SUSTAINABLE LESSON STUDY AND PARADIDACTIC INFRASTRUCTURE: THE CASE OF DENMARK

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Abstract

Most widely published books and papers on lesson study emphasise that a main attraction of this format is that it enables teachers to build and maintain a shared knowledge base for the teaching profession. This, however, clearly requires that lesson study is established as more than an occasional experiment, which is new to most teachers involved and involves only exceptional teachers. Therefore, research on lesson study increasingly focuses on the question of sustainability of the format as a practice-based form of professional development. We present and expand the recent theoretical idea of paradidactic infrastructure and analyse sustainability from the institutional perspective of the Anthropological Theory of the Didactic and especially based on the idea of paradidactic infrastructure. Our data come from the Danish context where lesson study in mathematics has been experimented with for more than 10 years, first by individual researchers and now, increasingly driven by institutions, such as schools or municipalities.

Key words: Lesson study, sustainability, paradidactic infrastructure, professional development

INTRODUCTION

Lesson study (LS) was essentially introduced to the Western world by a rare "best-seller" in the educational science literature authored by Stigler and Hiebert (1999). In both the USA and Denmark, the dissemination of LS has to some extent, been based on individuals' first-hand experience of it in Japan and in video recordings that relay some of what LS may contain and produce in terms of mathematics teaching. Slowly, several Danish scholars began to take interest in LS as described by international (especially American) scholars, such as Stigler and Hiebert (1999).

The purpose of this paper is to investigate the sustainability of LS in the Danish context, drawing on theoretical tools that have so far only been used in the context of Japan. Danish efforts to implement LS began relatively late compared to those of other Western countries. Such efforts first happened in the United States, and Lewis (2002) already reported on a number of "barriers facing LS" in this particular country and noted that "research is needed to understand the supporting conditions that have enabled lesson study to succeed at some sites" (p. 19). Newer empirical research has studied the effects of efforts to bring about such conditions, such as providing additional resources like prepackaged LS kits with templates, LS steps and mathematics resources (Akiba & Wilkinson, 2016; Lewis & Perry, 2014), involving leaders of schools or bigger districts in adapting LS in organisational structures and routines (Akiba, Howard & Wilkinson, 2019)

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or introducing the "knowledgeable other" to ensure the effectiveness of LS (Takahashi, 2013; Takahashi & McDougal, 2016).

The literature also contains case studies of small- and large-scale implementations of LS in dozens of other countries, including Denmark, Norway and Sweden. We note here that most such studies report some kind of immediate "success" of LS. For example, teachers have reported on such success and what they learned from the experience; but in this paper, we will not be concerned so much with the effects of LS on teachers and students as with the institutional conditions that allow or hinder the continued practice of LS.

What are the necessary and sufficient conditions for LS to become a regular and teacher-led activity, particularly in a Danish context? What could these conditions be, in general? The question was reformulated and approached during ICME13 in Hamburg as, *What makes lesson study "viable" in some contexts, where the activity is regularly done by all teachers with no external initiative?* (Qauresma et al., 2018, p. vii). To examine an indeterminate "what" in a research context, a theoretical framework, which specifies the boundaries and kinds of the research object, is required. The contribution of this paper is mainly to introduce and exemplify the recent theoretical construct of *paradidactic infrastructure* as a means to frame studies that investigate the above question more precisely. Within the Anthropological Theory of the Didactic, this construct models the conditions under which the teaching profession operates (in particular, learns) outside of teaching situations (Winsløw, 2012; Miyakawa & Winsløw, 2019). We present this theoretical framework along with the more precise research questions on the paradidactic infrastructure for LS in Denmark after the background section. Then, we proceed to investigate our questions through two case studies, which illustrate important constraints and potentials for LS in Denmark.

BACKGROUND: LESSON STUDY IN A DANISH CONTEXT

Danish researchers following the international literature became aware of LS starting in the late 1990s, along with everyone else (cf. introduction). It began to be disseminated to Danish teachers and teacher educators some years later via popularisation texts (Winsløw, 2004, 2006, 2009) and workshop presentations. The first implementation of LS in Denmark was in 2009 at the National Knowledge Centre for Mathematics Teaching (NAVIMAT, in existence until 2012). Led by Kaj Østergaard and Erik Bilsted, the elaboration of LS involved seven mathematics teachers, and Bilsted (2010) reported on this elaboration by presenting the lesson plans of seven "super lessons" together with a reprint of the work of Winsløw (2009). Since then, several other professional development and research projects based on LS involving both in-service and pre-service teachers, have been initiated. We focus here on LSs which have been documented in national and international academic journals.

A good amount of attention has been paid to LS as part of teacher education. Nyboe and Rasmussen (2015) described LS as a formal framework for practice elements in teacher education, which allows teaching to become a common and concrete object of concern for pre-service teachers, teachers and teacher educators. Jørgensen, Rostgaard and Mogensen (2016) presented a two-year research project in which LS was carried out by pre-service teachers in collaboration with their internship supervisors (who are experienced teachers). One conclusion was that LS fostered collaboration that aligned the goals of in-service teachers and internship supervisors as the internship became an avenue for knowledge sharing and a rewarding collaboration. These studies are mainly descriptive, but others are more theoretical. Skott and Østergaard (2015) used the *knowledge quartet* to categorise Danish pre-service teachers' mathematical *knowing* for teaching as it developed through LS during their internships. Østergaard (2016) used the the Anthropological Theory of the Didactic to explore the potentials of LS to improve pre-service teachers' learning during internship and concluded that LS constitutes an interface between teacher education and the internship programme.

Moving beyond the setting of teacher education, Skott and Møller (2016, 2017, 2019) used *Patterns of Participation* to conceptualise and understand teachers' *knowing* and participation from a social perspective. One of their aims was to contribute a deeper theoretical understanding of individual teacher learning and meaning-making derived from participation in LS. Another aim was to explore the conflicts that emerged when Danish teachers engaged in LS. The authors concluded that it is necessary to address broader issues of culture, such as teachers' collaborative work and power relations between the teachers in order to adapt LS in a Danish context. They also identified cultural norms as an explanation of implementation problems. Bahn (2018) used the Theory of Didactical Situations in the analysis of experienced teachers' work using open-ended lessons in the context of LS. He found that the open-ended approach was useful to some extent and that LS was effective in developing teachers' knowledge about the connection between teaching and learning. He also found that considerable external support was necessary for teachers' progress in this regard. Finally, in a mostly theoretical paper, Winsløw, Bahn and Rasmussen (2018) argued that the use of theoretical frameworks is crucial for systematic research on LS. They exemplified the use of the Theory of Didactical Situations and the Anthropological Theory of the Didactic using two Danish cases of pre- and in-service teachers' development.

We have not found documentation of Danish LS projects (involving the same teachers) which lasted for more than a few years or of research analysing the sustainability of LS in Denmark. Sustainability is an implicit concern in several of the documented implementation projects and in some of the more theoretical analyses of these projects, such as the identification of cultural issues as important conditions for implementing LS (Skott & Møller, 2019). We now propose a general and explicit framework.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

In the Anthropological Theory of the Didactic (Chevallard, 1999, 2006), we model knowledge in terms of praxeologies. Praxis refers to practical knowledge and "know-how" and encompasses tasks and techniques for solving tasks, and logos is explicit knowledge about the praxis and consists of technology, including explanations and justifications of the techniques, and theory, which is an abstract discourse clarifying or justifying the technology. In line with the work of Miyakawa and Winsløw (2019), we use the term *paradidactic* to refer to phenomena related to school teaching but found outside of teaching situations themselves. *Paradidactic praxeologies* refer to the practices and explicit knowledge (Chevallard, 1999), which teachers deploy and develop as they interact with other teachers, prepare or evaluate a lesson, and so on. As the examples show, some of these praxeologies are very similar to concrete *didactic praxeologies*, which occur in the classroom and are aimed at developing *mathematical praxeologies* among the pupils.

Other paradidactic praxeologies, such as meetings or courses uniting all teachers from a given school, typically do not address any specific didactic praxeology and certainly not any mathematical praxeology.

The professional praxeologies of mathematics teachers constitute a very complex composition of mathematical praxeologies, didactical praxeologies and paradidactic praxeologies. LS activity demonstrates that this matter cannot be reduced to the forms of practice and logos, which are visible in mathematics classrooms, especially because in the latter, teachers are usually alone and cannot share their experiences and inferences with colleagues. The construction of professional knowledge requires, more generally, an infrastructure (frameworks, settings, organisations, etc.) that goes beyond the professional practice itself. In some professions, this infrastructure is prestigious and visible, for example, the medical profession with its international scientific journals, conferences, organisations, etc. In the case of mathematics teachers, there is also such an infrastructure - the *paradidactic infrastructure* (cf. Winsløw, 2012; Miyakawa & Winsløw, 2018) - which consists of the shared conditions (institutions, levels, media, etc.) that govern and nurture teachers' work outside the classroom, whether they work in large or small groups or alone (as is often the case for the day-to-day work related to teaching). Certainly, some of these conditions are quite visible and almost trivial, such as the time and workspace which teachers are given to prepare their teaching and the official documents and other resources, such as textbooks, that are available for this work. Other conditions may be less visible and more difficult to describe. LS is but one of the elements of conditions that are shared by Japanese teachers, and it depends on multiple other elements of the Japanese paradidactic infrastructure.

Miyakawa and Winsløw (2019, Table 1) displayed a wider field of the paradidactic infrastructure in Japan. The consideration of such conditions drew our attention to the fact that paradidactic infrastructure is often *national* in nature. Unlike the medical profession, the profession of mathematics teaching does not possess a solid and internationally shared knowledge base, with the possible exception of that which pertains strictly to the mathematical praxeologies that it exists to disseminate. The experiences, results and assumptions of the profession are still mostly shared at a national level or even more locally. The internationalisation of mathematics education research over the past century or so has the potential to produce a wider knowledge base, but unlike what holds for the medical profession, it remains mainly known to a small segment of scholars, despite their efforts to interact with teachers and contribute to the paradidactic field of activity, including pre- and in-service teacher education. The national differences of paradidactic infrastructure also imply that the investigation of the conditions for conducting LS has to be done at a national level.

In particular, we are interested in *sustainable LS*, meaning LS that is conducted regularly over several years and with resources normally available to schools (i.e., not as an initiative or with external funding). This does not entirely exclude external support (even in the form of funds); for example, in Japan, "knowledgeable others" are usually present (Takahashi, 2014), but the activity must not be merely an initiative by external institutions and should not, in principle, be limited in time.

We focus on the paradidactic infrastructure of Danish mathematics teachers in primary and lower secondary school and its implications for the possibility of sustainable LS. The reason for limiting our scope to these levels is that these teachers share a number of important institutional conditions: their initial education takes place at university colleges and is very different from the university programmes that prepare upper secondary teachers. In Denmark, the primary and lower secondary levels (grades 1–9) are mandatory and delivered in the same school institutions unlike higher-level education. The relative homogeneity of the teachers is also reflected in common work conditions, a common union for teachers and the fact that the schools in which they teach are either administered by one of the country's 98 municipalities or state-sponsored private institutions (catering to roughly 20% of the children in grades 1–9).

With these preliminaries and assumptions, we present in Table 1 a rough overview of the main institutions and elements of the paradidactic infrastructure in Denmark (PID) in analogy with the Japanese case (Miyakawa & Winsløw, 2019, Table 1). Our first portrait of the PID is based on the authors' brainstorming sessions and informal conversations with colleagues. It is thus more a rough picture than a final map. We note that the institutional levels involved in Table 1 can be roughly connected to the levels of society and school in Chevallard's (2002, 2019) hierarchy of didactic codetermination.

We now explain, in a little more detail, the elements of the table that are the most important (indicated in boldface in the table). The School Department of the Ministry of Education, currently The Department for Teaching and Quality, is in charge of the national, centralised regulations and management of schools. In particular, it produces the national "Common Goals", which specify the learning content for each subject at a given grade level, and creates (in part by delegation to committees or companies) a number of mandatory tests corresponding to these goals, including the exams at the end of lower secondary school. There is no centralised production or control of teaching materials. Most teachers rely on textbooks published by commercial editors usually authored by 2–4 teachers or teacher educators. Teacher educators are generally employed at University Colleges, where primary and lower secondary teachers are initially educated. These institutions also engage in development projects and in-service education related to school teaching, often

Table 1 Elements of Paradidactic Infrastructure in Denmark

Institutional level	Institutional frameworks	Settings for teacher study and research	Media used or produced by teachers
National	**Ministry of Education** **- School department** Ministry of Higher Education and Research **- University colleges** - Universities National Teacher Union - Danish Math Teacher Association (MTA) **- MTA Publishing Co.** Math counsellors' network **Private foundations** Private textbook publishers	EMU (webpage with national curriculum and research) Teaching material centres Masters programmes Annual conference Theme conferences Book fairs	"**Common Goals**", **tests** Skolekom lists Folkeskolen.dk "Mathematics" Magazine Books and other resources **Commercial textbooks**
Region	Regional sections of MTA	In-service courses	
Municipality	School commission **School administration** **Educational Centre/ consultants** Math Resource Centre / Counsellor teachers	In-service courses New teaching materials, meetings and courses on specific themes	
Local school	**School principal** School management Math team(s)	Meetings (for all teachers) Math team meetings	School Intranet

financed by the Ministry of Education or by private foundations. Individual researchers at universities (such as the second author) may, exceptionally, be involved in such activities. Primary and lower secondary school teachers do not usually interact with universities except for the minority who (like the first author) pursue a masters or doctoral degree in addition to regular teacher education at the bachelor's level.

The public schools are funded and managed by the 98 municipalities. Each school has a board (appointed by the municipality) which appoints a school principal who is responsible for the day-to-day operation of the school, including the hiring of teachers and regulation of their working conditions. The principal appoints other members of the school management, who may take on specific pedagogical responsibilities. At the municipal level, within the school administration, there is also a unit - sometimes called the Educational Centre - which supports teachers through in-service courses and the like, usually staffed by consultants who may be specialised in some school subject(s). The organisation of this service depends on the municipality (size, political choices).

As noted in the previous section, LS in Denmark has so far mainly been initiated by individuals from university colleges and universities, often in relation to short-term projects, such as PhD dissertations. Such projects, and indeed the two cases considered, could offer some insight into conditions which favour or inhibit LS and enable us to hypothesize on sustainability requirements. More precisely, we will use our model of PID (Table 1) and the two cases to investigate the following questions:

RQ1: How do the different elements of PID affect (support, hinder) specific efforts to implement LS?

RQ2: What can be learnt from such specific cases regarding the potential for sustainable LS activity in Danish schools, for instance in terms of necessary changes in PID or in the method of implementing LS?

The two case studies are largely explorative (Streb, 2012) in the sense that we use them to generate hypothetical answers for the above questions, which are supposed to illustrate the meaning and pertinence of the proposed framework and of the concrete model of PID presented above. The hypothetical responses to RQ1 are summed up after each of the cases, and RQ2 is discussed in the discussion and conclusion.

The two cases are not comparable with regards to the length of the studies, numbers of participating teachers, academic support or the mathematical content studied in the classroom. However, both are cases of LS implemented over a long period in a Danish context and are aimed at making LS a viable and sustainable part of teachers' professional development. The cases were selected to exemplify the different aspects of how PID affects the efforts to implement LS.

Both cases draw on documents, pictures and field notes collected by the authors while observing and participating in the two contexts. In the first case, audio recordings of meetings were also used. The data is used to support the generation and illustration of hypotheses and not as "proof" of their general validity.

CASE 1: LEARNING TO DO LESSON STUDY IS NOT SUFFICIENT

ABC (pseudonym) School is a large public school near Copenhagen, and it is funded as a so-called resource centre of mathematics education in its municipality. In early 2018, the mathematics consultants of the municipality and the school management decided to establish a "LS environment" at ABC School in collaboration with two mathematics educators from the nearby University College (including the first author). Two explicit aims for this activity were cited: to increase teachers' explicit focus on students' mathematics competences (a key notion in the current curriculum) in the classroom and investigate how teachers' collaboration can improve the quality of mathematics teaching. Given the role of the school, another aim was for teachers to subsequently help other schools establish a LS environment. LS was totally new to the teachers, who were often teaching alone; at the time of the study, they had not experienced collaborative situations, including observing teaching and students' learning together.

The project began with an introductory workshop on LS, given by the mathematics educators, during which a teacher from another municipality (the one considered in Case 2) gave an open lesson, which was followed by a common reflection session for the participants (as a first experience of this part of LS). At the end of the workshop, the principal of the school that the presenting teachers came from narrated his experiences with LS. In short, the introductory workshop gave teachers of ABC School a first impression on how LS works. Later in the spring of 2018, two teams of mathematics teachers (grades 3 and 7) started their own LS with the two mathematics educators as facilitators. After a common workshop on "reasoning competence" and "problem solving competence" (cf. Niss and Jensen, 2002), they planned a research lesson and completed three cycles of teaching (observing), reflecting and revising the lesson plan.

In the autumn of 2018, one new LS team joined the first two, and the spring workshop was repeated, but this time, a teacher from one of the first teams gave the open lesson. Again, LSs were done within the team, and one teacher from each of the teams focused on how to facilitate and guide a LS process.

We will now discuss one of the research lessons for grade 3, taught in the autumn of 2018. The presentation is based on lesson plans, field notes from classroom observations, student worksheets (as shown in Figures 2 and 3), audio recordings of reflections sessions, project meetings with teachers and municipality partners and interviews with teachers. Reflections, evaluations and interviews relating to the PID were all transcribed and coded (Table 1).

The research lesson was the last of three, and it was the second time that the team of the three teachers participated in LS. In the research lesson and the following reflection, the three teachers, one of the mathematics educators, an invited "knowledgeable other" from a university college and a journalist from a major national teacher magazine, were present. The lesson plan had been revised twice based on the experiences from the first two trials and reflections.

The target knowledge was students' understanding of fractions and their representations.

In the textbook used at ABC school, the following type of tasks on fractions were covered: 1) draw the half and the quarter of a figure, 2) put a ring around ½ and ¼ of things, 3) fill out the form: whole, half, quarter, three quarters and 4) Is this right or wrong? (see the different tasks in figure 1). The suggested techniques included 1) making drawings of the correct size, 2) counting the total amount of elements in a group and marking ½ and ¼ of the element, 3) drawing whole-part relations and 4) ticking the correct box. The level of students' activities focused on practical knowledge and know-how techniques. The students were not asked to reflect on their techniques. While planning the research lesson, the teachers expanded the focus from praxis to also include logos (e.g., reasoning about the size of a fraction and the notation of a fraction).

The teachers' experience suggested that it was difficult for students to encounter fractions other than ½ and ¼ in their surroundings and everyday lives, where integers and decimal numbers appear more frequently. The teachers wanted the students to visualise fractions in different ways and formulated the fraction problems:

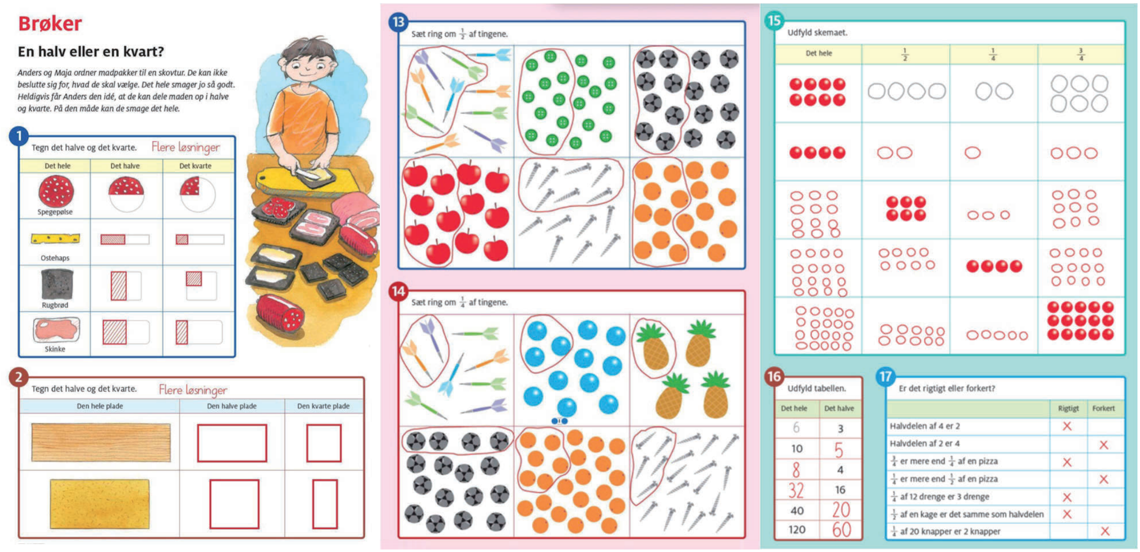

Figure 1. Grade 3 textbook: fraction tasks and examples of techniques (Lindhardt, Jensen & Møller, 2021)

Why is ⅛ smaller than ¼, while 8 is greater than 4? The broader goals of the research lesson related to students' inquiry skills and mathematical reasoning, involving work with different kinds of representations. As part of the planning of the lesson, the teachers designed a worksheet to support the students' work with drawings, written explanations, calculations and everyday stories (see Figures 2 and 3). The teachers found the fraction problem simple, but by working on the problem themselves, they came up with several potential ways for the students to reason and use representations to support their reasoning.

During the observation of students' work, teachers focused on their use of the different representations,

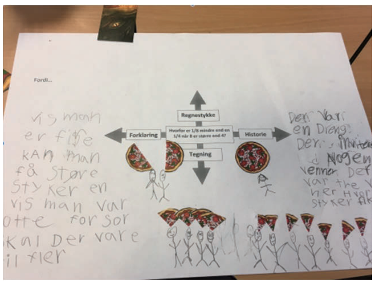

Figure 2. Worksheet with students' reasoning about sharing a pizza: "if there are four [people], you can have bigger slices [of pizza] than if there are eight [people] because you would need more pieces for more [people]"; the explanation is illustrated with drawings of people and pizzas. Everyday story: "A boy invites three friends; how big a slice of pizza does each of the children get?"

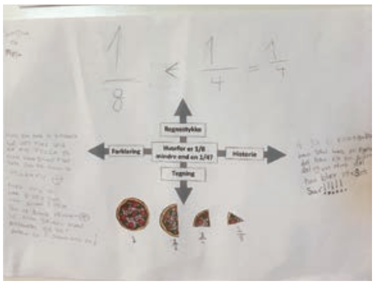

Figure 3. Worksheet with students' reasoning: "if you have four pieces of pizza, you need four pieces to form a [whole] pizza" (illustrated by a circle divided into four pieces), "but if you have eighths you need 8 pieces to form the pizza" (illustrated by a circle divided into eight pieces); "then you need more pieces and because of that, 1/4 is bigger than 1/8". On top: "1/8 < 1/4 = 1/4".

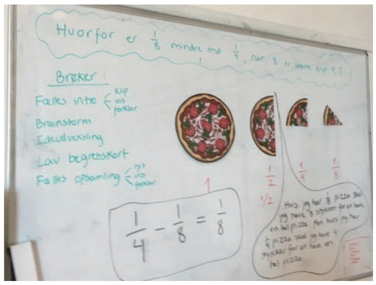

Figure 4. White board with students' reasoning and responses: "If I have 1/8 of a pizza, I need 8 pieces to make a full pizza, but if I have 1/4 of a pizza, I need 4 pieces to make a full pizza".

especially the drawings and the explanations, to help students understand the problem. The teachers also noticed that students progressively used more formal mathematical language during the entire class discussion (partly reflected in Figure 4). During the following reflection meeting, the teachers stated what they had learned from observing students' dialogues, especially regarding their reasoning. Many of these points were quite generic and could be relevant beyond the research lesson. The participants also discussed the difficulties of orchestrating a whole class discussion and, in particular, drew substantial connections between different students' solutions.

The "knowledgeable other" stated that the lesson plan was robust and described the lesson as a complex lesson: "a delicious bite, where we get special knowledge about children's thinking on fractions, part and whole, in the process". He further complemented the lesson because the focus was on students' reasoning, and the teacher did not tell them what to do or think: "I categorise this as *reasoning without authority*".

In an oral evaluation of the experience, all teachers from the different teams indicated that they wanted to continue doing LS. They described LS as a certain kind of "noise" that, in a good way, forced them to be curious, experiment and make challenging changes in their teaching. This kind of "noise" was absent when teachers worked alone. In short, the teachers explained that they learned mathematics and teaching mathematics when they observed students' learning and discussed new ways to orchestrate mathematics lessons. However, the teachers also described the LS project as challenging; they felt that it "doubled" their work. Some were confused due to organisational obstacles, such as a substitute teacher not showing up. The teachers stated that they would like more "completed" lesson plans in Danish.

In the evaluation, the consultants from the municipality and the school management discussed how to support the project in a better way and how to construct a new structure around the LS. The consultants and principals had attended a workshop or a research lesson and the subsequent reflection, and they described the potentials of the project. Their focus was not on the teachers' learning to teach mathematics, including learning mathematics, but on their learning of how to do LS. They asked the teachers severally whether they were ready to start facilitating LS processes at some of the other schools in the municipalities and whether it was possible to organise open lessons, etc. In several ways, it seemed like the management and consultants had different goals from those of the teachers and mathematics educators.

In spite of the positive evaluation, the project stalled during the following winter as the municipality and school management became engaged in other projects. They cancelled several LS meetings and explained that it was difficult to find teachers who had time for the project. The mathematics educators' interpretation was that the municipality and principals saw the first two rounds of LS as a completion of the goal of learning how to do LS and had moved on to other goals.

To sum up this case, at the beginning of the project, LS was strongly supported by three institutional levels of PID: the university college and the "knowledgeable other" from the national level, consultants from the municipality level and the school management from the local school level. During the project, the teachers had mixed experiences with LS. On one hand, they were confused about organisational obstacles (e.g., extra math team meetings), which were not scaffolded by the school management and on the other hand, the teachers experienced professional development during LS. The teachers' development was not recognised by the school management or the consultants from the municipality. During the second year, the support for the two last levels disappeared, and the project was no longer viable. What will happen in the future is hard to predict, but the teachers' positive stance on LS may influence the school and municipality in the long run.

CASE 2: LEARNING DURING LESSON STUDY REQUIRES EXTERNAL INPUT

Lyngby-Taarbæk is a municipality north of Copenhagen, with about 55000 inhabitants and nine ordinary public schools. As in other large municipalities, the central school administration employs "consultants" to support the schools' teaching in specific subjects, for instance, in relation to new national initiatives and more generally, cross-school collaboration and teachers' in-service education. The present case concerns an in-service course organised by the consultant for mathematics and science, Dr Jacob Bahn. He took up this position after completing his PhD thesis (2018) on LS as a method for implementing inquiry-based mathematics teaching. The actual experiments for the thesis were carried out at three schools in the Lyngby-Taarbæk municipality, where teachers and pre-service teachers had experimented with LS occasionally in relation to pre-service teacher education and sporadic events. In 2018 and 2019, Bahn organised two in-service courses, which were attended by about 60 of the mathematics teachers in the municipality; some attended both courses. The courses lasted for about 8 weeks, involving five 3-hour sessions and some homework for participants. The idea was to engage all teachers in one LS, focusing on a common theme, which in 2018, was "mathematical inquiry" (with examples mainly from counting problems in relation to simple geometric shapes). Teachers signed up for the courses as "teams" of at least three teachers from each school.

The courses started with classical coursework related to the theme and subsequently, teamwork to design a research lesson with some support from course teachers. Following the first two or three sessions, all teams experimented with their lesson at their own school, with one of the members teaching (usually one of their own classes) and the rest of the team observing and participating in the reflection meeting. In most cases, Bahn also participated as an observer and "knowledgeable other" (*koshi* in Japanese). After the experiments, a course session was devoted to the teams for them to give short PowerPoint presentations of their lessons and observations (often involving pictures and short video clips from the lesson experiment). At the end of this session, the participants chose which lessons to observe together in the final sessions. These sessions were organised as "open lessons"; the selected teams invited all course participants to observe, and the teams discussed their research at their schools.

We now focus on the work carried out by one of the teams selected to present an open lesson in the 2018 course; we call this team "T". The analysis is based on field notes from the research lesson, the lesson plan and student worksheets (as shown in Figure 6). The description and analysis of the case was verified by the organizer of the course and lesson study in question; only a few adjustments were needed, mainly concerning the context of the in-service course.

Most of the teachers from T had grade 3 classes and initially decided to design their research lesson for this level. They initially discussed the difficulty of having students do "mathematical inquiry" at this level, where they were still working on basic operations in addition to other basic skills, such as reading. They then selected a problem from teaching material that they had initially studied together and that one of them had previously used at this level. It concerned what the participants called "investigating the coordinate system", which is a new and difficult topic in grade 3. The problem selected was based on a figure from the textbook material, which was also used as the first handout in the lesson (Figure 5). In Figure 5, there was also a question: "How far is it from the car to the house?" In the lesson plan, the teachers formulated a more elaborate problem:

Carla's parents have bought a new house. Carla's mother says that Carla must find her way home from the car. How many routes can Carla choose (the shortest route)?

Figure 5. First handout (copy from Teglskov & Kristensen, 2012, p. 50).

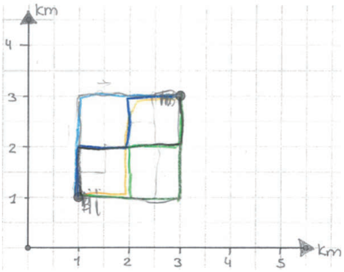

Figure 6. Coordinate system handout with student writing in colour.

Figure 7. Whiteboard writing, in part by a group of students (arrows also present in their personal writing).

In the open lesson observed, the teacher first devolved the question in Figure 5, and after about 10 minutes, the students' answers were formulated and validated in a short whole-class session. Here, the teachers only talked about the results and methods while pointing at a smartboard showing Figure 5 (and no other visual support). Some students made measurements (with rulers) along the line segment from the car to the house; the need for the car to stay on the road and hence, to measure distances along these was institutionalized by the teacher, based on formulations by students. With this convention, students agreed that the distance was 4 km. Then, the teacher devolved the question from the lesson plan (quoted above) and distributed the handout shown in Figure 6. The students were seated in groups but got individual handouts; some students worked together but still drew on their own handouts. The teacher realised that some students considered all grid lines in Figure 6 to be roads and interrupted the group work to explain how the roads appeared in the coordinate system using a whiteboard as shown in Figure 7. The last five minutes of the lesson was a whole-class episode during which the whiteboard was used again (see Figure 7; the arrows were added during the episode). During this episode, coloured routes were drawn, and proposals that were not 4 km long were rejected but not drawn; the group who invented the arrow notation explained their reasoning while the teacher summarized on the whiteboard (to the right). The fact that there were six routes was not questioned or validated, but based on the handout and observations, only the group that used the arrow notation had found as many as six, while others found three or four. The lesson ended with the teacher reiterating the explanation of the six routes (Figure 7).

During the reflection session, teachers provided many interesting observations. Some pertained to the handouts, which contained some confusing aspects for students (in Figure 4, the car looked 1 km long, the unclear location of the car, etc.; in the second handout, the grid lines could be confused with roads, as seen in Figure 6). Others questioned whether the two problems were pertinent to inquiry as they had unique answers. Many comments pointed out that the teacher did not manage to guide all 29 students or ensure they got feedback on their work. Some questioned the clarity of the tasks, including the rules to follow. For instance, students drew routes that were longer than 4 km, and some even talked about "finding as many routes as possible".

Dr Bahn pointed out that important points of the two tasks were not made clear to the students: most students had measured the "bird fly distance", and the reason given for the rejection of this was just a rule ("cars run on roads"). Similarly, whether the result (six) for the last task was actually correct, was not discussed or explained, although reasoning is an important part of inquiry. The knowledgeable other pointed out the richness of the last task and the need for systematic reasoning to get more than "some examples of routes", as many students did. He focused on the "arrow notation" invented by one student group and explained how it can be developed into a systematic way of keeping track of different routes even for the generalized problem.

The episode illustrated several more general challenges with LS that were also noted by Bahn (2018). Teachers often deviated substantially from the lesson plan in actual lessons, for instance, by changing the way that the main problem was explained to the students (here, in two steps instead of one, as the plan suggests) and by adding explanations in the middle of a period of problem solving. While T had foreseen (and in the first experiment, observed) the tendency of students to measure distances diagonally, they developed no ideas to deal with it other than institutionalizing the "taxicab" distance as correct. Finally, the last counting task, which was rich in potential for inquiry, was not thoroughly analysed by T. In the lesson plan, the solution ("six") was not even mentioned or justified, and we found no similarities to the earlier experiment in terms of possible student solutions. All of these shortcomings are of course important to notice in order to improve the teachers' experience with LS and their mathematical and didactical knowledge.

In summary, in this case, the school and the municipality, along with the (national) university were major factors in enabling and sustaining the LS activity. The course organised by the municipality was an element of PID that could (if pursued over several years) make LS viable as an activity, assuming strong coordination with the school management and input from external individuals educated at the postgraduate level in practice-related didactics and mathematics. We have reason to believe that the long-term effort with LS in the municipality has considerably affected a number of leading teachers but analysing individual teachers' experiences is outside of the scope of the current paper.

DISCUSSION AND CONCLUSION

Both cases demonstrate the decisive role of the municipality and school in PID in terms of initiating, modifying or even cancelling LS activities. Takahashi and McDougal (2017) made a similar observation in the U.S.: to ensure sustainability, "enthusiasm for lesson study" must be clearly established at the school level; purely organisational support is not enough. It is crucial for the decision makers to be informed about the long-term goals and functions of LS in countries, such as Japan, where such formats are a regular part of teacher development. While LS does not have a strict format, almost all of these limited experiences of Danish teachers were attained in collaboration with other teachers who were equally inexperienced with the format and with educators or researchers who usually also had very limited experience (occasionally, including some observations in Japan). It is also worth noticing that Danish teachers are novices in developing lesson plans and do not have the support of the Japanese mathematics education literature on LS. So, as in other countries where the format is new, "learning to do LS" in Denmark is a considerable challenge.

We propose that sustainability should not merely be taken to mean an autonomous LS entity where teachers, after intensive support with professional development and new resources, can continue without external initiatives. The cases we have examined suggest, rather, the importance of external academic support from entities, such as the "knowledgeable other", as well as organisational support from the management at the local school and consultants from the municipality. This vision of sustainability is also consistent with other experiences and positions expressed in the research literature. Clivaz and Takahashi (2018, p. 161) note that "Lesson study cannot thrive in a vacuum; the greater its support, the greater its impact will be". In our case, we consider that PID must provide more than substantial conditions and knowledge on "learning to do LS" and must provide professional nourishment and possibilities to share knowledge. This suggests an inclusive expansion of PID, more substantial than what is suggested by, for example, Akiba and Wilkinson (2016) in the form of prepackaged LS kits and LS steps.

Yet, LS is not an end in itself; its *raison d'être* in countries like Japan is to develop teachers' professional knowledge over the duration of their careers and at the same time, make this knowledge shared among members of the professional community (cf. Stigler & Hiebert, 1999). Although there have been reports on (usually and broadly speaking) "novice LS experiences" as successful and beneficial for teachers, such limited experiences with LS will at best, show the teachers how the activity works. With no infrastructure to engage in LS afterwards, such endeavours are as useless as a train with no tracks.

The paradidactic infrastructure can, thus, not be reduced to punctual opportunities to engage in a LS. The learning gained from LS depends on many factors, and our limited experiences and most studies on similar endeavours in other countries suggest that more or less seasoned "guides" of the activity are needed in order to create a meaningful, professional experience. In Japan, the "knowledgeable others", who are crucial participants in any LS, often come from the profession itself (Takahashi, 2014). As expert teachers who lead and guide the planning of lessons, they have simply learned the craft from participating in LS with more experienced peers. It may look like the hen and the egg, but even in Japan, LS grew out of the experience of learning from external experts (Isoda, 2007). The two cases show how the inclusion of such experience also make a difference to the outcome in the Danish context. The development of PID to make LS an important and useful format for professional development will thus require a focus not only on the creation of the time and place for teachers to engage in LS but also on the training of a number of "experts" who can contribute to guiding the activity and who can act as knowledgeable others in reflection meetings.

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