006) are presented. The strategy is illustrated with the help of an elementary geometry problem, which was analyzed jointly with students from the Bachelor of Mathematics Education at the University of Holguín.

THE HEURISTIC STRATEGY AND ITS COMPONENTS

The heuristic strategy is supported by a cognitive framework consisting of six stages: Selecting, Classifying, Associating, Searching, Verbalizing, and Transforming, with the acronym SCASV+T (Figure 1). Although the first five stages express an apparent linear path, the process becomes more complex when transforming is included.

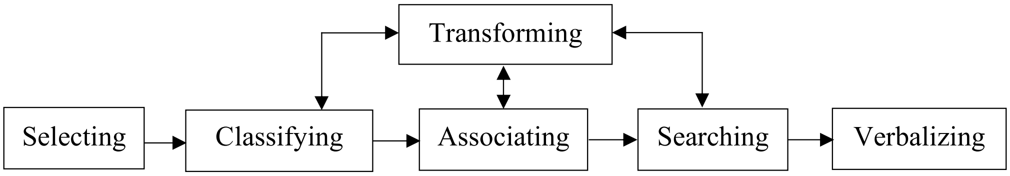


Figure 1. Cognitive framework of SCASV+T heuristic strategy

SELECTING

Structurally, this strategy begins with the selection of a given object or phenomenon which corresponds with "choosing a starting point" described by Brown and Walter (2005) and expresses the intentionality of posing problem as a motivated aware cognitive activity. Silver (1994) states that problem posing involves the generation of new problems about a situation or the reformulation of given ones, so the starting point in the Selecting stage can also be a previously solved problem, or even a problem that is in the process of being solved. Then, the subject breaks up the object or phenomenon (a problem, a situation, a geometric figure, a set of objects, etc.) through an analytic-synthetic process, which is similar to the heuristic strategy "decompose-recompose" described by Polya (1957) in problem solving process.

CLASSIFYING

The second stage is called Classifying, which is a cognitive process that implies listing, comparing and organizing attributes according to certain criteria (Inhelder & Piaget, 1969). Although the possibilities for listing attributes are limitless in mathematics, there are barriers for each person on their individual level. From the personal, institutional and socio-cultural point of view, classification schemes are formed. Under the restrictions of these schemes, each subject or group selects the most familiar attributes within their cognitive patterns. Jacob (2001) pointed out that classification schemes provide a powerful cognitive scaffolding, as this minimizes the perceptive load on the individual by providing tools, selection strategies, as well as criteria for selecting the most likely alternative. However, classification schemes also hinder creative thinking, which requires not only originality, elaboration, and flexibility, but also broad fluency in reasoning (Guilford, 1956).

ASSOCIATION

The next stage comprises the association of related concepts, with elements involved in the classification. For example, if an element resulting from the classification is a segment, then there is a set of related concepts that can be activated with the help of memory processes, such as length, bisector, and midpoint. But instead of looking for concepts of properties, if one prefixes two or more objects, then it is possible to think of relational concepts. By prefixing one segment and a certain angle we can think of the concept of a capable arc, and if we take two lines we can connect our thinking with the concepts of perpendicularity, parallelism, the angle between lines, and the existence or not of an intersection point. The depth and plurality in the associated concepts will be greater, to the extent that the objects extracted from the classification are more complex. If the prefixed object is a triangle, then measurement concepts such as perimeter and area emerge. Similarly, pieces of classification resulting from the lengths of their sides and the amplitudes of their angles come to our mind, as well as proper objects such as heights, medians, inscribed and circumscribed circles, Euler's line, and so on. The fluency and diversity of concepts that emerge are in direct correspondence with our mathematical skills and culture.

SEARCHING

In the fourth stage, the student looks for relationships and dependencies, explores conjectures, establishes analogies concerning already known situations, among other processes of high cognitive complexity. This stage is complex from a psychological point of view since it is directly related to creative processes and divergent reasoning. On many occasions, this stage has been masked with enigmatic terms such as insight. However, the didactic problem consists in modeling what actually happens when the subject intelligently searches for new patterns, relationships and ideas, in order to establish plausible conjectures. Therefore, the teacher's role here is to teach students to think mathematically.

Therefore, the best effort should be focused on investigating how to find connections between concepts and properties that really make sense. The use of mathematics software is a great opportunity, as they help us find and explore promising hypotheses (Abramovich & Cho, 2015). First, exploration can be done using the computer, followed by proving or disproving one's own hypothesis using mathematical tools. For these reasons, the searching stage is very closely related to problem solving activity, because the mere fact of considering the relevance and meaning of a question implies a glimpse of possible ways of solution.

VERBALIZING

Verbalizing appears in the final part of the process. Under Vygotskian epistemology, this stage involves the idea that language is the material wrapping of thought (Vygotsky, 1962). Although this stage may have a communicative purpose, its primary function is to summarize the problem in our own thinking. This is a synthetic process in which what is given or required to prove or find can be specified. This idea is directly connected to the well-known taxonomy of Polya (1957), in which he differentiates problems to find and problems to prove. Once we have specified the problem, then we can try to refine it, and also modify its levels of complexity, establish an inventory of possible solutions, find a real situation that masks it (in order to provoke mathematical modeling), and even find an interesting way to communicate it, and so on. These last actions are eminently didactic and go beyond the cognitive process of posing new problems. By taking them into account, the pedagogical importance of teaching mathematical problem posing in the teacher training curriculum will be realized.

TRANSFORMING

The transforming stage interacts with the previous sequence, Brown and Walter (2005) observed that problem posing process is not linear, since it involves certain cycles where the “What-if-not” strategy emerges. Kilpatrick (1987) pointed out that both this strategy and the “What-if-more” (suggested by Jim Kaput in a personal communication), are typical examples of a more general type of reasoning that he calls “contradiction”. Contradiction underlies the epistemological basis of critical mathematical thought. This aspect is intrinsically linked to the epistemic sources of mathematical knowledge, which “…it is necessary, stable, and autonomous but that this coexists with its contingent, fallibilist, and historically shifting character” (Ernest, 1998, p. 259).

From the perspective of social constructivism, mathematics is part of human culture. Ernest (1991) points out that mathematics is not neutral but laden with the values of its makers and their cultural contexts. In particular, Ernest (1991) states that: “Mathematics consists primarily of human mathematical problem posing and solving, an activity which is accessible to all. Consequently, school mathematics for all should be centrally concerned with human mathematical problem posing and solving, and should reflect its fallibility” (p. 265).

Contradiction is a situation that activates thought and motivates transformation, encouraged by a creative need to search for new ideas. However, there are other reasons that lead subjects to carry out transformations during this process. On the one hand, there is the case in which one encounters a problem, but senses or realizes that it is excessively complicated. Then one can try to transform it into a simpler problem, setting the value of certain parameters or abandoning one idea to undertake another. On the other hand, the poser may want to make more complicated things and finds that his/her finding is too trivial, or maybe uninteresting. Then he or she also has the opportunity to change things through transformation. The context in which a reasoning by contradiction takes place constitutes an expression of the sociocultural environment. This justifies the fact that one person can provide a kind of personal stamp to the problem. Although the reflexively critical and fallibilist spirit constitutes a catalyst for reasoning by contradiction, this is regulated by own barriers of each individual cognitive development.

If during the classification process one does not find any interesting aspect or some suggestive idea, there is the option of transforming the mathematical object. Then one can associate properties, or reclassify them in search of new components. The same is true in both Associating and Searching stages in which regressive subprocesses are admissible. From our point of view, the transforming phase is intrinsic in the three intermediate stages. If one sees the transformations in the object, problem or phenomenon to be selected, at some point one will decide to choose something to start with. So, this would be Selecting stage itself, and one must avoid a vicious circle in modeling this process. On the other hand, transforming the results of verbalization leads to previous stages. In this closing moment, the individual has conceived a question with mathematical meaning. An eventual reformulation of an algebraic problem in another geometric one would imply that the previous stages happen again. Likewise, aspects such as the refinement of the conceived question, the clarity in the approach and the aesthetic retouch, are less cognitive and more didactic.

AN EXAMPLE OF ELEMENTARY GEOMETRY

As Silver (1994) points out, the discovery of new problems can occur before, during, or at the end of the resolution of a problem. This idea is well connected with observations by Sharygin (1991a, 1991b), related to the invention of problems for mathematics Olympics (cf. Kontorovich, 2020; Poulos, 2017). Sharygin suggests the importance of looking for reformulations for a problem that has already been solved, as if one idea were encapsulated within another ("matryoshka" problems). For example, looking for a geometric interpretation of an algebraic result leads to interesting problems, which not only help the development of reflective thinking but also form a more interconnected conception of mathematical knowledge.

Below we present an example, which was discussed collectively with students of the Bachelor of Mathematics Education at the University of Holguín. During the data acquisition process, the professor was the third author of this work. The group was made up of 12 students gathered in a problem solving session, which lasted two class hours. The students were previously familiar with the heuristic strategy, both regarding structure and interrelationships. The analysis takes place in a professional practice session, where students can combine their mathematical and didactic knowledge.

SELECTING

This stage consists of choosing the situation or mathematical object that serves as a starting point. Specifically, we start from the following problem already solved by the students.

Suppose that the side *DB* of a square *BEFD* is the diagonal of a second square *ABCD*. Calculate the ratio of the area of the first square to that of the second square.

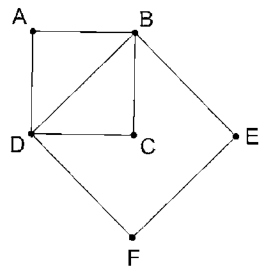


Figure 2.Problem selected as starting point

CLASSIFYING AND ASSOCIATING

The determination of attributes and components already has an advance, since the original problem directly refers to two squares and a diagonal. The concepts of area and ratio are also associated, and the calculation of the quotient between two areas is demanded. For the solution, it is assumed that the small square is of unit length, hence the side of the large square measures $\sqrt{2}$, and finally it can be concluded that the ratio between the areas is 2. The following are the different attributes that can be determined:

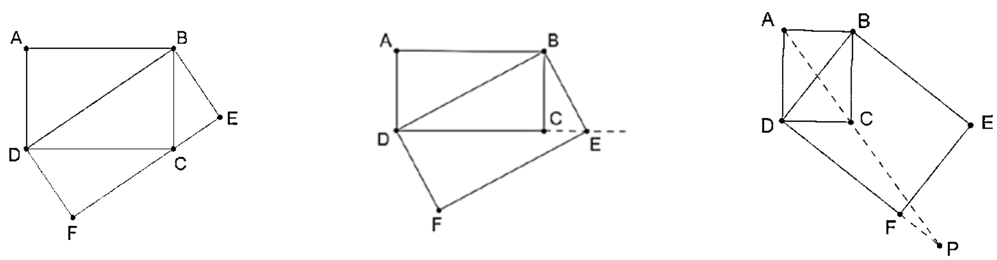
- In the initial problem, two plane geometric figures appear.
- The two geometric figures belong to the same class of quadrilaterals.
- The two geometric figures are squares.
- One side of a figure is a diagonal from the other.
- There is a relationship between the areas of geometric figures.

In the discussion with the students, it is highlighted that the best benefits are obtained when many elements of the mathematical object are imagined. Heuristic thinking works best when a plurality of components not drawn in the original figure is perceived, so that related concepts can be established. For example, that the point *C* is the center of the square *BEFD* may motivate one to imagine the center of the other square, at the midpoint of the diagonal *DB*. Before looking for relationships, it is fruitful to envision a variety of possibilities, which can potentially raise interesting questions.

SEARCHING, VERBALIZING AND TRANSFORMING

The possibility of establishing transformation is inherent in the whole process of Classifying-Associating-Searching. However, it is especially effective when the search does not produce interesting results, or when the subject does not find appropriate questions. Transforming is more feasible with the help of the "What-if-not" strategy, developed by Brown and Walter (2005). A consistent way to implement this strategy consists of the generalization-specialization technique, in Polya's sense. An immediate example is to replace the concept of square with the concept of a rectangle, which is more general. Now, based on this general case, it is possible to examine particular special cases. In fact, the original problem is the result of imagining the special case where both rectangles are squares. The students proposed numerous variants to analyze, of which three were primarily interesting. Figure 3 illustrates these three variants, worth exploring.

I	II	III



What if vertex *C* lies on side *FE*?	What if the vertex *E* lies on the ray *DC*?	What if the rays *AC* and *DF* are concurrent at a point *P*?

Figure 3. Three special cases after generalization

In the first variant (Figure 4), the students noticed that after drawing the segment *CG*, perpendicular to the diagonal *BD*, both rectangles have the same area. Indeed, this follows directly from the equalities $\Delta CBE = \Delta BCG$, $\Delta DCF = \Delta CDG$ and $\Delta BCD = \Delta DAB$. In fact, Figure 4 constitutes a kind of "proof without words" of this assertion. Therefore, verbalization focuses on expressing a question whose solution the student already knows. This consists of verifying that under the given conditions both rectangles have equal area. However, one student suggested verbalizing like this: Which of the two rectangles has a greater area? Give reasons for your answer.

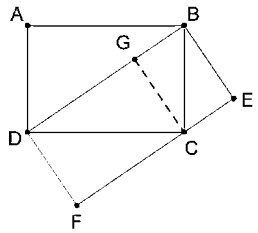


Figure 4. Equality of rectangles in the first variant

Regarding this last reformulation, some students raised objections. If the context of the problem responds to an affective and motivated environment with open reflection, then the question about which of the two rectangles occupies the largest surface is intended to show that the rectangles have the same area, contrary to what is expressed in the text of the problem. This requires the solver to act confidently and answer that neither of the two rectangles covers a larger surface, since they both have the same area. If the question is asked in a tense environment, this way of presenting the problem can lead to confusion, and even fear of refuting the demand expressed in the question.

In the second graph II of Figure 3, it can be seen that according to the position of point *E*, the segments *AB* and *DE* are parallel, so the right triangles *DAB* and *DBE* are similar. Therefore, $\frac{DB}{DE} = \frac{AB}{DB}$ and $DB^2 = AB \cdot DE$. A student observed that this property is present in the original problem, where obviously points *D*, *C*, and *E* are aligned. Therefore, the student discovered that another interesting question is to show, from the two squares in the original figure, that *DB* is the geometric mean of *AB* and *DE*. This fact suggests that the return to the starting point was not static but dialectical, since a new problem was perceived in the same object.

Next, the professor proposed that the students reflect on what would happen if the figures were not squares or rectangles simultaneously. For example, the equality $DB^2 = AB \cdot DE$ connects the lengths of both bases of the right trapezoid *ABED* with one side of the rectangle *BEFD*. If the lengths of *AB* and *DE* are given, then it is possible to calculate the length of *DB* and then *BE*, from Pythagorean relations and also from the equality $CE = DE - DC = DE - AB$. Finally, if the lengths of the bases of the right trapezoid are known, then all the areas of the triangles and rectangles represented in the second variant can be calculated. The previous observations were summarized in the following problem. In Figure 5, *ABED* is a right trapezoid at *A* and *D*. *BEFD* is a rectangle, *DE* = 13.0 cm and *AB* = 4.0 cm. Find the area of triangle *ABD* and also of rectangle *BEFD*.

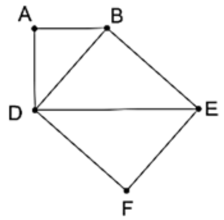


Figure 5. Geometric object resulting from the analysis in the second variant

There are two important aspects, related to the way ideas are verbalized. In the problem comfortable numbers are used, which facilitate the calculation since their Pythagorean nature provides the problem with greater elegance. For example, the results of the calculation of both areas constitute whole numbers. The complexities are not centered on numerical calculation, but on geometric reasoning. At this time, it is useful to discuss with the students about the veracity or impossibility of the figure. The default numerical values are not always appropriate, as the geometric figure described could be impossible to construct. On the other hand, Figure 5 hides segment *BC*, and by not drawing said element, the person who solves the problem is expected to draw it, as part of a heuristic reflection.

Regarding the variant represented in III of Figure 3, a student noticed that the triangles *CDA* and *FDB* are similar. The flexibility that transformation provides allows us to return again to the search and association process. In this case, it was feasible to imagine what would happen if the sides of these two triangles were extended. One possibility is to analyze relationships between lines *AC* and *FB*, which contain a diagonal in the corresponding rectangle. However, exploration was more successful using GeoGebra. Indeed, collective discussion led to exploring the relationships between the rays *AC* and *DF*, since both are parallel in the original problem. This is an example of the importance of considering special cases.

At this time, a high level of motivation was perceived in the students. The time consumed for the activity did not allow to continue exploring, so the professor advised to continue investigating this situation at home. In a subsequent session, the students presented various ideas, which were appropriately discussed. The most ingenious proposal corresponded to a student involved in the Mathematics Olympics. She noted that under condition $0° < \measuredangle CBF < 90°$,, which implies that $AB < BC$, the rays *AC* and *DF* intersect at a point *P*, as illustrated in Figure 6. In the original problem, $AB < BC$ and $\measuredangle CBF = 0°$, so *P* is the point at infinity, corresponding to the direction of the parallel lines *AC* and *DF* (see a special case in Figure 2).

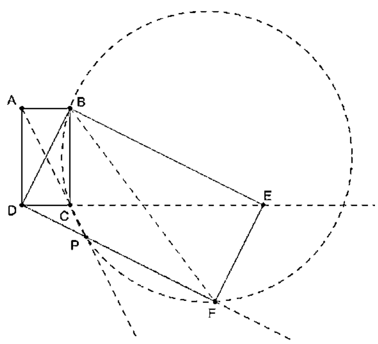


Figure 6. Exploring the third variant with GeoGebra

The most interesting observation was that the points P, C, B, and F are concyclic. The student explained that she had reached this conclusion by noting that $\measuredangle PCB$ and $\measuredangle BFP$ turn out to be supplementary when she enlarged the length of the segment AD in GeoGebra. This fact reflects the great heuristic value that underlies the ability to move and compare items. The dynamic geometry software, in this case, became a kind of catalyst, due to its wide possibilities in this sense and with an economy of time. After adding the circumference and testing her hypothesis experimentally in GeoGebra, the student also presented her proof of the property. Indeed, since the triangles FDB and ABC are similar, it turns out that $\measuredangle BFD = \measuredangle ACB$. Therefore, we have the following: $\measuredangle PCB + \measuredangle BFP = \measuredangle PCB + \measuredangle BFD = \measuredangle PCB + \measuredangle ACB = 180°$. Finally, a new problem consists of proving that under the conditions of the third variant, the points P, C, B, and F are concyclic (see Figure 3 and a complimentary animated GIF in Cruz, 2021).

Again, the analysis of the hypothesis and its verification, constituted an opportune space for debate and discussion in class. Another student observed that if instead of increasing the length of segment AD, this length decreases approaching the length of segment AB, then the quadrilateral $PCBF$ is no longer convex. This is precisely the case that appears in part III of Figure 3. Although this fact does not influence the proof, the idea served to establish a new open question: What happens if in the third variant III of Figure 3 the point P and the vertex F are coincident? Again, the professor pointed out the importance of specializing, which does not mean, in general, the tacit identification of a particular case. Although specialization is a particularization, its primary purpose is to select relevant cases. In other words, particular aspects that are notable and significant. This is the sense in which Polya (1957) describes generalization and specialization. Right at this moment, the professor suggested continue exploring further variants, even starting with other problems already solved.

DIDACTICS

In our strategy, it is difficult to separate the cognitive framework from the heuristic reasoning. The cognitive framework is structural and it underlies on the abstraction that we make about the thinking process itself. On the other hand, the heuristic strategy constitutes an expression of the cognitive framework at the didactic level. It is well known that Polya's scheme (1957, pp. xvi-xvii) shows us an ideal model, is made up of stages that demarcate the problem solving process. With the help of this scheme, the teacher, then, shows a general path to follow. Similarly, the structural component of our framework speculates about what the process should ideally look like. However, the functional component reveals the challenge of how to teach mathematical problem posing.

If we draw on components and relationships of the model, then we can provide a way to organize thought. One can start by selecting a mathematical object, and then suggest enumerating several of its components to establish relationships. It helps a lot to ask questions such as: What visible elements appear in the figure? What properties can we associate with these elements? What non-visible elements could we draw? What relationships could exist among certain elements? Let's try the computer, see what happens! Does this question make sense? These types of questions are heuristic in nature so they do not guarantee anything, instead they favor the search for original problems. The new questions are important, especially when these make mathematical sense and when these are the result of a process of conscious reflection. However, the most important aspect is the imprint that this heuristic reasoning leaves on thought.

During the discussion of the strategy, the students recognized that the framework provides them with a way of guiding their thought with a creative sense. They recognized that these ideas are useful in their training as mathematics teachers. However, this framework could hardly be explained in a school context. It is necessary to follow a more expeditious path, where guidance is specified in a synthetic and enjoyable way. As Newton asserted: "Truth is ever to be found in simplicity, and not in the multiplicity and confusion of things" (translation from Manuel, 1974, p. 120). An example is the apparent simplicity of Polya's scheme (1957), set out in four phases. *How to Solve It* is a book designed for students, and for this reason the author presents his profound ideas in a comfortable way. The "looking back" itself provides the scheme with a fertile conception. In addition, the phases are accompanied by suggestions, heuristic questions, argumentation and exemplification. This is the didactic mark that Polya leaves in the mathematics classroom.

Similarly, it is convenient to present the SCASV+T heuristic strategy in a simpler way. Table 1 contains six stages of heuristic reasoning, which can be developed in math class. The elements and relationships of SCASV+T heuristic strategy emphasize the cognitive level, while succeeding stages constitute a didactic expression. Figure 1 reflects one way of thinking, while Table 1 summarizes one way of doing it in class. Both aspects are useful for the teacher, since their professional training requires cognitive and didactic knowledge.

Table 1

A didactic expression of the heuristic strategy

Stages	Heuristic suggestions
Choosing a starting point	– Select one or more familiar math object – Consider a real and interesting phenomenon, that can be mathematically modeled – Start from a problem already solved – Finding problems within other problems
List explicit and non-explicit components	– Consider the essential elements of the situation – List elements of math object
Establish for concepts associated with each component	– Determine what concepts can be associated with each listed item – Think of other similar or analogous concepts – In addition to concepts inherent to an object, also consider relationship concepts between two or more objects
Search relationships and dependencies	– Remember problems in analogous situations – Transform elements of the situation or object – Consider special cases – Explore more general variants
Ask questions	– Distinguish what is most interesting – Present ideas clearly and rigorously – Assess whether the question could be interesting to other people
Analyzing the problem	– Use computational resources to establish relationships and dependencies – Analyze if the question makes sense. Data and figure are possible? – Find other more appropriate or attractive ways to approach the question. Can elements of the figure be hidden? – Assess the real possibilities to establish a solution path. Try to solve the problem

The first and last stages in Table 1 respectively correspond to levels “0” and “IV” of the strategy developed by Brown and Walter (2005, p. 64). The intermediate stages are based on SCASV+T heuristic strategy framework. “What-if-not” question is not part of the sequence of stages, since it is possible to think about other problems even without changing or varying the initial attributes. However, the possibilities of transformation are always present, which favors the emergence of cycles in reasoning. This freedom to transform affords the process with flexibility, which is very closely related to creative thinking (Silver, 1997). In this regard, Brown and Walter (2005) point out: “The process of varying one attribute followed by varying another suggests a systematic technique we could employ for brainstorming new problems. We call this technique cycling. Here we have a systematic way of generating new forms by combining the preceding two What-If-Nots” (p. 60). As can be seen, Figure 1 illustrates several cycles that are established with respect to the possibility of transforming the attributes of the problem. The identification and analysis of these cycles form an interesting aspect of experimental studies, where Schoenfeld’s episodes (2016) can be useful as they have already been in problem solving studies (Cruz, 2006).

On the other hand, it is necessary to pay attention to the regulatory processes that occur during the implementation of the strategy. In a classical model of cognitive monitoring, Flavell (1979) refers to a wide variety of cognitive activities that occurs through the actions of and interactions among metacognitive knowledge, metacognitive experiences, goals, and strategies. For example, metacognitive knowledge is related to the self-perception of strengths and weaknesses to perform a task or to fulfill an objective. In the case of problem posing, this aspect involves planning, monitoring, evaluation and self-regulation during the creative process as a relevant dimension (Baumanns & Rott, 2021). In particular, beliefs and affects are established as relationships between individuals and mathematical knowledge (Schoenfeld, 2016), so it can be expected that these processes also occur during problem posing, such as the belief that in geometric objects one finds greater diversity of problems (Cruz, 2006). However, some typical beliefs about the resolution of problems can influence and affect the formulation of problems, as in the case of “problems are designed to test procedural knowledge and be solved quickly” and also “the solitary source of mathematical problems is textbooks” (McDonald, 2017). Identification and enquiry of beliefs and conceptions that affect the problem posing process constitutes a complex challenge that goes beyond a typical teaching/learning error, as it also brings up certain training deficiencies in the field of the philosophy of mathematics itself.

It is also important to reflect on the epistemological premises that support the strategy at the didactic level. In a setting where mathematics is presented rigidly and dogmatically, it is difficult to promote the creative posing of new problems, since critical reflection and fallibilist conception of mathematical knowledge are blockaded (Ernest, 1991; Lerman, 1990). It is necessary to accept the possibility of mistakes, an aspect that not infrequently alarms the teacher and confuses the student. In the training of mathematics teachers this is especially important, since the student is also learning to teach. Transforming the attitudes of the prospective teacher requires not only the learning of didactic resources, but also the apprehension of adequate epistemological bases. One consistent path is to promote inquiry-based learning, where problem posing is central.

CONCLUSION

SCASV+T heuristic strategy provides a theoretical framework, which suggests how to organize the reasoning to pose new mathematical problems. The structural and functional components of the strategy reflect complexity, cyclicality, and flexibility. However, the framework expresses what happens on the cognitive level, and that is why it requires didactic recommendations. The latter are presented in the form of stages complemented with heuristic suggestions that can be enriched.

The content of Table 1 is a didactic expression of the heuristic strategy modeled in Figure 1. This means that the stages can constitute a teaching content in teacher training. However, the framework depicted in Figure 1 is too complex to be used in elementary and secondary school teaching. The didactic actions are reformulated to facilitate teaching. This is what the stages in Table 1 consist of, which are complemented by heuristic suggestions. Therefore, teachers in training can prepare themselves to teach mathematical problem posing, following the stages of Table 1 (explicit aspect oriented towards didactic action), but aware of the foundation provided by the model in Figure 1 (implicit aspect oriented towards didactic foundation)".

The example developed reflects the wide possibilities of imagining new problems, where verisimilitude and mathematical sense are essential aspects. These ideas may be useful in other contexts of mathematics teacher training. The process of posing problems not only helps to achieve a better mathematical education, but also to the equipping of professional tools. Indeed, knowing how to ask interesting questions, establishing a variety of solutions, promoting reformulation and problem posing in students, and favoring the development of self-regulatory mechanisms, are important components of the teacher's professional competence.

The cognitive component of the strategy reflects a certain link between the activity of elaborating a problem for students and the process of conceiving a problem for oneself. There are differences regarding the purpose of formulating and the way of presenting a problem. Making a new problem for students, although it may be open to creative imagination, has limits related to the teaching objectives. This problem constitutes an open field for research, which was pointed out by Silver (2013). The example that we have shown is not an experimental but an experiential result, however it makes us think that the common aspects are in the cognitive component, whereas differences are manifested in the didactic and professional components.

The education of creative reasoning encounters numerous obstacles in mathematical problem posing, which constitute challenges for teaching. A motivating environment is required, where the fear of making mistakes is minimized. Collaboration and collective work are very helpful, as well as the use of computational tools that facilitate exploratory work. On the other hand, the deployment of this heuristic strategy also requires a high mathematical and didactic preparation of the teacher. The teacher must be aware that the variants are infinite, which underlies the very nature of mathematics: contingent, fallible and historically changing. However, the challenge is to foresee the main opportunities for the conduct of reasoning and thus leave a favorable mark on the mathematical thinking of students. As Halmos (1980) pointed out, problems are the heart of mathematics, so the art of solving them must be combined with the art of posing them.

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HISTORICAL COMPARISON AND ANALYSIS OF PROBLEMS AND PROBLEM-POSING TASKS IN CHINESE SECONDARY SCHOOL MATHEMATICS TEXTBOOKS

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Abstract

In this paper, the problems and problem-posing tasks in six series of secondary school mathematics textbooks were analysed, the distribution of the number, location and types of problems from the perspective of historical comparison, as well as the types of problem-posing tasks and the distribution of problem-posing tasks across content areas were studied. It was found that although the number of problem-posing tasks has increased, the percentage is still quite small with a maximum of 0.4%, and the distribution of problem-posing tasks across content areas is uneven. It was found that a large number of problems had been included in the content text section since 1990s. The distribution of these problems across grade levels and content areas are well balanced, indicating that problem-guided learning has become a new feature of the textbooks. From the perspective of types, these problems provide rich mathematical learning opportunities for students to acquire knowledge ("knowing" and "understanding") and to go through the thinking process ("to abstract and generalize" "to explore and discover" "to reflect and summarize"). However, the distribution of each type of problems across different grades and content areas are both uneven.

Key words: Textbook, Secondary school, Problem, Problem-Posing

INTRODUCTION

Tao Xingzhi, a well-known educationalist in China, once said "the starting point of invention is to ask". American mathematician Halmos believed that "problem is the heart of mathematics". The importance of problem is self-evident. Teaching through questions, as one of the effective methods to improve students'mathematical ability, has a long history. How to pose problems to effectively promote students' learning has always been an important aspect of mathematics education research. As an important carrier of curriculum ideas and important materials used by teachers and students, how are problem-posing tasks included in textbooks? To this end, we collected the data of problems and problem-posing tasks in six series of secondary school (grades 7-12) mathematics textbooks. From the perspective of historical comparison, we studied the following two topics: one is about the changes in the number and the distribution of the problems across topics, and the analysis of these changes from the perspective of learning opportunities provided by the problems; the second is the change of problem-posing tasks.

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BACKGROUND

Problems and Mathematics Learning

The psychological explanation of "problem" is that people have a problem when they are faced with a task and have no direct means to complete it (Wang, S. & Wang, A., 1992). As for the connotation of mathematical problems, there is no relatively unanimous and generally accepted view at present, and their denotation is mostly determined by classification. The most common way to classify them is by dichotomy, such as closed or open problems, conventional and exploratory problems, etc. (Nie, 2001). Mathematical problems can direct students' attention to specific learning content and prompt them to think, to understand and apply mathematics actively. Problems with different cognitive requirements often bring different learning opportunities. Problems with high cognitive requirements require students to think in a connected and comprehensive way, and these problems can often provide students with more opportunity to understand mathematics deeply. Therefore, many mathematical education researchers also classified mathematical problems from the perspective of cognitive level. For example, Getzels (1975) divided mathematical problems into three types from the perspective of "existing problems" and "discovered problems". Both Type-Case 1 and Type-Case 2 problems are ready-made problems, the former mainly involves the process of memorizing and extracting knowledge, while the latter requires students to analyze and infer by themselves. Type-Case 3 problems are not ready-made, requiring learners to discover and create problems, and these problems themselves are learning objectives. Stein and Smith (1998) classified mathematical tasks as low level tasks and high level tasks according to their level of cognitive demand. The low-level mathematical tasks involve two categories of cognitive demands: memorization and procedures without connections. The high-level math tasks involve two categories of cognitive demands: procedures with connections and doing mathematics.

Problem Posing and Mathematics Learning

Taking problem posing as a basic characteristic of mathematical activities, some outstanding mathematicians and educators, such as Polya, have proposed that it should be an important aspect of mathematics education for students to put forward mathematical problems autonomously. However, it was only in the last thirty years that problem posing has been widely concerned by mathematical education researchers, the main reason being the requirement of "innovative talent education" (Xia, 2005). According to Silver (1994), problem posing refers to both the generation of new problems and the re-formulation of given problems, and posing can occur before, during, or after the solution of a problem. Brown and Walter (2005) believed that it is impossible to solve a new problem without first reconstructing the task by posing new problem in the very process of solving and one did not appreciate the significance of an alleged solution without generating and analyzing further problems or questions. Thus, problem posing is not only a means to solve mathematical problems, but also a relatively independent mathematical activity. Many studies have shown that problem posing can improve students' conceptual understanding, creativity and mathematical attitude. For example, Silver (1997) argued that inquiry-oriented mathematics instruction, which includes problem-solving and problem-posing tasks and activities, can assist students to develop more creative approaches to mathematics. Through the use of such tasks and activities, teachers can increase their students' capacity with respect to the core dimensions of creativity, namely, fluency, flexibility, and novelty.

Problems, Problem Posing and Mathematics Curricula

In the early 1950s, the syllabuses for secondary school mathematics have shown that, whether it is the problem related to mathematical concept or techniques (skillful calculation and drawing, operating by formula, using mathematical table), it is of great significance to learn mathematics well (Chinese Ministry of Education, 1954). Accordingly, selecting appropriate materials to compile mathematical problems as the carrier of mathematical learning activities has always been the key consideration in the compilation of secondary school mathematics textbooks in China, especially when compiling the problems in examples and exercises. Many questions in examples and exercises have become classic questions and are still used today. Teachers also used these classical problems to compile variant problems for teaching. This has become one of the characteristics of secondary school mathematics teaching in China (Zhang & Wang, 2015).

As for problem posing, it did not appear in the mathematics curriculum documents until the 1990s. In the Mathematics Teaching Syllabus for Full-time Ordinary High School, problem posing was mentioned for the first time in the explanation of the ability to solve practical problems: "the ability to solve practical problems refers to the ability to propose, analyze and solve mathematical problems with practical significance or in related disciplines, production and daily life" (Chinese Ministry of Education, 1996). In 2000, in the revised versions of the syllabus of compulsory education and high school education, it was mentioned in the explanation of the sense of innovation that people could discover and pose problems from the perspective of mathematics (Chinese Ministry of Education, 2000a, 2000b). In 2003, in the high school mathematics curriculum standard, for the first time, problem posing was independent of problem solving, and was officially included in the curriculum objective: "to enhance the ability to pose, analyze, and solve problems from mathematical perspectives" (Chinese Ministry of Education, 2003). The latest high school mathematics curriculum standard clearly listed the "four abilities" objectives of mathematics curriculum: the ability to find and pose problems from mathematical perspectives, and the ability to analyze and solve problems (Chinese Ministry of Education, 2018).

Since problem posing has become an independent and important ability in mathematics curriculum's objectives, it is important to have problem-posing activities in the curriculum materials that teachers regularly use. Yet, there is at present a lack of research that focuses on problem posing in the mathematics textbooks used by students and teachers. Cai and Jiang (2015) found that the Chinese primary mathematics textbooks did contain problem-posing tasks, but the percentage of such tasks in each of the textbook series they examined was quite low and the distribution of problem-posing tasks across different content areas and different grade levels in these textbook series was uneven. Then the natural questions that arises is : how has the inclusion of problem posing in curriculum standards impacted secondary school textbooks? Are there enough problem posing tasks in current secondary school mathematical curriculum materials to realize the goals in the curriculum standard? Given the variety of ways to engage students in one form or another of problem posing, how exactly do secondary school textbooks include problem posing? What kinds of choices have textbook writers and curriculum developers made in creating existing materials? This paper attempts to answer these questions through analyzing the data of problems and problem-posing tasks in secondary school mathematics textbooks in China.

METHODS

Selection of Textbook Series

Given that problem posing was first explicitly raised in the 1990s' curriculum documents, we selected six series of secondary school mathematics textbooks published by People's Education Press (PEP) in the 1990s and after 2000, including three series of junior high school (grades 7-9) textbooks (PEP, 1992, 2004a, 2012) and three series of high school (grades 10-12) textbooks (PEP, 1997, 2004b, 2019), here we marked each series of textbooks with the year of publication of the first volume. The problems and problem-posing tasks were compared and analyzed. In this way, the trend of the change and focus becomes visible. In particular, for problem posing, we can see both its germination and its development.

Task Analysis

We first studied all the problems in the six series of textbooks to identify those that were problem-posing tasks. The basic standards are as follows:

(1) Operational definition: the term *problems* refer to tasks requiring students to complete by certain means; *problem posing* refers to the task that requires students to generate new problems based either on a given situation or on a given mathematical expressions or diagram (Cai & Jiang, 2015).

(2) Scope: compulsory part, excluding selected part, comprehensive practice activities , chapter introduction and chapter summary.

(3) Counting rules: an example or exercise was counted as one problem, and the additional question matching an example was included in this example and not counted separately; each of the questions in the columns of "thinking", "observation", "exploration" and "induction" were counted as one question by column.

The research was divided into two studies. In study 1, based on the existing research on the comparison of problem-posing tasks in elementary mathematics textbook, the percentages of types of problem-posing tasks and the distribution of problem-posing tasks in different content areas were compared. In study 2, according to the position of the problems in the textbook, we divided the textbook into three sections: content text, examples and exercises, and counted the number of problems in each section. Then we found that the problems in the content text increased most. Thus we further analyzed the specific manifestations and key characteristics of the problems of the content text section in the latest two series of textbooks one by one. Referring to the existing studies on the classification of mathematical task (Stein & Smith,1998), we divided these problems into five categories according to the opportunities they can provided for students: knowing(k), understanding(u), to abstract and generalize(a), to explore and discover(e), to reflect and summarize (r). Based on this taxonomy, we compared the percentage distribution of the five types of problems in different grade levels and across different content areas.

As for the types of problems, we analyzed and classified them according to the specific performance of the problem (the type of learning opportunities it can provides for students) and the key characteristics (the type of knowledge needed to solve these problems). Current cognitive and constructivist views of learning emphasize what learners know and how they think about it. In view of this, in the analysis of each problem, we focused on whether the problem enables students to possess a piece of knowledge or to experience a thinking process. If it is the former, according to the cognitive level, it can be divided into two categories:

knowing and understanding; If it is the latter, according to the way of thinking, it can be divided into three categories: to abstract and generalize, to explore and discover, to reflect and summarize. Table 1 shows the specific manifestations and key characteristics of each of the five problem types, as well as the sample questions we chose from textbooks.

Table 1

Specific manifestations, key characteristics and sample questions of five types of problems

type	specific manifestations	key characteristics	sample questions
k	Memorizing (concepts, formulas, theorems, methods, etc.) (see sample questions 1);Imitating (algorithms, reasoning, etc.); Judging (a concept, formula, theorem, etc.) directly or intuitively; Analyzing details and supplementing key elements, special cases, etc. (see sample questions 2).	Single, isolated	1. Do you remember how to draw an image of a function by tracing points? 2. Is the angle between lines *a* and *b* related to the position of point *O*?
u	To illustrate (concept, principle, algebraic expressions, etc.) by example (see sample questions 3); Multiple representation (meaning) (words, symbols, images, tables; Translation and interpretation between vectors, trigonometry, algebra, and geometry (see sample questions 4); Establishing connections in form and structures between different knowledge; Integrated application of knowledge to solve problems (Using multiple knowledge, involving a variety of situations and transfer of methods, requiring multi-step logical reasoning, etc.)	Associated, integrated	3. Parallel lines are very common in our daily life. Can you give some other examples? 4. Example 5 gives a property of arithmetic sequence. Can you explain this property of arithmetic sequence from a geometric point of view with the help of figure 4.2-4?
a	Analyzing (or verifying) multiple examples to abstract essential features, and generalize general mathematical concepts (or mathematical facts or principles) (see sample questions 5, 6); By analogy with similar mathematical objects to abstract essential features, and generalize general mathematical concepts (mathematical facts or principles); To abstract and generalize mathematical structures (graphic structures, reasoning structures, etc.) from the specific to the general (see sample questions 7).	General, structural	5. Is 30 + (–20) equal to (–20)+30? Please find some more numbers to try. What can you conclude from the above calculations? 6. What are the common features of the functions in questions 1 to 4 above? Can you summarize the essential characteristics of a function? 7. By summing up the generalities of the reasoning processes described above, can you arrive at a general structure for such reasoning?
e	To seek or create an idea or method (including algorithm, reasoning method, statistical method) in a given situation; Constructing a strategy or solution to solve the problem in a given situation and execute it (see sample questions 8); To discover (or guess) relations, laws, or properties by induction (or observation, operation, analogy, comparison, etc.) or deduction (such as specialization, generalization, etc.) (see sample questions 9).	Unknown, extended	8. If you want to know the ratings of a certain TV program in your area, can you help design a sampling plan? Discuss with your classmates based on the actual situation in your area. 9. Draw a parallelogram by definition, and look at it to see what else relationship of it's sides of it except for the fact that the opposite sides are parallel. What's the relationship between its angles? Measure it. Does it match your guess?

type	specific manifestations	key charact-eristics	sample questions
r	To improve the understanding of knowledge and methods by reviewing, comparing and summarizing their characteristics and functions (see sample questions 10); To achieve knowledge systematization by comparing mathematical objects (definitions, formulas, etc.) learned in different stages; To summarize the experience of learning a certain kind of knowledge and reflect on one's own thinking process (for example, how to study mathematical objects, how to discover mathematical properties, etc.) (see sample questions 11).	Systematic, metacognitive	10. Write the equations and solve it. Does the result agree with your previous estimate? What new insights do you have about the application of equations to practical problems by solving this problem? 11. Based on your previous experience in studying functions, how do you think we should study these functions?

It should be noted that these five categories are only a division based on the analysis of problems in the content text section, they cannot cover all types of problems. They are not and should not be separated from each other, and they may even have non-empty intersections. Although there is overlap between the different types, each category has its own focus that sets it apart from the others. For example, when a question focuses on asking students to explore and discover unknown knowledge or methods, abstraction or generalization is also indispensable in many cases. However, the problem of "to abstract and generalize" is more focused on drawing general conclusions. As an illustration, consider the question: "Take a number of different values of base a (a>0, and a≠1) and draw the graph of the corresponding exponential function y=a^x in the same plane rectangular coordinate system. Look at the locations, common points, and trends of these graphs. what do they have in common? Can you generalize from this the domain, range, and monotonicity of the exponential function y=a^x (a>0, and a≠1)?". In this question, although it is necessary to abstract and generalize the common characteristic of different functions, its main focus is to engage students to choose specific examples and drawing methods by themselves, and find specific conclusions through observation and comparison. Therefore, this task was classified as the type "to explore and discover".

To ensure the inter-rater reliability for the coding of the types of problems, a total of 104 problems, including 47 problems in junior high school textbook series (PEP, 2012) and 57 problems in high school textbook series (PEP, 2019) were randomly selected and coded by two researchers. We reached 96% in agreement with respect to the types of problems.

RESULTS

Historical Comparison of Problems and Problem Posing

The total number of problems and problem-posing tasks in junior high school textbooks and high school are shown in Table 2 and Table 3. In both junior high school textbooks and high schools, the number of problem-posing tasks is minimal. Nevertheless, we further studied the distribution of problem-posing tasks in different content areas and the percentages of problem-posing tasks of different types. Overall, the percentages of problem-posing tasks are quite small with a maximum of 0.08% in the latest junior high school textbook series (PEP, 2012) and 0.4% in the latest high school textbook series (PEP, 2019). And the distribution of the few problem-posing tasks in the four content areas (Function, Algebra, Geometry, Probability and Statistics) is very uneven.

Table 2

Total number of problems and problem-posing tasks in junior high school textbooks

Year	Total number of problems	Total number of problem-posing tasks
1992	3402	1
2004	2505	2
2012	2485	2

Table 3

Total number of problems and problem-posing tasks in high school textbooks

Year	Total number of problems	Total number of problem-posing tasks
1997	2151	0
2004	2496	2
2019	2629	11

In both junior high school or high school, the number of problem-posing tasks in textbooks is very small. We classified the problem-posing tasks in the six textbook series by the content area in which they were situated: Function, Algebra, Geometry, Probability and Statistics. Consistent with the research of Cai and Jiang (2017), we divided the problem-posing tasks into four categories: (1) Posing a problem that matches the given relationship expression(equation or function analytic expression) or image. (2) Posing variations of a question with similar mathematical relationship or structure. (3) Posing additional questions based on the given information and a sample question. (4) Posing questions based on given information. We found that the problem-posing tasks in junior high school mathematics textbook series only appeared in the field of Algebra, focusing on integral expression, equations, inequalities and so on, and only two types are involved, namely type (1) and type (4). Table 4 shows the distribution of 11 problem-posing tasks in the latest high school textbook series (PEP, 2019), and Table 5 gives some specific examples of the 11 problem-posing tasks.

Table 4

Distribution of 11 problem-posing tasks in the latest high school textbook series

	Function	Algebra	Geometry	Probability and Statistics	Total
Type (1)	4	0	0	1	5/45%
Type (2)	1	0	0	0	1/9%
Type (3)	2	0	1	0	3/27%
Type (4)	0	0	0	2	2/18%
Total	7/64%	0	1/9%	3/27%	11/100%

Table 5

Some specific examples of the 11 problem-posing tasks

SN	Content area	Type	Examples
1	Function	(1)	Try to construct a problem situation in which the relationship of variables can be described by y=x (10–x).
2	Function	(1)	As shown in the figure, the graph of function y=f (x) is composed of curve segment *OA* and straight line segment *AB*. Please put forward a practical problem that coincides with the graph of function y=f (x).
3	Function	(2)	By analogy with the above generalization, write down a generalized conclusion related to "the image of function y=f(x) is symmetric with respect to the y axis if and only if the function y=f(x) is an even function".
4	Function	(3)	If $\sin\beta + \cos\beta = \frac{1}{5}$, $\beta \in (0, \pi)$. (1) Evaluate the expression *tan β*; (2) Can you construct more problems for evaluating an algebraic expression by yourself according to the given conditions?
5	Geometry	(3)	If $a \perp \boldsymbol{\alpha}$ and the straight line *b* outside of plane $\boldsymbol{\alpha}$ is perpendicular to the straight line a, what conclusion can you draw? Can you pose additional questions and find more conclusions by yourself?
6	Probability and Statistics	(1)	Can you construct a practical problem to explain the meaning of equation $C_n^k \cdot C_{n-k}^{m-k} = C_n^m \cdot C_m^k$?
7	Probability and Statistics	(4)	The student union of one school wants to investigate the opinions on the student activity plan for this semester. You volunteer as a researcher and plan to sample 10% students of the school. What problems might you encounter in the survey sample? What will these problems affect? How are you going to solve these problems?

It can be seen that 64% of the problem-posing tasks belong to the area of Function, and these tasks cover type (1), (2) and (3); problem-posing tasks in the field of Probability and Statistics occupy the second place, including types (1) and (4); problem-posing tasks in Geometry only covers type (3); there is no problem-posing tasks in Algebra. Although the percentages of problem-posing tasks in textbooks is very small, and the distribution of the few problem-posing tasks across different content areas is uneven, all the four types are involved. In the Function area with the largest proportion of problem posing tasks, one problem posing task is arranged in the example section of the latest high school textbook series: "*Try to construct a problem situation in which the relationship of variables can be described by y=x (10–x)*". In the solution of the question, the textbook gives the demonstration process of how to construct a problem situation, and similar questions are set for students to construct the problem situation in the exercises section. Accordingly, posing a problem that matches a given expression has become the main type of problem-posing task. For type (4) not found in Function, the seventh problem-posing task under Probability and Statistics in Table 5 falls into this category.

In general, the number of problem-posing tasks in textbooks is very small, even in the latest textbook

(0.4%). Moreover, the distribution of the problem-posing tasks across different content areas is also uneven: there are the most problem-posing tasks in the field of Algebra in junior high school textbook series, while there are the most problem-posing tasks in the field of Function in high school textbook series. As for the distribution of types of problem-posing tasks, the latest textbook series involves all four types, but the distribution is also uneven.

DISTRIBUTION OF THE PROBLEMS

Distribution of the Number of Problems in Different Sections.

As can be seen in Table 2, the number of problems in the three series of junior high school textbooks shows a decreasing trend. Compared with the textbook series published in 1990s (PEP, 1992), the number of problems in the textbook series published in 2010s (PEP, 2012) decreases by 27%. As can be seen from Table 3, the number of problems in the three series of high school textbooks shows an increasing trend. Compared with the textbook series published in 1990s (PEP, 1997), the number of problems in the textbook series published in 2010s (PEP, 2019) increases by 22%. In order to further see the position distribution of these changing problems in textbooks, we divided the textbook into three parts(content text section, example and exercise) according to the position of the problems in the textbook, and counted the number of questions in the three parts respectively, as shown in Table 6 and Table 7.

Table 6

Percentages of problems in different sections of junior high school textbooks

Year	Mainbody		Exercises
	Context text	Example	
1992	140/4%	508/15%	2754/81%
2004	648/26%	230/9%	1627/65%
2012	536/22%	250/10%	1699/68%

Table 7

Percentages of problems in different sections of high school textbooks

Year	Mainbody		Exercises
	Context text	Example	
1997	39/2%	379/18%	1733/81%
2004	527/21%	360/14%	1609/64%
2019	523/20%	372/14%	1734/66%

In terms of quantity, the number of exercises in junior high school textbooks after 2000 decreases by more than 1000, the number of sample questions decreases by more than 200, and the number of questions in content text section increases by about 400, so the overall trend of the number of problems in junior high school textbooks is reduced. Since 2000, the number of exercises and examples in high school textbooks has not changed much, and the number of problems in content text section has increased by nearly 500. Therefore, the overall trend of the number of problems in high school textbooks is increasing.

It can be seen that a common change in the number of questions in junior high school textbooks and high school textbooks is that a large number of questions have been added in the content text section. So what are the new problems? Considering the consistency of the compiling concepts of different versions of textbooks since 2000 (the textbook series of PEP, 2012 is the revised version of the textbook series of PEP, 2004a, textbook series of PEP, 2019 is the revised version of textbook series of PEP, 2004b), we performed a detailed analysis on each problem in the content text section of the latest junior high school textbook series and high school textbook series, and classified these problems into five types according to their specific manifestations and key characteristics showed in Table 1. The number of these five types of problems in different grades and content areas respectively was then counted.

Percentages of types of problems across grades.

Table 8 and Table 9 show the number and percentage of different types of problems in the content text section of each grade in junior high school and high school respectively. We found that the type "to explore and discover" accounted for the highest proportion with nearly 40% in junior high school textbook, followed by type "knowing" and "understanding", the sum of the three types accounted for about 85%. While in high school textbook, the percentage of type "to explore and discover" is almost as high as "understanding", they accounted for 70% in total. No matter in the latest junior high school textbook series (PEP, 2012) or high school textbook series (PEP, 2019), the proportion of the type "to reflect and summarize" is very low.

Table 8

Percentages of types of problems across grade levels in the latest junior high school textbook series

	k	**u**	**a**	**e**	**r**
Grade 7(214/40%)	67/13%	46/9%	24/4%	70/13%	7/1%
Grade 8(190/35%)	30/6%	54/10%	29/5%	72/13%	5/1%
Grade 9(132/25%)	26/5%	23/4%	13/2%	68/13%	2/0%
Total	123/23%	123/23%	66/12%	210/39%	14/3%

Table 9

Percentages of types of problems across grade levels in the latest high school textbook series

	k	**u**	**a**	**e**	**r**
Grade 10 (302/58%)	69/13%	79/15%	15/3%	122/23%	17/3%
Grade 11 (221/42%)	42/8%	98/19%	6/1%	61/12%	14/3%
Total	111/21%	177/34%	21/4%	183/35%	31/6%

Combined with the amount of class hours of each grade, it's about 2 questions per class hour in the content text section of the six grades. The most significant difference in junior high school textbook was the types "knowing" and "understanding". There are more "knowing" problems in Grade 7 than in Grade 8 and grade 9, and the "understanding" problems in Grade 8 are significantly more than those in the other two grades. In high school textbook, there are significant differences in the two types of "understanding" and "to explore and discover". The "understanding" problems in Grade 11 are significantly more than those in Grade 10, while the "to explore and discover" problems in Grade 10 are significantly more than those in Grade 11.

In general, the number of problems in the content text section of the six grades of secondary school mathematics textbooks is very balanced. This is an indication that the textbooks not only pay attention to impart students the mathematical knowledge, but also engage them to think mathematically. On imparting knowledge, the junior high school textbooks stress equally "knowing" and "understanding", while high school textbooks focus more on "understanding" of knowledge; Pertaining to thinking mathematically, both junior high school and high school textbooks emphasize the importance of engaging the students "to explore and discover". Moreover, according to the classification of Stein and Smith (1998), about 80% of the problems in the content text section are of high cognitive level, and more than 50% of the problems are for students to do mathematics.

Percentages of types of problems in different content areas.

Table 10 and Table 11 show the number and percentage of problems in content text section in different content areas. Combined with the class hours in each area, the number of questions in the four content areas is evenly distributed, with about 2 questions per class hour.

In junior high school, the five types are different in terms of field distribution; Problems of the type "knowing" are the most in the Algebra area and the least in the Function area; Problems of the type "understanding" are the most in the Function area and the least in the Algebra area; The type of "to explore and discover" is most in the fields of Geometry and Probability and Statistics; The type of "to abstract and generalize" and "to reflect and summarize" are most in the field of Algebra.

In high school textbooks, according to the type, the distribution of "knowing" type in the four fields is more balanced, and there are a little more problems in the field of Algebra; The distribution of "understanding" type is slightly different in the four fields, with the most in Probability and Statistics and the least in Function; Problems of "to abstract and generalize" type are more in the field of Function and Geometry; The distribution of "to explore and discover" type in the four fields is relatively balanced, and slightly less in the field of Geometry; The distribution of "to reflect and induce" type in the four fields is slightly different, with more problems in the field of Geometry and Function.

Table 10

Percentages of five types of problems in different content areas in the latest junior high school textbook series

Content area (class hours)	k	u	a	e	r	Total
Function (29)	6/1%	16/3%	8/1%	16/3%	1/0%	47/9%
Algebra (107)	54/10%	38/7%	39/7%	58/11%	10/2%	199/37%
Geometry (123)	56/10%	60/11%	19/4%	119/22%	3/1%	257/48%
Probability and Statistic (20)	7/1%	9/2%	0/0	17/3%	0/0	33/6%

Table 11

Percentages of five types of problems in different content areas in the latest high school textbook series

Content area (class hours)	k	u	a	e	r	Total
Function (81)	32/6%	47/9%	10/2%	63/12%	12/2%	164/31%
Algebra (59)	34/7%	47/9%	0/0	49/9%	4/0.7%	135/26%
Geometry (47)	23/4%	35/7%	8/2%	30/6%	11/2%	107/20%
Probability and Statistic (52)	20/4%	48/9%	3/0.5%	41/8%	4/0.7%	117/22%

In general, the distribution of content text problems in the four content areas is very balanced from the perspective of curriculum time, with about two problems per hour of instructional time.. The four content areas in the junior high school textbooks all favor "to explore and discover". In the high school textbooks, the problems in Algebra tend to let students to acquire knowledge, while the problems in Function area treat acquisition of knowledge and experiencing the thinking process equally.

CONCLUSION

According to the data in this paper, the number and position of problems in secondary school mathematics textbooks have changed significantly since 2000. Before 2000, the problems in secondary school mathematics textbooks were mainly questions presented as exercises; questions for students to practice, accounting for 81% of the total. Since 2000, the location of the problems has changed, with about a few hundred new questions appearing in the content text section.

As for these "new" problems, according to Table 1, the main function of these new problems serve to guide students' learning of mathematics. Textbooks not only focus on imparting students knowledge by means of "knowing" and "understanding" types of problems, but also engage the students to experience the process of mathematical thinking through "to abstract and generalize", "to explore and discover" and "to reflect and summarize" types of problems. In the six grades of secondary school, the problems of "knowing" and "understanding" account for about 50%, and the problems of "to abstract and generalize" "to explore and discover" and "to reflect and summarize" account for about 50%. The proportion of "knowing" type problems decreased slightly with increasing grade levels: the proportion of "understanding" type problems increased slightly, the proportion of "to abstract and generalize" type problems decreased slightly, the proportion of "to explore and discover" type problems remained basically the same, and the proportion of "to reflect and summarize" increased slightly. Secondly, we find that new problems are mainly of high cognitive level, and the types of problems are different according to the characteristics of knowledge. Although the new problems cover different cognitive levels, the proportion of high cognitive problems above the "understanding" level is as high as 80%. In particular, the type of "to explore and discover" is unanimously preferred by all knowledge fields. This also shows that textbooks focus on engaging students to "do mathematics". Therefore, "problem-guided learning" has become a new feature of secondary school mathematics textbooks in the 21st century.

As for problem posing, although the type of problem "to explore and discover", which occupied the highest proportion in secondary school textbooks, show students how to discover and put forward mathematical problems through inquiry to a certain extent, there are still few opportunities for students to discover and put forward a question by themselves. Even in the latest textbook series, only 0.08% and 0.4% are problem-posing tasks. This can be interpreted in two ways. The first is, compared with the long-standing concern about problems and problem solving in mathematics curriculum, it has only been more than 20 years since the emergence and independent existence of problem posing in curriculum standard documents, which may be the reason that problem posing has not been in a dominant position in textbooks. The second plausible reason could be seen from the perspective of textbook editors, although the added questions in the content text section focus on guiding students' learning, it is also a demonstration of "how to find and pose problems" for students. As mentioned in the "words of the editor in chief" of the second set of high school textbooks:"we will pose problems whenever we have the opportunity. We hope that after reading many questions, you can pose new questions even if you can't solve a problem" (PEP, 2004b). This may be the subjective reason why the percentage of problem posing tasks are very low, namely, the editors focused on demonstrating how to pose problems, while the awareness of letting students pose problems by themselves was still relatively weak, so problem posing did not become a feature of textbooks.

In this case, we believe that it only needs a little effort to change from the status quo of problem posing in textbooks to letting problem posing be a feature of textbooks. We suggest three ways to make this change: first, to increase the number of problem-posing tasks by modifying existing questions. This is relatively easy, for example, by deleting some existing information or adding statements like "can you ask similar solvable questions" or "can you use this information to ask other questions" to turn a problem into a problem-posing task. Second, by designing high-level problem-posing tasks to provide students with more active and in-depth learning opportunities. This point requires the efforts of the editor, such as the carefully constructing a situation or designing a problem-posing task, so that students can put forward a series of questions of different difficulty, etc. Thirdly, by systematically considering the distribution of problem-posing tasks in terms of types, fields and grades, the balanced and continuous distribution of problem-posing tasks in the whole set of textbooks can be realized.

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FIFTH GRADE CHINESE AND U.S. STUDENTS' DIVISION PROBLEM POSING: A SMALL-SCALE STUDY

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Abstract

This study reports a small-scale international comparative study investigating rural elementary students' mathematical thinking on division, through analyzing the similarities and differences between division problems posed by elementary students in Inner Mongolia in China and Montana in the United States (U.S.). Bruner's (1985) paradigmatic and narrative modes of thought served as an analytic framework in this study. The primary data source for this study was students' responses to the open-ended prompt, "Write two different types of division problems." Each student's responses were coded according to the perspectives of paradigmatic and narrative modes of thought. The structures and contexts of posed problems and students' characterization of different division problems were examined. Our findings show that most students in both countries posed problems involving partitive (i.e., group size unknown) and equal groups division. No students in either country posed array/area problems. Of the ten common structures for division problems, students in China created problems aligned with six structures while the students from the United States used only two structures. An examination of the contexts used in each problem revealed that different types of food were the most common context used by students in both countries, although with unique cultural contexts. None of the students in either group situated their story problems in a rural context.

Keywords: problem posing; division problem

INTRODUCTION

This study analyzed the similarities and differences between division problems posed by upper elementary school students in the rurual, northern border regions of China and the United States (U.S.), Inner Mongoliaand Montana respectively.. This study investigated and compared their conceptual structures and contexts in the division that can be represented in the symbolic expression "a ÷ b = ?." In mathematics education, there is an increased emphasis on the development of a connection between conceptual and procedural forms of mathematics (Kobiela & Lehrer, 2015). Beyond a traditional emphasis on symbolic mathematical language, such a development requires the use of multiple representations to express thinking, make meanings, and demonstrate a deep understanding of the same mathematical procedure.

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Among various representations, writing has been recognized as a way to "boost learning in mathematics, develop mathematical understanding, change the pupil's attitude towards mathematics for the better, and help the teacher's evaluation" (Joutsenlahti & Kulju, 2017, p. 2). Problem posing, or using problem writing to represent symbolic mathematical problems, provides a unique window to develop a connection between conceptual and procedural forms of mathematics. It has also been recognized as an effective instructional strategy and assessment tool, mostly in mathematics and prose comprehension (Cai & Leikin, 2020; Mishra, & Iyer, 2015).

Previous studies (e.g., Cai, 1998; Cai & Hwang, 2002) explored how problem posing relates to problem solving and found a positive relationship between problem posing and problem-solving skills. Palmér and van Bommel (2020) recently showed that tasks posed by the children "shed light on their interpretation of what the original problem-solving task was really about" (p. 743). Problem posing has been recognized as a way to unfold new knowledge. Problem posing has beensuggested to be assessed to determine the extent to which creativity, including the constructs such as fluency, flexibility, and originality, is present (Leung & Silver, 1997; Shriki, 2013). Recently, research on how affective factors such as curiosity, interest, and enjoyment are associated with problem posing has increased attention (Cai & Leikin, 2020).

While problem posing can be used in many mathematical contexts and may have an impact on cognitive and affective domains, research has shown that posing problems for number sentences involving division is more challenging for students than number sentences involving other operations (English, 1997). Division is an indispensable arithmetical operation in the elementary school mathematics curriculum. It is at the uppermost level of elementary school mathematics operations. In China and the U.S., students are introduced to division in their third grade soon after they learn multiplication. Prior studies have identified many challenges that both students and teachers face in understanding the concept of division (Joutsenlahti & Kulju, 2017), especially while translating symbolic division problems into words (Ball, 1990; Jansen & Hohensee, 2016; Lo & Luo, 2012; Simon, 1993; Tirosh & Graeber, 1990). We are thus left with an incomplete account of children's understanding of division situations.

Several comparative studies (e.g., Cai,1998; Cai & Hwang, 2002; Ma, 1999) which involved Chinese and U.S. students can be found in mathematics education. However, as addressed by Wang and Lin (2005), cross-national comparisons are often ambiguous. In terms of sampling, for example, Chinese and U.S. students are often categorized as a homogeneous group without consideration of the similarities and differences among geographical locations within each country. Students in rural regions in China were also often excluded from these studies. With such lack of differentiation, cross-national comparisons may "mask underlying ethnic and cultural differences and thus prevent adequate interpretation of differences related to student performance" (Wang & Lin, 2005, p. 4). To develop a deeper and more discriminative understanding of how Chinese and U.S. students perform in mathematics, we need to consider whether sampling in the comparative study is comparable, and whether the comparative study targets and investigates specific topics or factors. In this study, we focused on the development of mathematical thinking and meanings on division problems among students in *comparable geographical regions* of China and the U.S. develop similar mathematical thinking and meanings regarding posing division problems.

THEORETICAL PERSPECTIVES

Two main theoretical perspectives guided the design of this study. They are (1) problem posing, and (2) division schema.

Problem Posing

There have been interests and efforts to incorporate problem posing into school mathematics (Cai & Leikin, 2020). Problem posing is "both the generation of new problems and the reformulation of given problems" and is considered to be a characteristic of creative activity or exceptional talent and a feature of inquiry-oriented instruction (Silver, 1994, p. 19). Stoyanova and Ellerton (1996) gave a framework that distinguishes forms of problem posing into the following three paths: (a) free problem posing, (b) semi-structured problem posing, and (c) structured problem posing. Free problem posing "provokes the activity of posing problems out of a given, naturalistic, or constructed situation without any restrictions" (Baumanns, & Rott, 223, p. 63). Although free problem posing is more demanding compared to the structured and semi-structured problem posing, it leads students to think independently and elicits authentic ideas. To optimize the exploration and analysis of students' thinking and ideas in problem posing, this study focused on free problem posing.

While students are given "the opportunity to construct their own representations of mathematical concepts, rules, and relationships" (Cai & Lester, 2008, p. 282), the variety of problem-posing types must be taken into consideration. In this study, Bruner's paradigmatic and narrative modes of thought served as an analytic framework to analyze the types of posted problems. A paradigmatic mode of thought is "context free and universal" (Bruner, 1985, p. 97). A narrative mode of thought focuses on "the broader and more inclusive question of the meaning of experience" (Bruner, 1985, p. 98). In relation to story problems, a paradigmatic mode of thought would require a focus on mathematical structures or models that are independent of a particular social context (Chapman, 2006). A narrative mode of thought in the context of story problems would require a focus on the social contexts such as the characters, objects, situations, actions, relationships, and/or intentions of the story problem (Chapman, 2006).

Division Schema

Piaget (1952) defined a schema as a conceptual representation of an associated set of perceptions, ideas, and/or actions. In Woolfolk's interpretation (1987), Piaget considered the schema to be the basic building block of thinking: a way of organizing knowledge. To describe an individual's schema in the meaning of arithmetic, Steffe and Cobb (1998) used Van Engen's (1949) operational theory of meaning that consists of three components (a referent, a symbol of the referent, and an individual) to interpret the symbol somehow referring to the referent. An individual's verbal interpretation, visual representation, or observable behavior can be taken as the referent for a symbolic operation and be considered as a demonstration of his/her arithmetic schema (Steffe & Cobb; 1998; Wilkins, Norton, & Steven, 2013). Steffe and Cobb (1998), for example, constructed the elementary division schema of a seven-year-old child through analyzing his verbal interpretations on whole-number symbolic operations like "24 ÷ 3." This study examined the division schema rooted in fifth-grade students' thinking through evaluating their verbal representations for symbolic division problems. Thus, story problems written for representing symbolic problems of division were examined.

From a paradigmatic perspective, most division story problems can be classified as either partitive (group size unknown) or quotitive (number of groups of unknown) division (Greer, 1992; NGA & CCSSO, 2010; Lo & Luo, 2012), and further classified into five sub-structures, as demonstrated in Table 1. This table was reorganized and revised based on NGA & CCSSO's (2010) "Common Multiplication and Division Situations" (p. 89) in the *Common Core State Standards for Mathematics* and Greer's (1992) summary table of "Situations Modeled by Multiplication and Division (p. 281). Some common types of division story problems cannot be classified into either *partitive* or *quotitive* structure. The *rectangular area* and *Cartesian product* types of problems are two types that cannot be classified.

(a) *Rectangular Area*: A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?

(b) *Cartesian Product:* If there are 18 different routes from A to C via B, and 3 routes from A to B, how many routes are there from B to C?

Table 1

Common Structures of Division Story Problems

Structure	A: Partitive Division Group Size Unknown	B: Quotitive Division Number of Groups Unknown
1: Equal Groups	If 18 plums are shared equally into 3 bags, how many plums will be in each bag?	If 18 plums are to be packed 3 plums to a bag, how many bags are needed?
2: Part-Whole	A college passed the top 3/5 of its students in an exam. If 18 passed, how many students sat on the exam?	A college passed the 18 out of 30 students who sat on an exam. What fraction of the students passed?
3: Arrays	If 18 apples are arranged into 3 equal rows, how many apples will be in each row?	If 18 apples are arranged into equal rows of 6 apples, how many rows will be there?
4: Comparison	A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A rubber band was 3 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
5: Rate	A boat moves 18 feet in 3 seconds. What is the average speed in feet per second?	How long does it take a boat to move 18 feet at a speed of 3 feet per second?

On the other hand, the classification of story problem contexts that deal with the narrative mode of knowing vary by the different interpretations of temporary situations and experiences.

METHOD

This study was designed to explore and compare the insights of division problem posing demonstrated by upper elementary students from China and the U.S. To this end, we examined the structures and contexts of division story problem posed by the targeted 5th grade students in each country. The following subsections detail the research method.

The theoretical and empirical literature discussed in earlier sections led to the following specific research questions:

1. To what extent can Chinese and U.S. students pose division story problems? What types of division problems do Chinese and U.S. students pose?
2. How similar or different do Chinese and U.S. students pose division story problems?

Participants

This study was an international comparative project investigating rural elementary students' mathematical thinking in the division. Purposive sampling (Leedy & Ormrod, 2005) was adopted to determine the subjects. The Chinese sample was from one typical public school in a big city in Inner Mongolia. The U.S. sample was from one typical public school in a small city in Montana. Although there is a significant difference in population size, both cities are located in rural, northern-border regions of each country with a sizeable minority population. Inner Mongolia is an autonomous region in China. Its two largest ethnic groups are Han (79%) and Mongol (17%). Montana has been home to seven federally recognized Indian reservations in the U.S. Its two largest ethnic groups are White (89%) and American Indian (6%).

Class sizes differ between the two countries, with about 58 students per class in China and 28 per class in the U.S. A total of 86 fifth-grade students (58 Chinese students and 28 U.S. students) participated in the study. Their homeroom teachers, who are mathematics teachers, administered the instrument to their students. No intervention was conducted in this study.

Instrument and Data Collection

A written instrument was chosen as the data collection tool for this study. The first three authors drafted and discussed the prompts assessing student division thinking. Both participating mathematics teachers from each country also reviewed the instrument to establish the face validatity of the instrument. Since this instrument is used in two different languages, we tried to ensure the two language versions are equitable. For example, we noticed that story problems in the U.S. are called "应用题" (application problems) in China. Therefore, we used the Chinese character "应用题" to represent story problems in the Chinese version of the instrument. The instrument consists of three prompts: (1) How would you define division? (2) Write two different types of division story problems. (3) Explain why these two division problems are different. Since this study focused on problem posing, only the results from the second prompt, "Write two different types of division story problems," was explored. The main source of data for this study is students' open-ended responses to the second prompt.

Data Coding and Analysis

Each student's response to the second prompt was coded according to the perspectives of paradigmatic and narrative modes of thought. The technique of content analysis was utilized to identify specific characteristics of collected data. Coding the collected data into categories relevant to the research objectives is an essential procedure of content analysis (Gall et al., 1996). In order to explicitly identify each posed story problem's division structure, the common structures of division story problems shown in Table 1 were used as a classification framework. Those collected story problems were classified into one of the following structures: *Partitive* division problems with the code "A," *quotitive* division problems with the code "B," other division problem structure such as *rectangular area* and *Cartesian product* problems with the code "O," and incorrect division problems with the code "NA." Each classified story problem was further labeled with one of the following categories of sub-structures: "1" for *equal groups*, "2" for *part/whole*, "3" for *arrays*, "4" for *comparison*, "5" for *rate*, "6" for *rectangle area*, "7" for *Cartesian product*, and "8" for any other exemplary story problem.

Different from paradigmatic types of structure, there is no prototype derived from the literature to serve as a classification framework for coding narrative context. This study adopted the method of open coding in which "the concepts emerge from the raw data and later grouped into conceptual categories" (Elo & Kyngas, 2008). Since the dividend is the quantity divided or grouped in a division problem, its context was coded to represent the narrative mode of thought in that problem. The open coding began with labeling the context of the dividend for each posed story problem. As open coding progressed, the codes were compared between problems to create concepts that further defined the categories.

The coding was developed through a series of stages. In the first stage, the first author coded the collected story problems using the classification framework for coding problem structures and the method of open coding for labeling and categorizing problem contexts. She examined each story problem and created a spreadsheet with a list of preliminary codes.

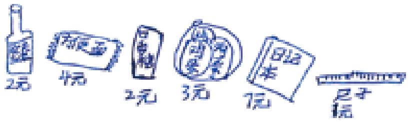
In the second stage, all of the authors later worked together to review, re-categorize, and adjust the preliminarily coded data. We tried to achieve coding consensus through this meeting, particularly on codes for those problem contexts that address the narrative mode of thought. The second author also paid particular attention to the equity issue of languages between English and Chinese. In the process, we also established a coding book.

In the third stage, the first author recorded the story problems using the established coding book and then compared her coded data with the coded data agreed or revised on the second stage. Any inconsistency was closely reviewed and discussed to ensure reliability before making the final decision. After completing the third step, we might still adjust based on a continued review of the coded data and literature. To accommodate the openness and variation of contexts, it is worth noting that a new contextual category might be added to the list of classification codes if necessary.

Table 2 provides the sample responses and their corresponding codes finalized after a series of coding stages. We analyzed the frequencies and percentages of division problem structures and contexts for each posed problem. The analysis results can allow us to know how students represent symbolic division problems in words and answer the research questions.

Table 2

Sample Responses and Their Corresponding Codes

ID	Story Problem	Structure	Context
C5.2	Xiao-Gang's mom and dad brought him to the beach to play. The temperature at the beach is 35.5 degrees Celsius. The temperature at Xiao-Gang's home is 20 degrees Celsius. What is the average temperature in degrees Celsius of these two locations?  	Partitive, Equal Groups	Measurement (Temperature)
C6.2	There are several products in Jia-Jia supermarket. Dad brought $60. What can he buy and how many can he buy?	Quotitive, Equal Groups	Money
U2.2	Joe has 40 M&M's and splits them between 4 friends. How many M&M's does each friend get?	Partitive, Equal Groups	Food (Candies)

Results

This section presents the results of the comparative analysis about the posed story problems, focusing on the problem structures and contexts along with descriptive statistics information. As shown in Table 3, among 116 story problems posed by 58 Chinese students, 95 (81.9%) are applicable division problems; among 56 story problems posed by 28 U.S. students, 48 (86.7%) are applicable division problems. Chinese students did not outperform their U.S. counterparts in posing appropriate division story problems.

Table 3

Distribution of Division Problem-Posing Performance across Chinese and U.S. Students

	China		U.S.	
Division Problems	Freq.	Perc.	Freq.	Perc.
Applicable Total	95	81.9%	48	86.7%
Not Applicable Total	21	18.1%	8	14.3%
Total	116	100%	56	100%

Problem Structures

In the study, except the story problems not written as an applicable division problem, all the posed story problem structures can be classified into one of ten common division structures shown in Table 3: (a) two primary division structures of *partitive* and *quotitive* and (b) five sub-structures of *equal groups*, *part-whole*, *arrays*, *comparison*, and *rate*.

As shown in Table 4, the story problems posed by Chinese students consist of six of the ten common division problem structures, while those posed by their U.S. counterparts only include two structures. The vast majority of the posed problems in both countries are A1 –partitive, equal groups division problems (China: 57.8%, U.S.:71.4%). Other applicable structures posed by students are A4 – partitive, comparison (China: 1.7%, U.S.: 0%), A5 – partitive, rate (China: 1.7%, U.S.: 0%), B1 – quotitive, equal groups (China: 15.5%, U.S.: 14.3%), and B4 – quotitive, comparison (China: 1.7%, U.S.: 0%) division problems. Students in China posed a more diverse range of problem structures than their U.S. counterparts. Additionally, no students in either country posed arrays problems. No *rectangular area*, *Cartesian product*, or other structures of division problems were generated by the participating students either.

Table 4

Distribution of Division Problem Structures across Chinese and U.S. Students

	China		U.S.		China		U.S.	
Structure	Freq.	Perc.	Freq.	Perc.	Freq.	Perc.	Freq.	Perc.
	A: Partitive Division				B: Quotitive Division			
1: Equal Groups	67	57.8%	40	71.4%	18	15.5%	8	14.3%
2: Part-Whole	0	0%	0	0%	2	1.7%	0	0%
3: Arrays	0	0%	0	0%	0	0%	0	0%
4: Comparison	2	1.7%	0	0%	4	3.4%	0	0%
5: Rate	2	1.7%	0	0%	0	0%	0	0%
Applicable	71	61.2%	40	71.4%	24	20.6%	8	14.3%

Problem Contexts

An examination of the context used in each problem revealed that different types of food were the most common context used by both groups of students. As shown in Table 5, nearly half (46.6%) of the story problems posed by Chinese students and over half (60.7%) of the problems posed by their U.S. counterparts used foods as the primary context. With such a significant number of the posed story problems written in the food context, it makes sense to classify this broad theme of food into smaller categories. This study refers to various food categories based on the elementary-aged children's consensus on distinguishing between fruit, snacks, candy, and foods served at meals. In Adams and Savage's (2017) study, food that children highly agreed upon as candy included mostly packaged food high in sugar and fat content, such as solid chocolate, lollipops, skittles; food they highly agreed upon as snacks included mostly non-perishable, often salty, convenience food, such as cereal, crackers, and pretzels. In Adams and Savage's study, children classified several "dessert-like food" such as cookies, ice cream, and cake-like foods less consistently as snacks or candy. In this study, we categorized those dessert-like food as a snack. Although most children in Adams and

Savage's study classified vegetables and fruits as snacks , this study classified these healthy and nutritional foods independent from snacks under a new subcategory. This study identified no context of meals or vegetables from the collected story problems. In summary, we broke down the broad category of food into three subcategories consisting of fruit, snacks, and candy.

A higher percentage (27.6%) of the story problems posed by the Chinese used fruit as context, specifically apples (21.6%), when compared to the percentage (10.7%) of those posed by their U.S. counterparts. Snacks, such as cakes, cookies, and chips, were the most frequently used food context by students in the U.S. (37.5%), but less by those in China (6.9%). Both groups of students posed similar percentages (China: 12.1%, U.S.: 12.5%) of their story problems in the context of candy. However, Chinese students only used the Chinese character "糖果"—a general word for candy in Chinese to represent candies, while their U.S. counterparts used more diverse terms, such as lemon drops and Starbursts.

Compared to the category of food, the rest of the context categories are smaller. The percentages of using daily necessities such as pens/pencils, books, and stamps between two groups of posed problems (China: 7.8%, U.S.: 7.1%) were closer than those in other categories. Chinese students were much more interested in splitting or grouping people than their U.S. counterparts (China: 11.2%, U.S.: 3.6%). Chinese students posed a higher percentage of monetary problems (6.9% versus 3.6%) than their U.S. counterparts. None of the U.S. students embedded the context of measurements, such as liters, temperature, and speed, into their problem posing as their Chinese counterparts did. None of the Chinese students embedded the context of toys into their problem posing as their U.S. counterparts did. The category of miscellaneous items consists of those contexts that occur only once.

Table 5

Distribution of Chinese and U.S. Students' Story Problem Contexts

Context	China		U.S.	
	Freq.	Perc.	Freq.	Perc.
Foods	54	46.6%	34	60.7%
Fruit	32	27.6%	6	10.7%
Snacks	8	6.9%	21	37.5%
Candy	14	12.1%	7	12.5%
Daily Necessities	9	7.8%	4	7.1%
People	13	11.2%	2	3.6%
Money	8	6.9%	2	3.6%
Measurement Attributes	8	6.9%	0	0%
Toys	0	0%	4	7.1%
Miscellaneous Items	5	4.3%	2	3.6%
Age	1	0.9%	0	0%
Balloons	1	0.9%	0	0%
Dog Treats	0	0%	1	0.9%
Fish	1	0.9%	0	0%
Pearls	1	0.9%	0	0%
Rocks	0	0%	1	0.9%
Terracotta Warriors	1	0.9%	0	0%
Applicable Total	95	81.9%	48	86.7%

Discussion

This study unfolds students' mathematical thinking from international perspectives through the lens of problem posing. Bruner's paradigmatic and narrative modes of thought were used as an analytic framework to examine the posted problems. As being conducted in two different rural regions, this study sheds light on the problem addressed in the introduction that other international comparative research simplifies countries into homogeneous entities.

The results provide the distribution of the posted problem types (structures and contexts). In this study, approximately 82% of the Chinese students and 86% of the U.S. students posed applicable division problems. The applicable problem-posing rates could confirm Cai's (1998) research results that both Chinese and U.S. students were able to formulate mathematical problems. Additionally, the results do not suggest that Chinese students outperformed their U.S. counterparts. On the other hand, different from Cai's study, this study found that Chinese students generated a more variety of problems than their U.S. counterparts. Thus, Chinese students' story problems are more diverse and inclusive than those posed by their U.S. counterparts. It is unclear whether or not the difference between studies was contributed by their distinction in research designs, such as sample sizes, research locations, and problem-posing tasks.

Both groups of students overwhelmingly posed the structure of *partitive* (group size unknown), *equal groups* division problems regarding division problem structures. The structure of *partitive, equal groups* division problems are often situated in an equal sharing scenario with a whole-number group as the divisor (Lo & Luo, 2012). Since the form of free problem posing was adopted in this study, its openness might make students intend to use a whole-number quantity as the divisor and consequently write more partitive, equal groups division problems. It is not intuitive to share a quantity among a fraction group such as the "1/4" group (Lo & Luo, 2012). It is unknown whether the structure of partitive, equal groups division problems would still be the most common problem among students when a divisor is a fraction group.

English (1998) and Lavy and Bershadsky (2003) stated that those problems without a clear mapping between the problem situation and the required operation are comparatively rarely posed. In English's study of addition and subtraction problems, such problems include comparison situations. Although this study focused on division operation, students also posed comparison division problems less than the equal-groups problems. Furthermore, none of the U.S. students posed a *comparison* problem. Given the limited range of problem types generated by the participating students from both countries, the question arises as to whether children could generate greater diversity in problem structures if their school experiences provide them more opportunities to consider multiple meanings of division. Thus, we wonder if some specific types of problems posing can be improved with some problem-posing instructions.

The contexts of the story problems collected in each country show the similarities and differences from narrative perspectives. Both groups of students overwhelmingly used food as context. Some contexts, such as Terracotta Warriors (China) and mooncakes (China) occasionally shown in those story problems, are indication of cultural contexts of the respective country. The contexts of rocks and dog treat in the U.S. might be related to the local biophysical environment or living habits. However, generally speaking, the inclusion of social-cultural contexts in the posed story problems was limited. For example, although the U.S. students lived in a spectacular valley adjacent to several stunning ski mountains and the well-known Yellowstone National Park, they did not incorporate any of those regional geographic features into their problem posing.

The results analyzed from the narrative mode of thought help reflect the true meaning and scope of the real-world contexts from the students' perspectives. We argued that students' problem posing might not be impacted by perfectly-phrased real-world contexts surrounding their communities. Instead, daily food such as fruit or various types of candy might better reflect their real-world contexts.

Several limitations exist while this study contributes to the ongoing efforts of understanding and explaining division problem posing. First, this study lacks sufficient information to generalize the posted problems in terms of learning settings and teaching approaches. This study did not collect information about the type of textbooks used by students and the type of story problems the students discussed or solved. This information could be relevant to the frequency of the type of problem around the examples provided by the students. In addition, this study did not investigate how teachers in both countries frame their teaching. The lack of investigation on teachers and their teaching approaches makes the contamination of teaching, a potential limitation regarding internal validity, unclear. If the teachers of the two participant groups joined in and finished an interview or a survey, we would know (1) the extent to which the teachers fostered the students' problem-posing activities in their teaching and (2) what contexts embedded the examples and problems they discussed in their classes. What teachers teach about division –for instance, types of division problems – would influence students' interpretation of division and writing of story problems.

Second, this study did not explore the relationship between problem contexts and structures. It is unknown whether using some contexts to pose a problem led to the way to pose a particular problem structure. It is unknown whether using some contexts to pose a problem led to the way to pose a particular problem structure. It is interesting to know, for example, whether the use of food contexts is associated with posing a partitive, equal groups, also called the equal sharing problem.

Third, this study did not relate the students' posted problems to their responses to the other two prompts: How would you define division? Explain why these two division problems are different. This study could characterize the students' problem posing more insightful by relating the types of posted problems to their responses to the other two prompts. It is interesting to know, for example, whether or not the problem contexts situated by the students would be less realistic if they defined division from a more procedural perspective, such as inverse multiplication.

Fourth, this study did not examine the relationship between problem-posing and problem-solving developments. It is unknown whether the problem contexts in which students situated their posed problems show consistency with the data involved and possible solutions. It is also unknown the complexity of numbers and steps to solve the posed problems. It is also unknown whether Chinese school students in this study were more likely to pose multiple-step or challenging story problems than their U.S. counterparts through more data analysis. Compared with U.S. students, Chinese school students have been more likely to choose complex tasks (Wang & Lin, 2005).

Lastly, this study did not provide students with sufficient opportunities to demonstrate their cognitive schemes. Since the students were only asked to pose two rather than as many different problems, the posted problems might only tell us the richness of problem types in their initial minds. In particular, this study collected and analyzed only a limited number of story problems from students in China and the U.S. It is crucial to take caution With such a small-scale sample size while interpreting the findings and recommendations generated from this study.

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DEVELOPING PROBLEM POSING IN A MATHEMATICS CLASSROOM

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Abstract

This is an exploratory study of 18 grade 9 students working on two problem-posing tasks involving the quadratic function. There are a variety of problem-posing strategies used by students, including the use of associated sub-topics, using the quadratic formula as a guide, working backwards, and adopting a trial-and-error approach. The free-posing task seems to help students to bring out more variety of sub-topics that they can connect, perhaps reflecting some confidence for such type of task. This is less so in the semi-structured task. It also appears that the number of sub-topics used is not dependent on student achievement type. Some implications for teaching and for teachers are also discussed. Specifically in the context of differentiated instruction in a classroom, problem posing activities can be one strategy to engage students. The findings of this exploratory study have the potential to add to the body of local knowledge about how problem-posing instructions can be engendered in the classroom to bring about deeper classroom engagement in mathematics.

Keywords: Mathematical Problem Posing, Problem Solving, Polya, semi-structured, free-posing

INTRODUCTION

Since the publication of Polya's (1945) *How to solve it,* problem solving has received much attention. This was especially so in the 1980s when there was a world-wide push for problem solving. Polya's problem solving model was in particular well-known as it is especially easy to "carry in the head" (Toh et al., 2008a, 2008b). In particular, Polya's four-phase model of problem solving, the fourth stage of problem solving is *Look Back* phase. This phase addresses the common phenomenon in school mathematics classrooms that the problem-solving process often ceases when a solution is reached. The *Look Back* phase extends beyond checking the reasonableness or correctness of the solution obtained; it includes adaptation, extension and generalisation of the original problem. Thus, Toh et al. (2008a, 2008b) renamed the *Look Back* phase as the *Check and Expand* phase. This refers to the stage of making the arrived solution of a problem as a starting point for further mathematics exploration, and marks the beginning of problem posing.

In this paper, we adapt the definition of mathematical problem posing operationally as referring to the process of generation of a new problem or a question by learners based on the given situation (Mishra & Iyer, 2013). It includes the generation of new problems in a mathematical context or the re-formulation of existing ones (Silver, 1994). In a classroom milieu, many of the students' mathematics experiences are solution-

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driven. Problem posing as a classroom activity can therefore offer a platform for students to transcend the fixation on problem solving where thinking is chained by prior knowledge and by set ways of seeing things. Teachers can also learn how much their students understand a mathematical topic through the problems that their students pose, since problem posing cannot be done without a context.

If learning mathematics is taken to involve creating meaning, then the ability to pose problems is an essential skill for creating that meaning to the learner. Meaningful learning will have a place to support arousing students' interest in the subject. Such a problem-posing classroom approach would have its place in supporting classroom enactment. In particular, in the Singapore context, we believe that problem posing would be able to support the recently released The Singapore Teaching Practice framework (Ministry of Education (MoE), 2017) by addressing the aspect of "arousing interest" for classroom instruction.

By a search of existing mathematics literature, there are relatively few studies on mathematical problem posing, and even fewer studies explore promoting student problem posing in the mathematics classroom (Chua, 2011). The exploratory study reported in this study attempts to add to the knowledge in the Singapore context through addressing the following research questions (RQs):

RQ 1 - What are the characteristics of students' problem posing in the semi-structured and free-posing tasks?

RQ 2 - What are students' selection of mathematical domain knowledge in problem posing?

RQ 3 - How does problem-posing performance vary across achievement levels?

Such knowledge on the characteristics of the students' posed problems is important in understanding how teachers can promote problem posing in classroom instructions. In answering the research questions, the products of the problem posing and the views of the tasks by students are examined.

RESEARCH BACKGROUND

Mathematical problem posing is identified as an essential mathematical activity and a companion to mathematical problem solving (Kilpatrick, 1987). Various studies have pointed to the importance of students' mathematical problem posing. This is related to students' exploration in mathematics (Cai, 2003) and the teaching and learning of mathematics (Crespo, 2003). Bransford et al. (1996) noted that developing students' ability to formulate their own problems is important for developing the mathematical thinking needed to solve complex problems. It has long been acknowledged that problem posing is an important intellectual activity in scientific investigation (Cai et al., 2015a). Brown and Walter (1993) noted that problem-posing activities in the classroom helped in lessening mathematics anxiety, in explicating misconceptions and in fostering group learning. Ellerton et al. (2015) noted that given that problem posing is closely interwoven with real-life situations, it can be seen as a natural link between formal mathematics instructions, problem solving and the world outside the classroom. Researchers have also found links between doing problem-posing tasks and students' being able to make connections and make sense of mathematics (Carrillo & Cruz, 2015). Hansen and Hana (2015) also argued that problem posing can give students the much needed ownership of their learning environment, since it is a natural component of inquiry-orientation and is grounded in the "belief of giving priority to the question over the answer." Elsewhere, Cai et al. (2015b) found that problem-posing activities can promote students' conceptual understanding, foster their mathematical communication and capture their interest and curiosity.

In the Singapore Ministry of Education (MoE) Mathematics Syllabus (2012), students are encouraged to "connect ideas within mathematics" (p. 8). The Singapore Mathematics Curriculum Framework (MoE, 2012) also places importance on the need for thinking skills and metacognition in mathematical problem solving. Such skills can be developed in the students through engaging students in problem-posing activities since posing a problem requires them not just to have the necessary concepts but also the ability to link these coherently together to form a problem. Students' responses to problem-posing tasks could provide a window through which to view students' ability to make connections within mathematics and a mirror that reflects the content and the character of their school mathematics experience. The importance of problem posing as a mathematical activity that could promote engaged learning provides the main impetus to the present study.

Depending on the purposes of the study, there are different problem-posing tasks in mathematics problem-posing research literature, ranging from free situation, structured situation, and semi-structured tasks (Stoyanova & Ellerton, 1996).These different contexts may result in different types of responses. This exploratory study aims to develop a description of what and how students problem pose in response to a semi-structured task and a free-posing types, and the mathematical domain knowledge that students' selectas they formulate their problems. This local description, though highly contextual, can provide insights into the problem-posing processes and provide clues about how students can be supported to problem pose.

The exploratory study can contribute to the knowledge of how students, being novice problem posers, can be taught heuristics that will build up their problem-posing skills. This will be helpful for the professional development of practicing teachers as such newly acquired knowledge on problem-posing will be able to facilitate teachers to stretch students beyond problem-solving and generate their own problems, which can be perceived as a form of empowering student learning. Through our collective experience with the mathematics classrooms, many teachers might not feel comfortable about problem posing, hence they will unlikely involve students in problem posing.

METHODOLOGY

Tashakkori and Teddlie's (1998) definition of a mixed methods design is adopted for the present study. It combines the "qualitative and quantitative approaches into the research methodology of a single study or multi-phased study" (p. 17). Green et al. (2006) and Denscombe (2007) argued that such a design besides providing a means to compensate the strengths and weaknesses of both approaches also provides a more complete picture of the research problem, hence in answering our RQs. Onwuegbuzie and Teddlie (2003) contended that mixed methodology allows one to make representation by way of extracting adequate information from the underlying data. It could also help in legitimizing "the validity of data interpretation" (p. 353).

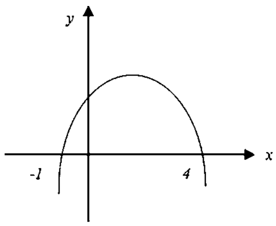
The problem-posing characteristics were inferred from the posed problems and the solutions. By engaging students to pose problems relating to a given stimulus and solve the problems they have posed, it is likely that the students would reflect within their posed problems and solutions, which is a crucial part of the learning experience which is an important part of the curriculum. This is supported by researchers such as Malara and Gherpelli (1994), who asserted that requiring students to solve their problems might prompt them to do deeper reflection on their own posed problems. This approach to engage the students to solve their own posed problems and to reflect on their problems is adopted by researchers on problem posing (e.g., Ellerton, 1986; Silver & Cai, 1996; Lowrie, 1999).

For practicing teachers, students' solutions are an indicator of how they have conceptualized their problems since the solving and posing processes are complementary (Contreras, 2007). Their problem-posing responses would therefore uncover the extent of their domain knowledge preference. Lowrie (1999) noted that the problem poser may not only focus on the underlying structures of the problem but also the extent to which the problem solver would be able to interpret the components of the problem. Students are therefore more involved in exploring the problems that they posed than when they were with the problems given by their teachers. The solution strategies and the use of the mathematical domain knowledge could give insights into the posed problems (Cai, 2003). Drawing from work by English and Halford (1995), problems were first classified as either solvable or non-solvable. A solvable problem has a well-defined initial state, a goal state and an inherent solution path.

The participants of this study were drawn from a sampling of a class of 18 Secondary 3 (grade 9) students from a secondary school, comprising 10 males and 8 females. The students were asked to complete two problem-posing tasks involving quadratic functions. The two problem-posing tasks were designed so that each could draw out a variety of problem-posing characteristics and to uncover the mathematical domain knowledge (sub-topics) used by the students.

Task 1: Free-posing Task

Write possible problems that are related to the quadratic curve given.



The quadratic curve for task 1 is commonly encountered by high school students in school algebra. In the syllabus documents, students are expected to be able to identify the geometrical properties and to solve problems involving the graph of a quadratic function. This free-posing task with a minimum context requires the problem poser to form the initial state, the goal state and the building of a context for the emerging problem. In this way, it attempts to uncover specific posing characteristics among the students.

Task 2: Semi-structured Task

The points (-2, 11) and (4, -1) lie on the curve given by the equation $y = ax^2 + bx + 3$.

Pose some problems arising from the problem stem.

Task 2 is more contextualized with elements of a given quadratic. The intention of the task is to investigate whether there are variations in posing strategies and in problem characterization in a more contextualized posing situation, with the given information about the two points and the form of the equation with a constant 3.

The students were given 20 minutes to complete each of task 1 and 2. After completing the problem-posing tasks, the students individually took a survey on their perceptions of the task difficulty, interest in the tasks, mental effort exerted and the extent in which they would like to attempt similar problem-posing tasks in the future. The students' performance in the two problem-posing tasks and the survey taken by them were used to answer RQ1 and RQ2.Three students were selected each from the different achievement levels (low, middle, high) with each student working on a new problem-posing task 3. A task-based interview was conducted with the three students. The interview provided a qualitative account of their problem-posing behaviour. The interview was audio-recorded and transcribed. The performance of this selected group of students on problem-posing task 3 and the task-based interview, together with the analysis of problem-posing performance across tasks 1 and 2 for the different achievement levels (low, middle, high) were used to answer RQ3.

Task 3:
Pose as many different problems where the solution involves:"the roots are $2+\sqrt{3}$ *and* $2-\sqrt{3}$*"*

FINDINGS AND DISCUSSION

The categories of the posed problems with the exemplars are shown in Table 1. Across the two tasks, the common characteristics of the posed problems are the use of turning points (*maximum / minimum*), the *line of symmetry*, and the range of values for x. There were two non-solvable/vague problems:

- There is another line cutting through the curve. Name one curve. (free-posing task)
- If $x = 11$, or -1, solve the equation $y = ax^2 + bx + 3$ (semi-structured task)

These were not further analysed.

Table 1

Descriptions of posed problems

Category	Exemplar of students' response under this category.
Equation of graph	Find the equation of the graph
Maximum / minimum point	Find the maximum point
y-intercept	Find the point where the curve cuts the y-axis
line of symmetry	Find the line of symmetry.
Find *a, b* (in Semi-structured Task)	Solve for *a* and *b*.
Sketch	Sketch the curve $y = \frac{1}{2}x^2 - 3x + 3$
1-sub-topic	Find the coordinates of the curve at the y-intercept
2 sub-topics	Find the equation of the curve and the maximum point
Range of values for x	Find the range of values for x when y is positive.

CHARACTERISTICS OF STUDENTS' PROBLEM POSING IN SEMI-STRUCTURED AND FREE-POSING TASKS

One limitation to this exploratory study is that the some of the students' responses do not afford themselves for more in-depth analyses. Many of the students' solutions to their posed problems were too brief or left blank (44% in the semi-structured task, 50% in the free-posing task). The characteristics of the students' problem-posing for the two tasks are described based on the number of sub-topics, process of problem posing, and the degree of perceived difficulties. These are sufficient to describe the characteristics of the problem posing and also to shed light into some possible implications for practice and further research.

a) Number of sub-topics

The distribution of the number of sub-topics across the two tasks is shown in Table 2. For the free-posing task, majority of the students used two or more sub-topics, reflecting perhaps their confidence in bringing items together to form the problems. In the semi-structured task, more than half of the students used two or less sub-topics. Student G2 reflected that (s)he took on sub-topics in the formulation by looking at what were familiar:

"*I was using most of the questions given by the teachers in my exam paper itself. By seeing where I know that I've seen familiar equations and familiar answers by using the same thing of what they have given me*."

Table 2

Number of sub-topics

	Number of sub-topics (%)		
	1	2	3
Task 1 (Free posing task) (total 17)	11.8	64.7	23.5
Task 2 (Semi-structured) (total 13)	69.2	23.1	7.7

Note: Some students did not complete at least one of the two tasks.

b) Process of problem posing

Across the two tasks, students used various strategies in problem formulation. Students drew on their knowledge about the associated sub-topics, for example, *turning point*, and the *line of symmetry*, to formulate problems. The variety of sub-topics that were used perhaps pointed to their understanding of the mathematical structure which they had embedded in their posed problems.

The quadratic formula appeared to have "guided" students to the questions that they wanted to pose. Student H1 noted that the formula guided her/him to the question that (s)he wanted to ask:

"*I was trying to find numbers that can actually get to the answers because it was square root 3 and it was quite difficult because there was a divide by over 2A which was, … so I have to make the number divisible by 2*."

"*So before I begin, I actually tried thinking of numbers that can actually put into this formula so that I* *can actually get to the answer. Like I have to find numbers that can actually be divisible by 2 and getting a number that can actually get to square root 3. So I had to, so that the number can be square rooted …*"

Some students also appeared to fit in the values to arrive at the final answer through a process of working backwards. For example, student H2 noted that:

"*… so I can check what is A, B, C by working backwards or I can write the equation first. I will first solve the equation by trying to work backwards so I will try to solve the equation by coming up with random numbers. Or I can use my calculator. ….*"

Student G3 also reflected that:

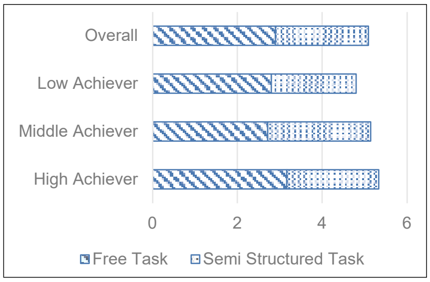
"*... when you pose a problem, you have to work backwards and create some numbers so that the questions can be solved. But as for problem solving, it is quite simple as you can just use some, add numbers and divide. And you are working forward and not backwards*."

There was also evidence of the use of a "trial-and-error" approach to problem formulation. From student G2 noted that:

"*I would have tried to sub(*stitute*) or treated an equation first to be able to solve. If I won't be able to do it I will restart it and try a different number that I can ... By going with guess and check method*"

c) Degree of perceived difficulty

The free-posing task seemed to elicit better perception of ease of posing compared to the semi-structured task, specifically with the high achiever group as shown in Figure 1. Student H1 felt that the problem (s)he had posed was suitable for her/his friends to answer because "*the teacher has already taught in class so they can use this as a practice question, maybe a possible question that can come out in the exam paper*." Together with student G1's earlier reflection, " *.... I was using most of the questions given by the teachers in my exam paper..*" students appeared to have the strong consciousness of posed problems as being linked to assessment. This perhaps can be indicative of their classroom mathematics experiences as being solution-driven.



Perception of Task as "Easy"

	Free Task	Semi-Structured Task
Low Achiever	2.8	2
Middle Achiever	2.71	2.43
High Achiever	3.17	2.17
Overall	2.89	2.22

Perception scale: 1(strongly disagree) to 4 (strongly agree)

Figure 1. Students' perception of task as being "easy"

SELECTION OF MATHEMATICAL DOMAIN KNOWLEDGE (SUB-TOPICS)

The distribution of mathematics sub-topics are shown in Table 3 and Table 4. The use of turning points (*maximum /minimum*) were most common in the responses in the free-posing task, and least likely was the choice of using y-intercept. Finding a, b was most commonly seen in the semi-structured task, and least likely are the use of *line of symmetry* and *range of values*. These perhaps reflected students' familiarity with these sub-topics in their problem-solving experiences, and specifically with their idea about finding unknowns. No students extended their posed problems to involve topics in other strands of mathematics. Perhaps, this absence of connecting to other mathematical ideas could be because they were novice problem posers. That almost all the sub-topics revolved around quadratics could be due to the 'recency' effect as they were just taught quadratics in their lessons prior to their problem posing exercises.

Table 3

Number of sub-topics in Task 1 (free-posing task)

	Task 1 (free-posing), total 17				
Sub-topics	Equation of graph	Max / min pt	y -intercept	Line of symmetry	Range of values
Overall (%)	64.7	82.4	17.6	29.4	17.6

Note: Students can use more than one sub-topic in their posed problems.

Table 4

Number of sub-topics in Task 2 (semi-structured task)

	Task 2 (semi-structured), total 13				
Sub-topics	Find a, b	Max pt	Sketch	Line of symmetry	Range of values
Overall (%)	84.6	7.7	30.8	7.7	7.7

Note: Students can use more than one sub-topic in their posed problems.

For the free-posing task, the use of *Equation of graph* and the *line of symmetry* in their posed problems was moderately corrected, with $r(15) = -.60$, $p < .05$. Students who used the *line of symmetry* in their posed problem would likely use the notion of *sketch* in their semi-structured task, with $r(15) = .56$, $p < .05$.

PROBLEM-POSING PERFORMANCE ACROSS ACHIEVEMENT LEVELS

Performance is described in terms of number of mathematics sub-topics in this exploratory study. Achiever types and number of sub-topics used in the two tasks are shown in Table 5 and Table 6. Achievement level was found to be not significantly correlated to the choices of sub-topics across the two tasks.

Table 5

Achiever types and number of sub-topics in free-posing task

	Free-posing Task, total 17		
	1-topic	2-topic	3-topic
Low Achiever (%)	20	28.5	40
Middle Achiever (%)	20	71.4	0
High Achiever (%)	0	57.1	40
Overall (%)	11.8	64.7	23.5

Note: 1 student did not complete the task.

Table 6

Achiever types and number of sub-topics in semi-structured task

	Semi-structured Task, total 13		
	1-topic	2-topic	3-topic
Low Achiever (%)	60	0	0
Middle Achiever (%)	60	20	0
High Achiever (%)	60	40	33.3
Overall (%)	69.2	23.1	7.7

Note: Five students did not complete the task.

Majority of middle achievers and high achievers posed two-topic problems in free-posing task. Low achievers tend to produce one sub-topic problems in the semi-structured task. More middle achievers produce one sub-topic problems in the semi-structured task. The choices of sub-topics used by the different achievement levels across the two tasks are shown in Table 7 and Table 8. Across the achievement levels, most students use two sub-topics for the free-posing task, and fewer use one sub-topic. This may suggest students' greater confidence in problem posing in the free-posing task. The finding is also consistent with an earlier study (Chua & Wong, 2012), which showed that students tend to posed problems that involved more topics in a free-posing task. Majority of middle achievers and high achievers posed two sub-topic problems in free-posing task. Low achievers tend to produce one sub-topic problems in the semi-structured task.

Table 7

Achiever types and sub-topics in free-posing task 1

	Task 1 (Free-posing), total 17				
	Equation of graph	*Max / min pt*	*y-intercept*	*Line of symmetry*	*Range of Values*
Low Achiever (%)	80.0	80.0	40.0	0	20.0
Middle Achiever (%)	57.1	71.4	0	42.9	14.3
High Achiever (%)	60.0	100.0	20.0	40.0	20.0

Table 8

Achiever types and sub-topics in semi-structured task 2

	Task 2 (Semi-structured), total 13				
	Find a, b	*Max pt*	*Sketch*	*Line of symmetry*	*Range of Values*
Low Achiever (%)	60.0	0	0	0	0
Middle Achiever (%)	80.0	20.0	40.0	20.0	0
High Achiever (%)	80.0	0	40.0	0	20.0

There were also variations in their perceptions about the problem posing across achievement types as shown in Figure 2. High achievers appear to have more positive orientation towards problem posing, specifically in having interest and trying it in the future.

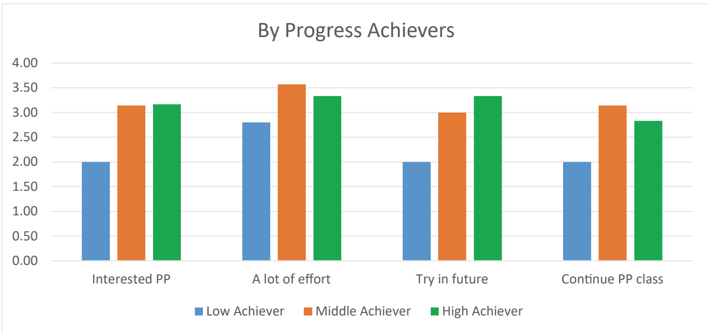


Figure 2. Students' perception of problem posing across achievement types

DISCUSSION AND IMPLICATIONS FOR TEACHING PROBLEM-POSING

Across the various problem-posing strategies used by students, namely, using of associated sub-topics, using the quadratic formula as a guide, working backwards, and adopting a trial-and-error approach, there appears that such an exercise is helping students to "think back" on the concept of quadratic function while posing. This may help students to in facilitating their recall of key mathematical concepts.Free-posing task seems to help students to bring out more variety of sub-topics that they can connect, perhaps reflecting some confidence for such type of task. This was less so in the semi-structured task. So a good starting point for teachers would be to introduce the free-posing type of tasks to students in the classroom.It appears that the number of sub-topics used is not dependent on student achievement type. In the context of practicing differentiated instruction in a classroom, problem posing can be one strategy to engage students in mathematical exploration

Student G2 noted that the whole problem-posing process itself can be "fun":

"*I believe such activities are fun for the students because the students will know how much … be able to solve the question if they were to form their own personal question. If they have to think on how to solve it, which actually gives them the idea of what teacher have to think ... then they would be able to guess more on which equation to use for certain questions ...*"

There are also others who thought about the "utility" of problem posing in their classroom experience, as reflected by student G8:

"*I learnt how to create questions and find this is quite useful as we have to do things reverse which helps me understand better. I've understood how to be a teacher too*."

Besides, both reflections of students G2 and G8 also point to the potential for problem-posing exercise as a way to engender interest and understanding in learning mathematics, for example, such activities

- as "*being fun for students ...*(G2),"
- bringing an understanding how the teacher poses questions (G8), and
- being able to bring about understanding because they "*have to do things reverse ...*(G8)"

These potential affective development that arise from problem-posing activities can bring about better student classroom engagement and may warrant further study beyond this exploratory study.

CONCLUSION

With the push for more subject discipline classroom engagement in the mathematics classrooms, there will be an increase in the emphasis in getting students to go beyond just problem solving and having the sense of inquiry in the real-world context which can be facilitated by problem posing. Given the importance of problem-posing as an emerging field of study in mathematics education research, the findings of this exploratory study have the potential to add to the body of local knowledge about how problem-posing instructions can be enacted in the classroom to bring about deeper classroom engagement in mathematics. Specifically, the study underlines the importance of planned approaches for the use of problem-posing activities in the classroom. Like problem-solving heuristics, teachers may have to teach problem posing explicitly as part of the classroom instructional programme. Students posing problems can bring about connections to real-life contexts, better student engagement, and may bring insights into students' understanding of mathematics. Specifically, problem-posing exercises can bring about greater awareness of the connections between the initial state and the goal state of a mathematics problem, and better cognizance of problem-solving strategies, mathematics content knowledge and processes.

Both problem solving and posing are influenced by affective factors, for example, beliefs, control, etc., and by the regulation of cognition. (English & Halford, 1995, p. 261). Besides this regulation of cognition, Schoenfeld (1985) also argued cogently about the importance of beliefs in solving. According to his model, beliefs referred to "one's mathematics world view, the set of (not necessarily conscious) determinants of an individual behaviour and about self, environment, topic and about mathematics" (p. 15). An area that warrants further study will be the effects of students' beliefs on problem-posing, their motivation and their background experiences in shaping how they construct problems. The poser, for example, had to be aware of when it was suitable to access a known problem and had to decide on the extent of correspondence between the two problem structures so as to determine the extent of procedural adaptation needed. Such a decision drew upon the poser's beliefs and the metacognitive control over the content knowledge.

A suggestion for a research agenda would be to further examine problem-posing responses across task types and involving other domains in school mathematics like statistics or probability. Such work could be useful in the design of intervention studies to promote problem-posing skills. More work may be needed to examine gender and problem-posing performance. There is still the need for more follow-up studies on a bigger sample. The present study nevertheless contributed to knowledge in this area.

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