TEACHING MATERIALS ON CALCULUS AS SEEN FROM THE APPLICATION TO ENGINEERING

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Abstract

The authors have developed teaching materials on calculus including multivariable functions for first-course undergraduate students in the science and engineering fields. The materials differ from commonplace textbooks in that they first introduce the topics encountered in engineering and then explain the mathematical aspects. This style aims to help readers understand mathematical concepts smoothly by identifying their interests and offering topics that appeal to intuition.

Key words: University mathematics education, teaching materials, multivariable functions, application to engineering, intuition

INTRODUCTION

Although many calculus textbooks have been published in Japan, only a few explain the concept of calculus in the context of applications such as engineering. This is especially true for the calculus topic multivariable functions. Generally, textbooks play an important role in learning. However, several researchers have pointed out the necessity of reconsidering the role of textbook usage in university classes. For example, Berry, Cook, Hill, and Stevens (2010) investigated textbook usage in finance classes and pointed out that, although students recognize the importance of reading the textbooks, most still do not spend sufficient time doing so. Randahl (2012) investigated an actual class to determine the characteristics of first-year engineering students' approach to using calculus textbook. The results showed that students preferred to use lecture notes rather than the textbook and that students increasingly tended to ignore the textbook, except in relation to the tasks the textbook offers. Randahl identified students' difficult experience of textbooks' formal treatment of concepts as one of the reasons for this trend.

Teachers might have to take special care of students because first-year engineering students may have serious difficulties due to big differences between learning mathematics in college and school. Robert and Schwarzenberger (1991) considered the differences between elementary mathematics in compulsory education and specialist mathematics at colleges and universities, highlighting considerable changes in the nature of mathematical content being taught. Concepts also differ substantially from what students previously experienced, often involving not only generalisation but also abstraction and formalisation (see also Tall, 1991; Dreyfus, 1991). It is hard to imagine that the engineering students discussed here can understand such

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abstractions and generalisations in the same way as mathematicians do. Robert and Schwarzenberger (1991) wrote that "The change in the ratio of quantity of knowledge to available acquisition time means that it is no longer possible for the student to learn all new concepts in class time alone; significant individual activity outside the mathematics class is now an absolute necessity." (Robert & Schwarzenberger, 1991, p. 128). In such situations, some students may rely on textbooks. However, if the textbooks provide formal definitions of calculus concepts and theorem proofs monotonously, students will have difficulty with concept formation because of their limited knowledge and ability. First-year engineering students relying upon such textbooks need sufficient mathematical and modelling skills to understand and smoothly apply theoretical calculus. However, this is an unreasonable expectation.

It is known that treating formal definitions of the concepts in teaching mathematics is difficult (e.g. Vinner, 1991; Cornu, 1991). Dreyfus (1991) mentioned that it is insufficient to merely define and exemplify an abstract concept. Raman (2002) illustrated the difficulties students encounter when coordinating the formal and informal aspects of mathematics. The gap between the two approaches taken by precalculus and calculus textbooks, namely emphasising the informal aspect at the expense of the formal aspect or the reverse, respectively, complicates coordination. Tall and Vinner (1981) described concept image as the total cognitive structure including all mental pictures and related properties and processes. Additionally, they described concept definition as a form of words used to specify the concept. Textbook content should enrich the concept image so that students understand formal definitions of concepts. Bingolbali and Monaghan (2008) pointed out that environments such as students' departmental affiliations can influence the acquisition of concept images, in the context of research on undergraduate students' understanding of the derivative. Harris et al. (2015) analyzed engineering students' problems with mathematics through interviews. Those scholars challenged both the pedagogical practice of teaching non-contextualized mathematics and the lack of transparency regarding the significance of mathematics in engineering. Moreover, some previous researchers pointed out that practical contexts or situations which could clearly show the necessity of extending the existing knowledge are absent and that strict and pure mathematical contexts cause the situation in which students experience textbooks as difficult to use (e.g. Randahl, 2012; Randahl & Grevholm, 2010).

The authors are currently preparing calculus teaching materials for use in classes. This paper specifies some requirements imposed on mathematics textbooks for engineering students and offers examples of teaching material pertaining to both partial derivatives and area integrals intended to aid students' concept formation of multivariable functions by employing topics in practical contexts. The use of such materials in calculus classes is also discussed.

We assume the following minimum requirements for mathematics textbooks for engineering students:

1. Introduce the concept not only with a formal definition but also by providing practical contexts that represent the characteristics of the concept.
2. Introduce the concept in practical contexts before providing a formal definition.
3. The practical contexts include the reasons for the necessity of extending students' knowledge.

The first and third requirements reflect previous research results that have pointed out the difficulties associated with providing a formal definition without practical contexts or situations. This has demonstrated the necessity of extending students' present knowledge, and it will help calculus teachers who are unfamiliar with the use of calculus in engineering. Preferable mathematics textbooks should include both formal definitions of concepts and practical contexts. If a formal definition of a concept is introduced before the practical context is given, most students will experience difficulty with concept formation, and their comprehension motivation might decline. Teachers should try to facilitate students' understanding of these concepts. Therefore, we adopted the second requirement, which will lead students to abstract and formalize concepts.

On the other hand, from the standpoint of an engineering course that the second author took, calculus is an extremely important topic in university-level mathematics. Derivatives and integrals of multivariable functions are new units for first-year engineering students, and understanding of these concepts is poor for the majority, even though these units are extremely important fundamentals of subsequent subjects in their course of study. Therefore, we prepared calculus teaching materials including derivatives and integrals of multivariable functions, for engineering students.

In this paper, we introduce some of our teaching materials concerning derivatives and integrals of multivariable functions. The materials fully utilize geometry, which is commonly used to explain functions. This is intended to minimize the complexity and, therefore, avoid the confusion associated with introducing dimensions other than length into calculus. For partial derivatives, the geometry of a river-bank, which is familiar to many people, is taken as an example to assist concept acquisition. Subsequently, Taylor series and total derivatives are introduced in relation to the topic of derivatives as employed in fluid mechanics, which students encounter in mechanical, civil, and chemical engineering studies. For integrals of a multivariable function, two examples of area integrals of the first and second moments of the area are introduced. The area integral is the basis for determining the center of an area. We were unable to find such an application in the ordinary mathematics textbooks that we perused. The first and second moments of the area arise as topics in hydraulics and structural engineering, which are subjects that mechanical and civil engineering study.

EXAMPLES OF OUR TEACHING MATERIALS

Partial derivatives

Partial derivatives are defined in the context of multivariable functions as the rate of increase in functions with respect to one selected independent variable with other independent variables fixed. One of the most effective ways to understand the concept of partial derivatives is to consider the geometry of a 3-dimensional body. If we consider a 3-dimensional body located in the x, y and z spaces, we can imagine an edge formed by cutting the body by an arbitrary plane. Any edge formed in this manner can be generally expressed as a curve in a 2-dimensional space. In this case, the partial derivative can be evaluated with respect to the variable whose axis is taken horizontally in the cutting plane. Here, we attempt to explain partial derivatives through the function expressing straight lines formed from a plane in 3-dimensional space as a special case.

Let us consider a river that conveys flood water in a channel (Jain, 2001) and a dike with a flat slope with a tilt angle θ built on the river's horizontal high-water floor, as shown in Figure 1. If the x-axis is taken along the line where the dike slope and high-water floor cross each other, and the y-axis is taken in the perpendicular horizontal direction, the height of the surface of the slope z measured from the high-water floor is expressed by a two-variable function as $z = f(x,y)$.

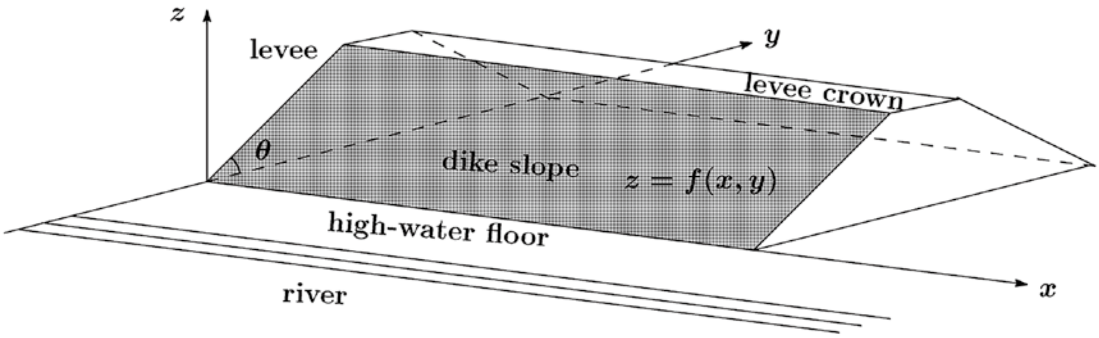
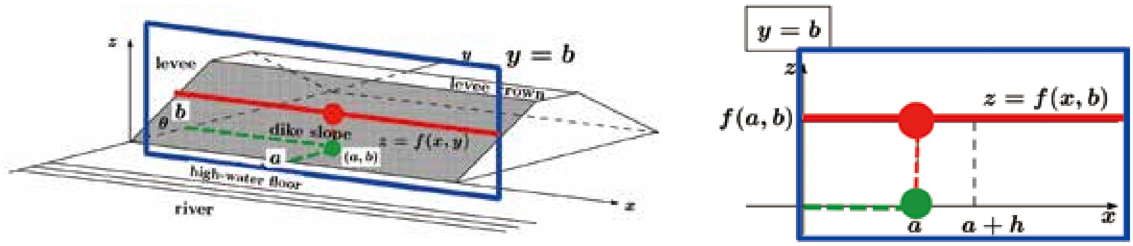


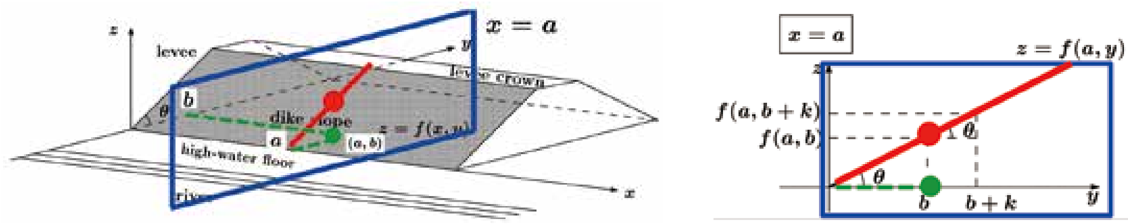
Figure 1: A bird's eye view of a bank with flat dike slope.

In this situation, it is clear that the partial derivative of z with respect to x is expressed as $\partial z/\partial x = 0$ and that the partial derivative of z with respect to y is expressed as $\partial z/\partial y = \tan\theta$ (see Figure 1). We believe that Figures 2 and 3 enhance understanding of these situations.



(a) Bird's eye view showing cross-section $y = b$. (b) Cross-section cut by $y = b$.

Figure 2: Cutting the dike slope by the plane $y = b$.



(a) Bird's eye view showing cross-section $x = a$. (b) Cross-section cut by $x = a$.

Figure 3: Cutting the dike slope by the plane $x = a$.

Next, let us consider the change in the value of the dependent variable z when the independent variables of x and y change, which is expressed by Taylor series. In the case of the situation indicated in Figure 1, the increase in the dependent variable z, dz, owing to infinitesimal increases in the independent variables x and y, dx and dy, is expressed using Taylor series as follows:

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy + O(dx^2 + dy^2)$$

where $O(dx^2 + dy^2)$ is the component of dz due to the derivatives of the second-order and higher, which expresses the effect of surface unevenness and is 0 in this problem. Therefore, the above expression is written as follows for the case shown in Figure 1.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

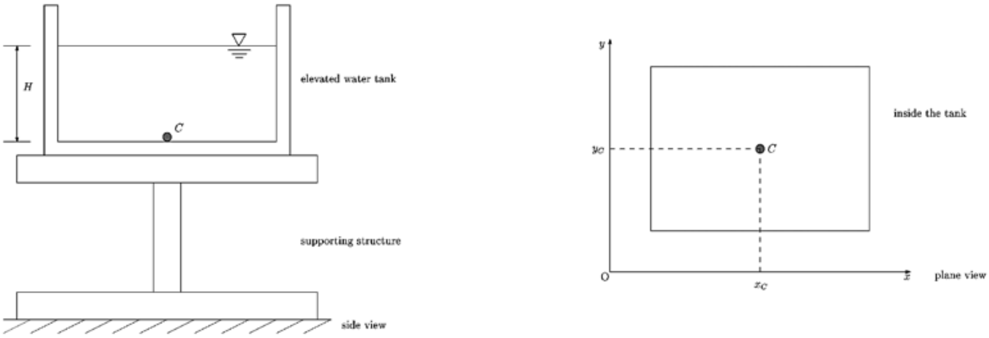
In this case, the above expression shows the height increment when a person walks on the slope via dx in the x-axis direction and via dy in the y-axis direction.

It should be emphasized that Taylor series' connection to total derivatives is an extremely important basis for studying engineering course subjects. In fluid mechanics, physical quantities indicating fluid properties are treated as functions of time t and the position in space, x, y and z. Additionally, the concept of a fluid particle is the minimum unit of the fluid, which is a conceptual matter meaning an infinitesimal element constituting the fluid and moving around in the flow. The time rate of change in physical quantities of a fluid particle, called material derivatives or substantial derivatives, is expressed using total derivatives. This concept is necessary to define the acceleration of flow and is one of the most important basic concepts of fluid mechanics.

Area integrals 1

Let us consider an elevated rectangular parallelepiped water tank with a horizontal bottom supported by a supporting structure, as shown in Figure 4. To maintain a stable condition, it is necessary to support the tank with a structure that can bear the total weight of the water at the position selected so that the central axis of the vertical part of the supporting structure lies below the line of action of the total gravity force exerted on the water, where the weight of the tank is considered to be negligible compared with the weight of water. This problem requires calculating the area integrals. First, the force that the structure should support is determined by integrating the weight of the water on an infinitesimal area of the bottom of the tank over the total area of the bottom. When a force acts normally on an area, the force per unit area is generally called "pressure" and is usually denoted by the character p. If we set the weight of the water per unit volume as w, the water depth in tank as H, and the area of the bottom of the tank as A, the force acting vertically downwards on the infinitesimal area of the bottom of the tank dA due to the pressure p is given by $p\,dA = wH\,dA$. Therefore, the force F the structure should support is given using the area integral, as follows:

$$F = \int_A p\,dA = \int_A wH\,dA = wH\int_A dA = wHA$$



(a) Side view of the entire system. (b) Plan view of the bottom of the tank.

Figure 4: Elevated water tank supported from below.

Second, the position of the center of the supporting structure is determined. Another requirement to ensure the stability of the system is that the axis of the center of the vertical part of the supporting structure should be aligned with the line of action of the total gravity force F indicated above. This condition is obtained by evaluating the moments about the x and y axes, as indicated in Figure 4. Force F expressed by the above equation should be considered as an equivalent concentrated force acting on a point, with the restriction that the moment due to force F about an axis is identical to the sum of the moments about the same axis due to the pressure acting on the area in a distributed manner. Let us consider the moment of the force about the x-axis using Figure 4. If we apply the force due to the pressure p on an area element dA located at a distance y from the x-axis as $dF = p\ dA = wH\ dA$, the moment of the force due to this force about the x-axis is expressed as $dM_x = dF \cdot y = p\ dA \cdot y = wH\ dA \cdot y$. Therefore, the sum of the moment dM_x for all components of the infinitesimal areas comprising the total area of the tank bottom is as follows:

$$M_x = \int_A py\ dA = \int_A wHy\ dA = wH \int_A y\ dA$$

From the above discussion, the quantity indicated by M_x must be identical to the product of $F = pA = wHA$ and some distance y_c from the x-axis (see Figure 4):

$$wH \int_A y\ dA = wHA \cdot y_c$$

The above expression reveals the following relationship:

$$\int_A y\ dA = A \cdot y_c \quad \text{or} \quad y_c = \frac{1}{A} \int_A y\ dA$$

The term indicated by integration in the above equations is called "the first moment of an area A with respect to x-axis", and y_c indicates the y-ordinate of the action point of total force $F = wHA$. The first moment of area A with respect to the x-axis is defined as the sum of the infinitesimal area of each area component dA multiplied by the first power of the distance between the component and x-axis. It is interesting to consider that the areas of the individual elements act as weights on the distance, which resembles the calculation of the mean value in statistics. The moment of force about the y-axis is considered in the same manner. The expressions corresponding to the forms in the former case are as follows:

$$M_y = \int_A wHx\ dA = wH \int_A x\ dA \quad ; \quad \int_A x\ dA = A \cdot x_c \quad ; \quad x_c = \frac{1}{A} \int_A x\ dA$$

The quantities x_c and y_c derived above indicate the ordinates of the centroid and the geometrical center of an area and are customarily written as x_G and y_G, respectively. That is,

$$x_G = \frac{1}{A} \int_A x\ dA \quad \text{and} \quad y_G = \frac{1}{A} \int_A y\ dA$$

Area integrals 2

Next, we consider the problem of hydrostatic pressure acting on an inclined plane submerged under the water surface, as shown in Figure 5. Similar to the former example, the water pressure at a point y vertically below the water surface is given by wy using the weight of the water per unit volume w. Consider a case where the inclined flat bottom of a body of water has a hole with the shape shaded, as shown in Figure 5, and the hole is covered by a hard plate with its same shape and area to prevent water outflow.

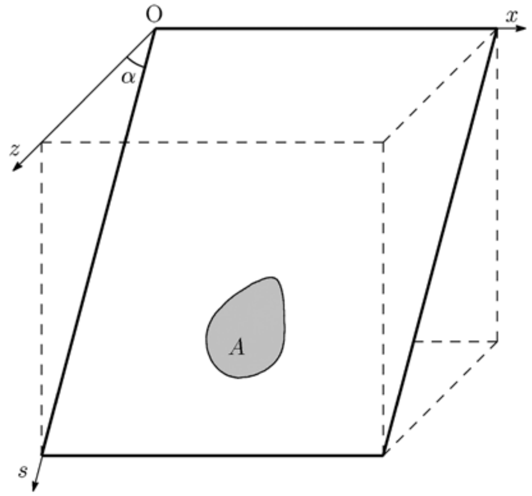


Figure 5: A plate covering a hole in the inclined flat bottom of a body of water.

The model for analysis of this problem is shown in Figure 6, in which the left side is a view alongside the flat bottom, and the right side is a view normal to the flat bottom. The shaded part in Figure 5 is exaggeratedly shown in Figure 6.

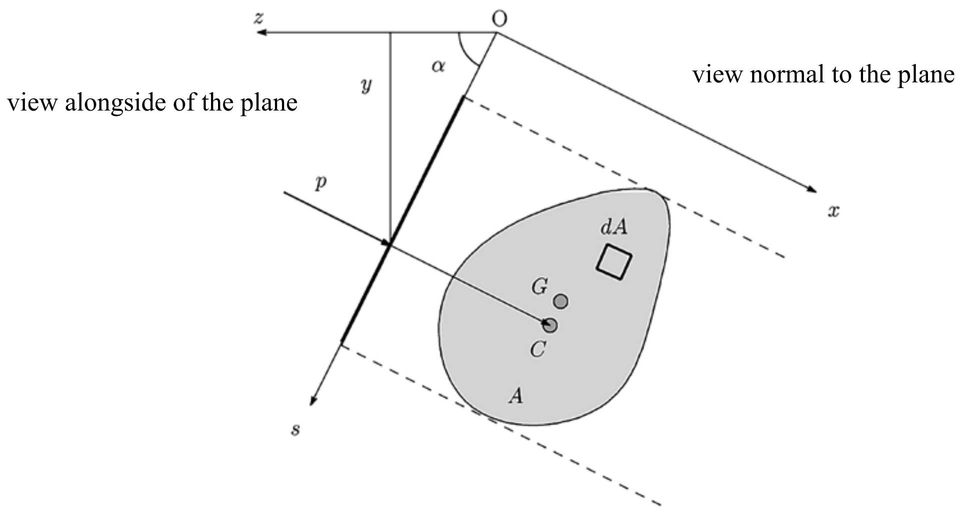


Figure 6: Model for analysis of hydrostatic pressure in Figure 5.

The water pressure at distance s taken downwards along the plane from the water surface is $p = ws \sin \alpha$. Therefore, the force due to the water pressure acting on the plane element of area dA at this position can be written as $p\, dA = ws \sin \alpha\, dA$. The total force F is the sum of $p\, dA = ws \sin \alpha\, dA$ over the entire surface of area A and is given by the following equation:

$$F = \int_A p\, dA = \int_A ws \sin \alpha \cdot dA = w \sin \alpha \int_A s\, dA = w \sin \alpha \cdot s_G A$$

Here, $s_G A$ is the first moment of area A with respect to the x-axis, and s_G is the s coordinate of the centroid of area A.

Next, we consider where to support plate A at the back to obtain a stable force balance. This point is referred to as the point of action of the total pressure and is usually indicated as C, while its coordinate is s_c. Let us consider the finding of s_c. When the moment of the force due to the water pressure acting on the surface element area dA at a distance s from the x-axis $dM = pdA{\cdot}s$ is summed over area A, the following form is obtained:

$$M = \int_A ps\, dA = \int_A ws^2 \sin\alpha\, dA = w \sin\alpha \int_A s^2\, dA$$

Here, $\int_A s^2\, dA$ is the moment of inertia of area A in Figure 5 with respect to the x-axis. The area moment of inertia is also called "the second area moment", which is the sum of the infinitesimal area of each area component dA multiplied by the second power of the distance between the element and axis in question. Subsequently, from the relationship $M = Fs_C$, the following is obtained:

$$s_c = \frac{w \sin\alpha \int_A s^2\, dA}{w \sin\alpha{\cdot}s_G A} = \frac{1}{S_G A}\int_A s^2\, dA$$

If we define a new axis taken from the centroid s_G in the s direction as u the area moment of inertia expressed by the coordinate s is rewritten in the following form:

$$\int_A s^2\, dA = \int_A (s_G + u)^2\, dA = s_G^{\,2} A + \int_A u^2\, dA = s_G^{\,2} A + I_G$$

Here, I_G is the sectional moment of inertia about the axis passing through the centroid s_G of plane A. Therefore, we obtain the position of C as follows:

$$s_c = \frac{1}{S_G A}\int_A s^2\, dA = \frac{s_G^{\,2} A + I_G}{s_G A} = s_G + \frac{I_G}{s_G A}$$

This indicates that action point s_C of the total water pressure is located below the centroid s_G by $\frac{I_G}{s_G A}$.

DISCUSSION

The example teaching materials proposed in this paper provide a clearer image of partial derivatives and area integrals. The sample materials also seem to satisfy the three requirements stated in the Introduction section of this paper. Indeed, these examples introduce concepts in practical contexts before the provision of a formal definition. Moreover, these practical contexts highlight the necessity of extending existing student knowledge. We expect that first-year undergraduate students in engineering courses will use these examples to establish a professional foundation.

Given that the topics the examples cover are all related to civil engineering, students in other fields, such as electrical engineering, may feel that these materials are unfamiliar. Therefore, we believe that materials that would be more widely accepted are required.

On the other hand, all engineering students, not just those in civil engineering courses, seemed to be familiar with the dike slope example given to explain partial derivatives. Indeed, when the first author used this material to deliver a lecture on partial derivatives to economics students, the results of a post-lecture questionnaire showed that most students understood partial derivatives with respect to the individual variables of two-variable functions in relation to the slope in each direction. Furthermore, due to the additional explanation provided, these students understood that partial derivatives play an important role in the extreme value problems of the two-variable functions.

This example, however, has a weakness because the function treated expresses the flat plane whose derivatives of the order higher than one all become 0. Therefore, this example fails to explain the more general aspects of partial derivatives. In this study, the authors attempted to explain partial derivatives through the exclusive use of geometry, that is, without employing any relationship between physical quantities, to avoid confusion arising from complexity. Although this constraint may reduce the potential for explaining concepts by reducing freedom of consideration, the geometry-based explanation proposed here is effective, as evidenced by the first author's experience, which is described in the previous paragraph.

Regarding the materials on area integrals, the authors assume that they are used in classes as an introduction when expanding from the integral of one-variable functions to the integral of multivariable functions. If non-engineering students use these materials, a small supplement to physics may be needed.

The authors believe that using these materials to introduce each math unit has the advantage of allowing students to understand the need for mathematical knowledge and how it can be applied in the real world or in their specialty. As another merit of using this type of material, the authors assert that students in engineering courses can not only understand professional subjects as applications of mathematics but can also deepen their understanding from a mathematical viewpoint. It is difficult for either mathematicians or engineering teachers to create these materials independently. Collaboration between teachers on both sides is necessary to accomplish this goal.

CONCLUSION

We have discussed problems encountered when teaching calculus to students in engineering courses and have tried to provide teaching materials for engineering education. The teaching materials covering the topics treated in this paper employ problems contextualized in daily-life situations and communicated using figures. We believe that the proposed examples will provide a clearer image of partial derivatives and area integrals and that the explanations given in this paper will deepen students' understanding of the technical problems they encounter in engineering courses. We plan to use the examples presented here not only in engineering but also in mathematics classrooms and take into account students' opinions and/or requests in order to refine and further develop the teaching materials. We are also working on creating teaching materials that support various topics of instruction. For example, regarding multivariable functions in calculus as specialized fundamental education in engineering, more exciting exercises are needed so that students in engineering courses can derive ordinary and partial differential equations that arise as the governing equations to evaluate important physical quantities. In engineering, it is necessary to analyze differential equations, particularly partial differential equations, based on key physical laws rather than ideal assumptions. We hope that students will acquire mathematical skills for practical problems, including physical elements, through derivations of the related differential equations.

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References

Berry, T., Cook, L., Hill, N. & Stevens, K. (2010). An exploratory analysis of textbook usage and study habits: Misperceptions and barriers to success. *College Teaching*, *59* (1), 31–39.

Bingolbali, E. & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics*, *68* (1), 19–35.

Cornu, B. (1991). Limits, In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Kluwer Academic.

Dreyfus, T. (1991). Advanced mathematical thinking processes, In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Kluwer Academic.

Harris, D., Black, L., Hernandez-Martinez, P., Pepin, B. & Williams, J. & with the TransMaths Team (2015). Mathematics and its value for engineering students: What are the implications for teaching? *International Journal of Mathematical Education in Science and Technology*, *46* (3), 321–336.

Jain, S. C. (2001). *Open-channel flow*, John Wiley & Sons, Inc.

Lindeburg, M. R. (2014). *FE review manual third edition*, *Revised*, Professional Publications, Inc.

Raman, M. (2002). Coordinating informal and formal aspects of mathematics: Student behavior and textbook messages. *The Journal of Mathematical Behavior*, *21* (2), 135–150.

Randahl, M. (2012). First-year engineering students' use of their mathematics textbook - Opportunities and constraints, *Mathematics Education Research Journal*, *24* (3), 239-256.

Randahl, M. & Grevholm, B. (2010). Learning opportunities offered by a classical calculus textbook to engineering students about differentiation. *Nordic Studies in Mathematics Education*, *15* (2), 5–27.

Robert, A. & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced level, In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 127–139). Kluwer Academic.

Shepherd, M. D. & van de Sande, C. C. (2014). Reading mathematics for understanding—From novice to expert. *The Journal of Mathematical Behavior*, *35*, 74–86.

Skemp, R. (1987). *The psychology of learning mathematics*. London, UK, Penguin Books.

Takagi, S. & Hadano, K. (2018). Educational materials on basic partial derivative which appeal to intuition, *Proceedings of the International Workshop on Mathematical Education for Non-Mathematics Students Developing Advanced Mathematical Literacy* (http://iwme.jp/pdf/Proceedings_IWME2018.pdf), 73-76.

Takagi, S., Yamaguchi, S. & Mizumachi, R. (2018). Designing 1st year calculus courses, *Proceedings of the International Workshop on Mathematical Education for Non-Mathematics Students Developing Advanced*

Mathematical Literacy (http://iwme.jp/pdf/Proceedings_IWME2018.pdf), 47-52.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 3–24). Kluwer Academic.

Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12* (2), 151–169

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics, In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Kluwer Academic.

Wylie, C. R. & Barrett, L. C. (1995). *Advanced engineering mathematics 6th edition*, McGraw-Hill, Inc.

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