ATTAINING MATHEMATICAL INSIGHT DURING A FLOW STATE: WAS THERE SCAFFOLDING?

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Abstract

This post-lesson video-stimulated interview study of the creative development of mathematical insight by a Year 6 student—during group activity in class—examines whether scaffolding occurred, and if so, what was its nature? This capable mathematics student who generally disliked school mathematics, became intensely engaged both intellectually and affectively during a problem-solving task in the research period. The task, its implementation, group interactions, and teacher actions all influenced, but did not cause his insight development. Scaffolding actions that differed in nature to those identified by Bruner did occur. This study could inform researchers, teachers, and professional learning providers intending to deepen mathematical understandings and increase student interest in mathematics.

Key words: social interactions, spontaneity, high positive affect, creative mathematical activity, flow, problem solving

INTRODUCTION

The value of problem solving for enabling deep learning of mathematics has long been recognised (Schoenfeld, 1992) as has the value of mathematical discourse in whole class and group activity (Smit, van Eerde, & Bakker, 2013; Wegerif, 2006), and mathematical thinking that develops (Hino, 2007). Curricula in the USA (Kaur & Yeap, 2009; Kilpatrick, 2002), China (Dello-Iacovo, 2009), Korea (Kwon & Cho, 2012), and Australia (Masters, 2013) for example, recognise the value of problem solving. The high quality of Japanese Lesson Study (JLS) as an approach that leads to deep mathematics learning has long been recognised (Stigler & Hiebert, 1997). The Engaged to Learn Approach (E2L) employed in this study has various features, and outcomes, consistent with JLS but has other features and outcomes that are not. In order to familiarise the reader with E2L which will be less familiar to most readers than JLS, similarities and differences between the development and purposes of these approaches to classroom mathematics learning are now discussed. Features of each approach are compared and contrasted in the Research Context and Discussion sections when elaborating the nature of E2L This study examines the scaffolding of *creative* mathematical thinking that is accompanied by high positive affect.

Japanese Lesson Study

For over six decades, educators in Japan have focused on ‘mathematical thinking’ which is considered different to problem solving (Hino, 2007). Koyama (2019) illustrated mathematical thinking during whole class activity, where a teacher, probed student ideas on how a triangle is defined, and used pre-prepared counterexamples to "'shake the students' recognition of a triangle" and to "deepen their understanding of the definition" (p. 43). Mathematical thinking is "considered relevant in creating and forming mathematical concepts, rules and algorithms" (Hino, 2007, p. 504). Problem solving on the other hand was considered appropriate for consolidation after rules and concepts had been developed. Open-ended questions, implemented before formal instruction, were more frequently employed as enrichment activities. They were found to "… arouse students' interest and foster … mathematical views and thinking" (Hino, 2007, p. 508).

JLS is an approach developed to support teachers experiencing difficulties implementing open-ended questions (Hino, 2007). It includes teacher participation in professional conversations, and resource development associated with developing the mathematical thinking of students (Hino, 2007). Open. ended tasks undertaken during JLS are observed by other teachers in the Lesson Study team. These team members contribute ideas during the development of the lesson plan. Shimizu (2009 in Funahashi & Hino, 2014, p. 424) emphasised that "valuing students' thinking as necessary elements to be incorporated into the development of a lesson is a key to the approach taken by Japanese teachers." Further research of JLS is encouraged to help to overcome difficulties Japanese teachers still experience in implementing open ended tasks (Funahashi & Hino, 2014; Koyama, 2019). The study reported herein could also have implications for teachers wanting to further develop students' mathematical thinking including the subset of *creative* mathematical thinking because it examines the nature of interactive processes associated student controlled mathematical thinking during students' creative development of mathematical insight accompanied by high positive affect, which occurs in a 'flow' state (Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002a) during mathematical problem. See for example Barnes (2000) and Williams (2002a, 2010) for illustrations of students who entered a flow state while developing mathematical insight within E2L (initially termed Class Collaboration).

Engaged to Learn Approach (E2L)

Williams, initially began to develop her group problem solving approach in her mathematics classes in the early to mid-nineteen eighties, to provide more opportunities for students, who struggled with mathematics, to engage with the subject (Williams, 2002c). It has since been taken up by various schools and individual teachers in various forms (e.g., Clarke, Duncan, & Williams, 2014). Williams found that E2L engaged not only students who struggled with mathematics but also other students, including highly capable students who wanted to understand mathematics deeply rather than only learn rules and procedures (Williams, 2000). Different to JLS where there is one key purpose, to develop students' mathematical thinking, E2L, has multiple, equally valued intentions. These are to increase inclusivity through an emotionally safe learning environment, to enable students to develop what were initially unfamiliar mathematical ideas, and to develop positive affect around the learning of mathematics (Williams, 2014).

Similar to Hino's (2007) descriptions of open-ended tasks employed before formal teaching in JLS, E2L employs accessible but complex unfamiliar problem-solving tasks, that are implemented prior to formal introduction of a topic. Different to JLS, the teacher in E2L does not provide *mathematical* input *during* task exploration. Mathematical input by the teacher occurs when student work on the problem-solving task has finished. The teacher then draws together, elaborates on, and extends the ideas of the students as needed to formally develop the topic. This enables student autonomy and spontaneity in developing mathematical ideas—crucial to enabling flow (Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002a).

High level positive affect accompanying mathematical thinking during problem solving and modeling (e.g., Galbraith, Stillman, & Brown, 2010; Williams, 2010) also has additional benefits. It has been found to build optimism (resilience) (Seligman, 1995; Williams, 2010) over time and optimism has been found to increase problem solving capacity (Stillman et al., 2009; Williams, 2010), and buffer against depression (Seligman, 1995). Recent research from Japan (Inoue, Asada, Maeda, & Nakamura, 2019) that studied the actions of eight teachers identified as highly effective teachers of inquiry based learning found that these teachers made decisions based not only on developing the mathematical thinking of students, but also on " 'students' whole person development and creat[ing] a collaborative and inclusive learning community" (p. 376). This points to the usefulness of the study reported herein for further reflections on JLS. Mayer's (1978, in Csikszentmihalyi, 1992) finding that student enjoyment of a course was a better predictor of final grades than previous scholastic achievement or aptitude supports the usefulness of the findings reported herein for mathematics teacher development more generally. This study, with its microanalysis of classroom and group interactions during the creative mathematical thinking of one Year 6 elementary school student provides insights into some of the types of social interactions that can enable such processes.

THEORETICALLY FRAMING THIS STUDY

Constructs in this section provide the theoretical lenses through which the data is analysed in this study: flow, creative mathematical thinking, Bruner's (1986) conceptualisation of scaffolding.

Flow

Flow is a state in which creative activity is accompanied by high positive affect (Csikszentmihalyi, 1992). It can occur when a group or individual spontaneously set themselves a challenge that is almost out of reach and develop new skills (Csikszentmihalyi, 1992) and concepts (Williams, 2002a) to overcome it. It occurs in many domains (e.g., art, sport, weaving) including where groups of Japanese Bosozoku bike riders develop new skills to escape the police as they terrorise the town each evening (Sato, 1992). Conditions for flow include autonomous setting of a challenge almost out of reach that requires the development of new skills to overcome it. During flow people lose of all sense of time, self, and the world around as all their energies are focused on the task at hand.

Flow can occur during the creative development of insight during mathematical problem solving, and in that case, both skills *and* concepts develop (Williams, 2002a). Flow conditions specific to mathematical problem solving include students:

- Discovering an unfamiliar mathematical complexity
- Inclining to explore it
- Spontaneously formulating a question to focus that exploration (a challenge to overcome)
- Developing new skills and concepts to answer that question (Williams, 2007).

Thought processes associated with creative mathematical activity have been described in different but consistent ways (Krutetskii, 1968/76, p. 292; Chick, 1998, p. 17; Csikszentmihalyi, 1997, p. 65 in Williams, 2007, p. 70). Creative mathematical thinking involves “not only choosing the cues and concepts — and often unexpected cues and concepts — but even the very question” (Chick), and is “not so much direct attempts at solving the problem as a means of thoroughly investigating it, with auxiliary information being extracted from each trial” (Krutetskii). There is a cumulative effect of small discoveries: “You may have only one big insight, but as you try to elaborate, as you try to explain what the insight is, you have small insights coming up all the time too” (Csikszentmihaly).

The term ‘high positive affect’ as employed in this study has been developed from a combination of Csikszentmihalyi’s (1992) properties of flow, and Quilliam’s (1995) body language indicators of several properties. Table 1 displays the properties of flow, indicators of these, elaboration of each, and the codes used to identify each.

Table 1 Body Language/ Voice Articulation Framework: Indicators of high positive affect developed from Csikszentmihaly (1992) and Quilliam (1995)

Property	Indicators	Elaboration	Code
Lose all sense of time, self, and world around	Unaware of anything other than task at hand	Does not hear interruptions	U Unaware
Energies focused on task at hand	Eyes on task and / or group.		Ey Eyes
	Body directed towards task activity.		D Direction
	Participates verbally / in writing / listening		P Participate
Creative ideas developing	Connect to ideas of others	Finish their sentence Extend their idea	L Latch
	Insights develop	Exclamation of excitement, surprise or pleasure on Emphasis words	Ex Exclaim

Positive affect can exist where few of these indicators in Table 1 are visible. What the **EyDUPLEx** Body Language including Voice Articulation Framework does offer is evidence of some but not necessarily all instances where positive affect occurred. Changes in the indicators displayed can raise awareness of changes in affective state.

Creative mathematical thinking

Creative mathematical activity, for the purposes of this study, is autonomous, spontaneous mathematical activity that may be influenced but is not caused by sources external to those exploring. Spontaneity which has previously been linked to student learning, is not considered to occur in the “absence of elements with which the student interacts” but rather to “refer to the non-causality of teaching actions … the self-regulation of the students when interacting” within “the student’s frame of reference” (Steffe & Thompson, 2000, p. 291).

Actions of Vygotsky’s ‘expert other’ eliminate spontaneity of student thinking with actions such as ‘telling’ and ‘hinting’. Such activity is not within the student’s frame of reference (the student does not select the focus, the mathematics to use, how to use it, and whether the mathematics generated is reasonable) but is rather undertaking learning the expert other is controlling.

E2L was developed to enable autonomous, spontaneous student activity to provide frequent opportunities for creative mathematical thinking during flow. JLS, on the other hand, with the inclusion of explicit teacher guiding and teacher linking of mathematical ideas for the purposes of eliciting students' mathematical thinking limits opportunities for spontaneity, thus *creative* mathematical thinking during these times. Where JLS develops mathematical reasoning skills during these teacher guided episodes, in E2L, the teacher draws attention to such skills as the students communicate their thinking to the class.

For the purposes of this study, processes associated with developing 'mathematical insight' are examined through the lens of Abstracting in Context (AiC) (Dreyfus, Hershkowitz, & Schwarz, 2001). AiC involves the process of constructing something mathematically profound that was not apparent previously. Williams (2007) integrated (Krutetskii, 1976) 'mental activities' with the 'observable cognitive elements' of AiC (Recognizing **(R)**, Building-with **(BW)**, and Constructing **(C)**) to examine complex student thinking. She operationalized the spontaneity of these observable cognitive elements through the sources of the social elements (Control, Explanation, Elaboration, Query, Agreement, and Attention) (Dreyfus et al., 2001) that influenced them. Where social elements that include new mathematical ideas (e.g., explanation, elaboration, query, and affirmation) are 'internal' (in this case sourced from individual student), the observable cognitive elements are spontaneous thus the mathematical thinking is creative (Williams, 2007). Spontaneous 'observable cognitive elements' are described as follows:

Spontaneous recognizing: student identified mathematics to use, or student identified mathematics within a context.

Spontaneous building-with: student employing known mathematics in new combinations / sequences (Dreyfus et al., 2001): simple analyse (breaking into component parts to examine); synthetic-analysis (simultaneous analyses of two of more representations, methods, and / or features); evaluative-analysis (judgment arising from synthetic-analysis) (Krutetskii, 1976; Williams, 2007).

Spontaneous constructing: student synthesis (insight) and/or recognizing usefulness of an insight for another purpose (Dreyfus et al., 2001; Krutetskii, 1976).

Scaffolding

The term 'scaffolding' as employed in the building industry is a temporary structure built on the outside of a building to support the builders as they climb higher (Fernández, Wegerif, Mercer, & Rojas-Drummond, 2001). That structure is not part of the building constructed so can be removed when building is completed. In education including mathematics education scaffolding is a metaphor for support to learning that enables the "child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (Wood, Bruner, & Ross, 1976, p. 90).

In this paper, in defining scaffolding of creative mathematical activity, both the common language meaning of scaffolding and D Wood, Bruner, and Ross' definition are combined as follows:

> "scaffolding creative mathematical thinking is enabling the student to develop mathematical insights that would be beyond his unassisted effort, where the scaffold must be such that when removed the mathematical construction remains standing: mathematical input is not provided".

Bruner (1986) examined the role of the 'expert other' (Vygotsky, 1978) who "demonstrates the task is possible", instructs, shows, starts the task for the child, hints, provides ideas, and "asks leading questions" (van der Veer & Valsiner, 1994, p. 337) when an expert tutor supported a child who was building a required block construction. Bruner also identified additional scaffolding actions, the tutor: "controlled the focus of attention", "segmented [the] task to control the size and complexity of task", and "set up the task so the child could recognize the solution and perform it later (even though he could neither use nor understand at time)" (Bruner, 1986, p. 75). This study will examine scaffolding through the lenses of the expert other and Bruner's additional scaffolding actions to find whether, and if so how, these scaffolding actions relate to a students' creative development of mathematical insight during flow. The analyses do not specifically focus on more recently identified scaffolding strategies such as: a) interactions with an individual, a group, a teacher, or the whole class (Smit et al., 2013); b) teacher actions within class and between classes (Smit et al., 2013); c) teacher encouraging of dialogue that increases the depth of student-student interactions (Wegerif, 2006); and d) teacher sequential processes such as diagnosis, responsiveness, and handover to independence (Smit et al., 2013). That said, the microanalyses undertaken are associated with such contexts.

The Research Questions

Given the potential usefulness of promoting creative mathematical activity with the accompanying high positive affect (Csikszentmihalyi & Csikszentmihalyi, 1992; Williams, 2002a) it is important to identify teacher actions that increase the frequency of such activity. For this purpose, this study identifies a case in which a student appears to attain a flow state and examines his interactions, affective state, and evidence of his thought processes to find whether there were actions that scaffolded his insight development. The research questions posed are: "Did this student experience a flow state?" "Were there scaffolding actions associated with his creative mathematical activity?" and if so, "What was the nature of these scaffolding actions?"

RESEARCH METHODOLOGY

This data is part of the final year of a three year longitudinal study of the role of optimism in group problem solving (see, Williams, 2014) and contributes to that research by examining other influences on mathematical problem solving activity. The Research Design is an extension of the Learner's Perspective Study Design (Clarke, 2002).

Context

This section includes descriptions of the school, the class, the subjects under focus, the task, and the pedagogical approach.

School and subjects

This elementary school class was selected because twelve of the 24 students had not previously participated in the first two years of the broader research study. It was considered that more explicit teacher actions would occur to introduce these ‘new’ students to the classroom culture. The researcher-as-teacher (RT, author) was the primary implementer of the task. She team-taught with the class teacher (Mrs J) who was familiar with the E2L approach because she had participated in the research study the previous year. RT and Mrs J are referred to collectively as teachers when differentiation between them is not necessary.

The focus student Jesse’s activity was studied as he participated with Clara, and Lexie in group problem solving in a composite Year 5/6 elementary school class in a Government schools in Australia. This small group was selected for this study because there were indicators that Jesse developed mathematical insights. In addition, Jesse and Lexie usually did not engage with mathematics lessons but in this lesson, each displayed (individually) intense interest during this task. This change in orientation to mathematics for these two students during this class was expected to provide greater opportunity to examine factors that influence the development of positive affect.

Clara and Jesse (but not Lexie) had participated in the E2L approach previously. Clara was conscientious and focused in class, and during research tasks. Mrs J described Lexie as frequently engaged in off task talk rather than focused on mathematics. Jesse reported in his interview that he found ‘maths’ easy but that he was generally not interested in his ‘usual’ mathematics lessons. In discussing the How Many Boxes Task in his interview, he stated: “I wanted to listen … If there is something enjoyable in it, I’ll listen, and I wanted to find all the combinations … … I have to be interested otherwise … I sort of just … fiddle!”. Mrs J’s descriptions of Jesse during his usual mathematics classes were consistent with Jesse’s interview report. She stated that Jesse was frequently late for class, and generally fidgeted and distracted others. Jesse was on time for the second and third research lesson. We teachers considered that Clara’s conscientious approach would help to involve Jesse and Lexie in the task. In Lesson 2, an additional student, who unsuccessfully tried to distract group members, joined the group. His activity was not relevant to the reported study.

Pedagogical approach: Engaged to Learn (E2L)

The Engaged to Learn Approach has previously been found to provide opportunities for flow conditions. It is employed before topics are ‘formally’ taught, for the purpose of building informal understandings that students can build upon later. This exploratory process includes every group briefly reporting some of the ideas they develop to the class as a whole, several times during work with the task (without query or affirmation from expert others). The student / group decide what they will use, or query in a later report. Given the nature of the task and approach which includes an expectation of justification when mathematical ideas are communicated, a ‘patchwork’ of understandings are developed and presented to the class to varying extents. After the problem solving, the teacher explicitly draws on ideas from various groups in developing the topic of teacher focus (not part of this study). These tasks frequently develop initial understandings associated with several topics and the teacher or students draw on these ideas as appropriate. E2L consists of multiple cycles, with each cycle including activities that occur in the following order:

- Groups of 3-4 brainstorming [group roles (reader and timer, recorder, reporter, and encourager), intended to increase inclusivity, change after each report]
- Multiple RT questions and comments of varying levels of complexity, with no requirement for students to consider them [occur after each group work interval to stimulate the development of reports]
- Groups priming reporters [intended to increase sharing of ideas, develop group understandings, and

keep reporter emotionally safe through group accountability]

- Reports from each group [two class rules keep reporter emotionally safe: reporter cannot be asked questions beyond what was reported or contradicted while at the board. Such questions / comments can be built into subsequent reports and contradictions presented in such contexts must be justified].

There is an expectation is that teachers do not provide mathematical input *associated with the mathematical structure* the student/s are building *during* the exploratory process. The intentions of the approach are to provide an inclusive environment in which students feel emotionally safe taking risks as they explore and communicate new mathematical ideas.

Types of Tasks Employed Within E2L

E2L requires the use of 'conceptual tasks', tasks intended to provide opportunities for *students* to develop mathematical understandings that were unfamiliar to them at the start of the task. See Williams (2002b) for more detail on how such tasks are constructed to achieve this, yet are accessible to all class members. Groups are constructed by the teacher. Groups make decisions on what, and how, they will explore these tasks. Groups report to the class each 10-15 minutes for approximately two minutes. They decide what they will share with the rest of the class in their reports to the class. As previously identified, the intention is not for students to finish the task, but rather for students to become familiar with new mathematical ideas within the topic under study. Sometimes this mathematics is associated with more than one topic in the curriculum over the subsequent eighteen months.

Task: How Many Boxes

The purpose of this task is to informally build understandings associated with the internal structure of rectangular prisms composed of one cubic centimetre blocks. Students are not explicitly directed or guided towards particular mathematics.

Mathematical ideas likely to develop in some form for various groups through the task include:

- "rectangular prisms are made up of layers (or verticals slices or columns) of blocks",
- "the meaning of factors",
- "the dimensions of the prism are factors of the total number of blocks in the prism give",
- "this is so because of the structure of blocks in the prism",
- "the total number of blocks in the prism can be found by finding the number in a layer or vertical slice (by counting or by recognising the array) and multiplying by the total number of layers, or by finding the number of blocks in a column of blocks in a prism and multiplying by the number of columns".

This task extended over three eighty-minute sessions. This study focused within Lesson 1, Part 2, and Lesson 2, Part 1: "How many 'our boxes' contain exactly 24 1x1x1cm little blocks?"

Lesson 1, Part 1: Examining features to identify 'our boxes' (rectangular prisms)

Lesson 1, Part 2: How many 'our boxes' contain exactly 24 1x1x1cm little blocks?

Lesson 2, Part 1: Continue to explore "How many 'our boxes' contain exactly 24 1x1x1cm little blocks?" With greater emphasis on "How do you know when you have got them all?" "Have you found any patterns?" "Why that pattern?"

Lesson 2, Part 2: Students prepare for a game where each group tries to be the first to find the dimensions and orientation of the 'our box' hidden in a colourful box. They will be told the number of blocks in the 'our box' and each group can ask a question that has a Yes or No answer to help find the 'our box'. Students have 24 little blocks and can explore different 'our boxes' (with more or less than 24 blocks) and questions that might be useful to ask.

Lesson 3: Students play the game then play other versions of it. For more information on different classes investigating this problem see for example, Williams (2007, 2014).

Soon after group work commenced, RT drew attention to ways of working and several possible areas of focus: "you are working out how to record what you find [so you can use the blocks] to make another one if you think there's more that could be made". "[Y]ou are working together" and "you are thinking about- what is the thinking you are doing to work this out because that's what we're really interested in- how you are working- what helps you to do it?"

As the group work progressed, teachers asked questions of individuals, groups, or the class to elicit more complex thinking. Questions included: "How many can you make?", "How do you know when you have got them all?", "Can you make a mathematical argument for how you know when you have got them all?", "Can you see any patterns?", "Why does the pattern you have found work?"

This task, like other tasks in the broader study, was accessible at multiple levels of mathematical sophistication through multiple representations (e.g., concrete, diagrams, numerical, verbal, mathematical arguments). The task was designed with opportunities for surprise which were expected to increase student inclination to explore (create flow conditions).

Research design

Research timeline

During Lesson 1, Part 2 and Lesson 2, Part 1 (How many 'our boxes' contain exactly 24 1x1x1cm little blocks?), two problem-solving cycles occurred over 2 consecutive lessons. This study focuses primarily on five minutes in Lesson 1 (four short intervals of Jesse's interactions with his group and his exploratory activity), and Jesse's report to the class and teachers' comments about it in Lesson 2, Part 1.

Data collection instruments

Four video cameras with audio leads to each group captured the activity of each group, and group reports to the class. In addition, Jesse's group's microphone on their desk was still active during the reporting sessions, and teacher talk with the class as a whole. Excerpts of lesson video and Jesse's video-stimulated post-lesson interview after Lesson 2 captured Jesse's exploratory activity and influences on it. Teachers' activity was captured on different cameras and microphones as they moved from group to group. Student interactions, gestures, other body language, voice articulation, and the content of their reports assisted in identifying intervals of creative thinking, and insights developed. This was achieved through attention to: 'observable cognitive elements', 'social elements' influencing them, and accompanying indicators of affect (displayed in group interactions, reports, and student interviews). Video-stimulated post-lesson discussions with Mrs J provided additional information.

Process of analysis

The four intervals identified in Lesson 1 were examined to identify cognitive, social, and affective elements. The three more complex intervals were used to produce Interpretive Visual Displays (see Figures 1, 3, 4) to make 'visible' Jesse's thinking, its spontaneity, and teacher and student influences on it. These displays captured Jesse's, Clara's and Lexie's activity over time. The five columns in the Interpretive Visual Displays show transcript line numbers [Column 1], times from start of lesson [Column 2], Jesse's 'observable cognitive elements' (Recognizing, Building-with, Constructing) [Columns 3c, 3b, 3a from left to right], social elements Jesse responded to (from Lexie, Jesse, Clara, Teachers) [Columns 4a, 4b, 4c, 4d]. Influences on Jesse's activity can be identified from the arrows pointing back from Jesse's activity to what he responded to [Column 4]. Evidence of positive affect were identified through group interactions self-reports, body language, and voice articulation using Williams' Framework of Quality of Experience (EyDUPLEx, Williams, 2003), see Column 5. This framework is based on properties of flow (Csikszentmihalyi & Csikszentmihalyi, 1992) where people become so engaged that they lose all sense of time, self, and the world around because all energy is focused on the task at hand. Column 5 (5a, 5b, 5c, 5d, 5e) display indicators of Jesse's quality of experience through 'eyes on task' (Ey), 'body directed towards task / group member/s' (D), 'unaware of world around' (U), 'participating' (P), 'latching to talk of others' or 'cutting across interviewer' with an urgency to respond (L), and intensity and exclamations articulated (Ex). Column 6 contains group transcripts and other talk the group might have heard. Complex cognitive activity occurring simultaneously with many indicators of high-quality experience can be used to identify possible flow states.

Jesse had access to blocks, lesson video, and worksheets, as salient stimuli to add validity to his interview reconstructions (Nisbett & Wilson, 1977). He controlled video selection and discussed what he thought was happening, and what he was thinking and feeling. Probing questions asked by the interviewer (RT) included: "Can you tell me what you were thinking about there?" "You appear to be really thinking there. Is that right?" These probes generated data about the mathematical content of Jesse's thinking and his affective responses as he developed ideas.

ANALYSIS AND RESULTS

Narrative: Jesse's activity

Jesse's group activity in Lesson 1

Jesse, Clara, and Lexie used trial and error to make an 'our box' (rectangular prism) with two layers. Clara and Lexie tried unsuccessfully to find the number of 'blocks' in this 3x4x2 cm 'our box' (notation from now on: 3x4x2 box). Jesse asked: "How many on one face?" leant over, inspected, soundlessly moved his lips as though counting before confidently stating "24". Lexie organised the group to commence: "Okay so … we start out with a cube" and Jesse enthusiastically offered to make one: "… I can make a cube" [see Figure 1, Lines 90, 91, Column 6 (notation to be used: Fig. 1, L90, 91, C6)]. Jesse built a 2x2x2 cube then began to add blocks one by one to this [Fig. 1, L96, 98; C6]. He was unaware of activity around him [Fig. 1, element of C5: Unaware (U)] with for example, Clara's comment [Fig. 1, L95, C6] and Mrs J's presence and question [Fig. 1, L97-99, C6]. Jesse focused only on his own exploratory activity [see black arrows Fig. 1, L93-105, C4]. His tentative statement at the end of this interval showed he wondered whether his goal was possible: "You can't really make an actual cube [with 24 blocks]". His decision to persevere indicated he was still not sure: "[to himself] there must be a way to make a cube" [Fig. 1. C6, L105].

Clara disagreed with Jesse's statement that a cube could not be made: "Oh no I think you can" [Fig. 3, L106]. During the approximately 2 minutes that the class remained silent as RT suggested various possible foci for group reports [see Table 2, Column 1; Fig. 3, L107-113], Clara placed her 3x3x3 (27 block) cube in front of Jesse. He examined it, counted various external features, attended only to the part of RT's talk that fitted his focus [see Table 2, Column 1, Item 3]:

> … some people have been able to say- that one- that one works because it's so many [blocks]- how are you doing that? How do you know how many are there? How are you working it out?

After listening to his self-selected excerpt of RT's talk [see Table 2; Fig. 3. L113, C6], Jesse removed and replaced several face layers of Clara's cube (one by one). He inspected and counted visible internal elements [Fig. 2a]. When RT finished, Jesse immediately stated: "*Clara*; there's *too many pieces* there. that's 26" [Fig. 3. L116], He displayed increased positive affect at the times when he exclaimed [Fig. 3. L116, 129-130, C5]: "you *can't make it a cube*".

Jesse continued to manipulate his own blocks while responding to Lexie who asked for assistance in making a 24-block cube: "You need more pieces to make a cube" [Fig. 4, L131]. Jesse appeared unaware of interactions between Lexie and RT [Fig. 4, L133, 134, 142, 143] when Lexie asked RT for affirmation and was directed back to the task requirements instead.

Jesse and Clara disagreed over whether there were 26 (Jesse) or 27 (Clara) blocks in Clara's cube [Fig. 4, L137-139, 144-147]. Clara made another one by one count of the blocks in the cube [see Fig. 2b; Fig. 4, L144]. Jesse interrupted: "What's three nines?". When Clara answered "27" Jesse exclaimed "oh yeah! I missed the middle one". He had developed insight into the internal structure of this cube [Fig. 4. L144-147]. He could see it was made up of layers. It was unclear how he knew the number of little blocks in a particular layer at that time. Did he recognise an array and the relevance of multiplication, or did he count them or remember his previous counting?

<u>Jesse generalised his insight in Lesson 2.</u>

Interpretive Visual Displays have not been constructed for Lesson 2 Part 1 because only small excerpts have been selected to provide evidence of different investigations different students and groups undertook, and of Jess's generalisation.

There was evidence that Jesse was more expert than his two group members with regard to cube structure by Lesson 2. Evidence included the nature of Lexie's frequent questions to other group members, and the different ways Jesse and Clara analysed the 27-block cube (Clara deconstructing it unsystematically [Figure 2b] whereas Jesse was aware of layers). Lexie and Clara individually, searched for patterns related to rectangular prisms made from 24 blocks, probably focused by RT's frequently made comment [e.g., Table 2, Row 8]: "are you seeing any patterns …?" Jesse, on the other hand, worked individually, sometimes interacting with other group members as an expert other.

Jesse's report to the class 47 minutes and 48 seconds [47:48] into Lesson 2 and his post-Lesson 2 interview showed he had developed further mathematical ideas. These included Recognizing factors were involved, beginning to generalise the internal structure of the cubes to cubes with different numbers of blocks, and also beginning to generalise from the internal structure of cubes to the internal structure of rectangular prisms. Jesse stated in his report to the class:

> … there's only … a few ways that you can make a cube- because you can make a cube with 27 blocks but not 24 … you have to find the factors of- if you find the factors of 24 you can … make the blocks [boxes] … [it] is either eight by three or six by four- it can be … either way … [number in a layer by number of layers]

To elicit clarification Mrs J reflected Jesse's words back to him: "you did mention something in there Jesse something about you can't make a cube out/" [Lesson 2, 51:04 sec]. Jesse cut off Mrs J's talk in his eagerness to respond: "Yeah you can't make a cube out of 24 blocks if you *have* to use 24" [51:12]. To elicit further complexity in the thinking Jesse had been doing, without requiring it, Mrs J posed her question in a non-confrontational way: "Are you able to explain that a little bit more?" [51:19]. Jesse gave a little more detail about his thinking: "… there's two ways of making a cube - well there's more but- you can make bigger cubes" [51:20].

Reading Interpretive Visual Displays
Figures 1, 3, 4 show three consecutive time intervals in Lesson 1 Part 2 [0:57:45-1:02:15]—4 minutes and 30 seconds in which Jesse's initial creative constructing of insight occurred. Each row of these figures links cognitive [Column 3], social [Column 4], and affective elements [Column 5] of Jesse's activity for small time intervals in the lesson [Column 2]. Column 6 contains transcript of Jesse's group interactions and other relevant class talk. By inspecting Figures 1, 3, and 4 simultaneously it can be seen that increased complexity of cognitive activity [Column 3] tended to occur simultaneously with more indicators of positive affect [Columns 5]. Column 4 in each of Figures 1,3, and 4 show social influences on Jesse's activity. Where the black arrows point back to Jesse's previous activity, Jesse was influenced by his own previous activity. Where arrows from Jesse's activity point back to the activity of others, Jesse's activity was influenced by other sources.

Transcription notation
In these displays is as follows: '…' omitted transcript does not alter meaning; '-' change in direction of ideas; '[text]' text added by researcher to include other relevant information (e.g., clarify, include gestures); '//': parallel talk, one person begins to talk before other finishes '[58:55-1:08:04]' time intervals in hours, minutes, and seconds or minutes, and seconds.

Interpreting Figure 2a
Figure 2a represents activity Jesse undertook as he inspected and examined Clara's cube. Diagrams 1-4 within Figure 2a illustrate four successive actions: 1) Jesse inspected the cube; 2) Jesse removed the top layer and inspected and counted visible aspects of internal parts of the cube; 3) Jesse replaced top layer; and 4) Jesse removed left hand side layer of cube and inspected and counted internal features. Jesse continued this activity with other external layers. When RT finished talking to the class, Jesse immediately indicated [to Clara] that she had not made the required cube: "*Clara* there's *too many pieces* there- that's 26" [Figure 3, L116].

No	Time	Cognitive *Jesse*			Social *Jesse*				*Jesse's* Body Language						Transcript
		C	B	R	L	J	C	T	Ey	D	U	P	L	Ex	
87	57:45					o			✔	✔		✔			Jesse: here give it to me [leans over, tries to take constructed box from Clara who draws it back towards herself again]
88	57:50				o				✔	✔					Lexie: [to Jesse] That's a rectangular prism though
89	57:53						o		✔	✔					Clara: I think that is what we are meant to make
90	57:57				o				✔	✔					Lexie: Okay so we can do- we start out with a cube [takes paper provided]
91	58:00					o			✔	✔		✔			Jesse: I can make a cube [leans over his pile of blocks and begins to make a 2x2x1 layer] // [looks at Clara, states more loudly] *I can make a cube*
92	58:05						o		✔	✔		✔			Clara: // Okay we can try to make one [begins to build with her own blocks]
93	58:08					o			✔	✔		✔			Jesse: *Hmm* [looks very briefly at Clara who is putting together 3 blocks (as start to 3x3x1 layer). Almost immediate: starts making another 2x2x1 layer]
94	58:10					o			✔	✔		✔			Jesse: [looks up, louder] I feel like just making a cube
95 96	58:12					o	o		✔	✔	✔	✔			Clara: //[makes a 3x3x1 layer, points to blocks as counts] (…) 4 5 6 7 8 9 Jesse: //[Appears unaware of Clara's activity. Adds his second layer to the top of his first to make a 2x2x2 cube, places it on the desk]
97	58:28							o	✔	✔	✔	✔			// Mrs J: [comes to the group and leans over behind Jesse and Lexie just as Jesse places his cube on the desk] Where are we at now?
98	58:30				o	o	o		✔	✔	✔	✔			// Jesse, Lexie, Clara: [each continue to build individually without indication they hear or see Mrs J. Jesse builds onto his 2x2x2 cube block by block. Clara constructs another individual 3x3x1 layer]
99	58:39							o	✔	✔	✔	✔			Mrs J: [leaves the group without making any further comment]
100-104	58:50-59:05	?			o		o		✔	✔	✔	✔			Lexie and Clara: [discuss who will report and report content] Jesse: [continues building without evidence he has heard Lexie and Clara]
105	59:07	?				o			✔	✔		✔			Jesse: [to group firmly and loudly] You can't *really make* an *actual* cube … [to self] there *must* be a way to make a cube

Key: Cognitive [Columns 3c, 3b, 3a] Recognizing, Building-with, Constructing, [Columns 4a, 4b, 4c, 4d]. Social elements Jesse responded to from: Lexie, Jesse, Clara, Teachers, [Columns 5a, b, c, d, e]. Indicators of Jesse's positive affect: Ey: eyes on task, D: body directed towards task, U: unaware of world around, P: participating, L latching, Ex: exclamations.

Figure 1. Jesse focuses his exploration: "There must be a way to make a cube [containing 24 blocks]"

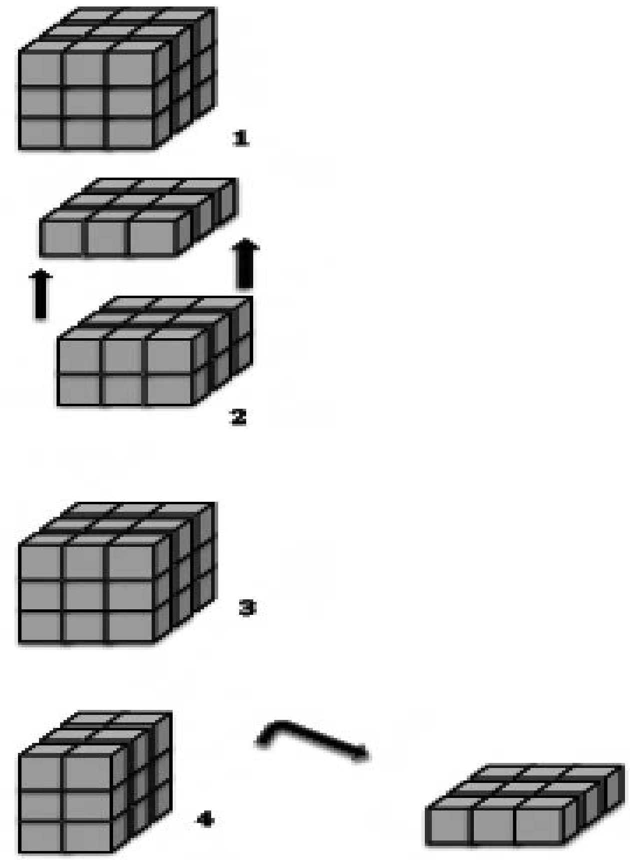

Figure 2a. Illustrations of Jesse's actions as he analysed Clara's cube

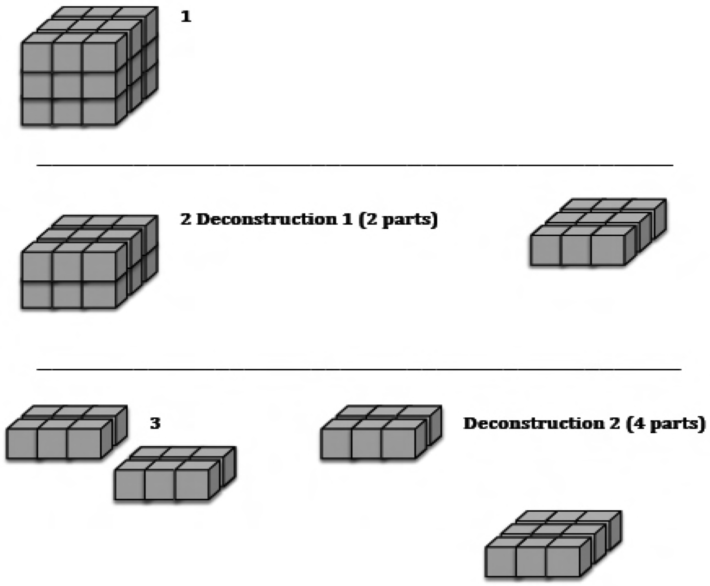

Figure 2b. How Clara deconstructed her cube to count the blocks

No	Time	Cognitive *Jesse*			Social: *Jesse*				Body Language: *Jesse*						Transcript
		C	B	R	L	J	C	T	Ey	D	U	P	L	Ex	
106	0:59:14					o	o								Clara: // Oh no I think you can ... see this is nearly a cube- watch [shows her construction which has one and the start of the second layer of 9 (3x3)]
107								o	✔	✔	✔	✔			Jesse: // [Continues to extend his own 2x2x2 cube. Does not look, nor appear to listen to Clara or the RT suggesting aspects of reports]
112	0:59:54		?				o	o	✔	✔		✔			Clara: //[places her constructed 3x3x3 cube in front of Jesse]
113	0:59:57	?				o o		o	✔	✔	✔	✔			Jesse: // [picks up cube, twists/examines, and successively pulls apart and replaces different 'face layers' (3x3x1). Twists cube, pointing one by one to blocks on a face] [RT 'talk' [59:29-1:01:00] attended to [1:00:11-1:00:25] by Jesse "some people have been able to say- … that one works because it's so many- … *How* do you know how many are there? How are you working it out?" Jesse returns to cube]
116	1:01:00	?				o			✔	✔		✔		✔	Jesse: *Clara*- there's *too many pieces* there- that's // 26
118	1:01:12						o								Clara: //[to Jesse] Okay [takes back cube from Jesse]
119	1:01:12					o			✔	✔		✔			Jesse: [picks up own 2 x 3 x 3 construction, starts adding blocks to make a cube]
121	1:01:18					o	o		✔	✔		✔			Clara: // [begins to deconstruct her cube by taking off a layer of nine and breaking the remaining 2x3x3 into three layers of six] Jesse: [attended only to his own cube except one very brief glance towards Clara]
124	1:01:13	?							✔	✔	✔	✔			Jesse: [has a 3x3x2 construction. Looks at it from several angles. Begins to build the third 3x3x1 layer a block at a time. Examines cube built, places it on the desk]
125	1:01:36	?					o				✔				Clara: [to Lexie] … so that means it's not a thingy [cube]
126 127	1:01:43	?			o				✔	✔	✔	✔			*Lexie: Naaaaa how do you do it?* Jesse: [Does not react. Remains focused on own construction].
128	1:01:47				o										Lexie [leans in towards others] … How do you make it a cube?
129 130	1:01:48	?				o	o		✔	✔		✔		✔	Jesse: //[keeps building own construction, as immediate to Lexie] you *can't make it a cube*! [from 24 blocks] Clara: // Nahh [shakes head, counts each deconstructed part] 1, 2, 3, 4, 5, 6, 7, // 8, 9, then … 9 … right … 9 [next part] 10, 11, 12, 13, 14, 15, [next part] 16, 17, 18, 19, 20, 21 [next part] 22, 23, 24, 25, 26, 27
131	1:01:53	?				o			✔	✔		✔			Jesse: [to Lexie] You need more pieces to make it a cube

Key: [Columns 3c, 3b, 3a] Recognizing, Building-with, Constructing, '?' possible Building-with/Constructing), [Columns 4a, 4b, 4c, 4d]. Social elements Jesse responded to from: Lexie, Jesse, Clara, Teachers, [Columns 5a, 5b, 5c, 5d, 5e]. Indicators of Jesse's positive affect Ey: eyes on task, D: body directed towards task, U: unaware of world around, P: participating, L latching to ideas of others, Ex: exclamations.

Figure 3. Jesse analysed Clara's 3x3x3 cube to find the number of blocks within

No	Time	Cognitive (Jesse)			Social				Body Language (Jesse)						Transcript
		C	B	R	L	J	C	RT	Ey	D	U	P	L	Ex	
133	1:01:55				o				✔	✔	✔	✔			Lexie: [to researcher/teacher] um are we meant to make a cube or this?
134 136	1:01:58							o	✔	✔	✔	✔			Researcher/teacher: [shrugs, points to board] //what are you meant to be doing? … (It's up to) you … how *you* (do it) [smiles, walks away]
137	1:02:04						o		✔	✔	✔	✔			Clara: //[to Jesse] Yeah there's 27 Jesse- that's a cube
138	1:02:05					o			✔	✔	✔	✔			Jesse: [looks up at Clara] *26*
139	1:02:06						o		✔	✔	✔	✔			Clara: 27
142	1:02:12	?			o				✔	✔	✔	✔			Lexie: [holds up another construction to RT as she walks away] //This is 24 as well
143	1:02:13	?						o	✔	✔	✔	✔			RT: [to Lexie] // Well (pause) talk in your group (…) work it out [gestures with hand to encompass whole group]
144	1:02:13	?					o		✔	✔	✔	✔			Clara: //No there's 27 Jesse 1, 2, 3, 4, 5, 6, 7, 7, 8, 9, … 10, 11, 12, 13, 14, 15, 16, //17, 18 [counting one by one each block in each deconstructed part]
145	1:02:14					o			✔	✔	✔	✔			Jesse: What's three nines?
146	1:02:15						o		✔	✔	✔	✔			Clara: Three nines are 27}
147	1:02:15					o			✔	✔	✔	✔	✔	✔	Jesse: {*Oh yeah!* I missed the middle one

Key: [Columns 3c, 3b, 3a] Recognizing, Building-with, Constructing, '?' possible Building-with/Constructing), [Columns 4a, 4b, 4c, 4d]. Social elements Jesse responded to from: Lexie, Jesse, Clara, Teachers, [Columns 5a, 5b, 5c, 5d, 5e]. Indicators of Jesse's positive affect Ey: eyes on task, D: body directed towards task, U: unaware of world around, P: participating, L latching to ideas of others, Ex: exclamations
RT: Researcher-as-Teacher
→ Arrows point to what Jesse responded to
┈→ Arrows point to what Lexie and RT responded to

Figure 4. Jesse constructs insight into internal cube structure

Table 2 Simultaneous activity of RT, Jesse, and Clara during RT's talk.

RT's Comments to Class	Group Activity Relevant to Jesse
1. "… remember your reporter is reporting about- what have you found, how did you find it-"	
2. "… what was your thinking- now what have you found? How did you think it out?"	Clara: [placed cube in front of Jesse, L112]. Jesse: [picked it up, examined it from various perspectives, pointed individually to blocks along edges of several faces, pointed one-by-one to blocks on one face, L113]
3 "I notice some people have been able to say- that one- that one works because it's so many- how are you doing that? How do you know how many are there? How are you working it out?"	Jesse: [looked up, listened to RT, L113,]
4. "… once you have got a way that you are thinking about can you explain to the others why that way of thinking is helping you?"	Jesse: [returned focus to Clara's cube, L113]
5. "… and it may be that somebody isn't sure how they are thinking it out yet"	Jesse: [pulled off and inspected a 3x3x1 face layer and replaced it, see Figure 2a, Steps 2, 3; Figure 3, L113, 1:00:32]
6. "… but they are sort of starting to see some pattern"	Jesse: removed a different 3x3x1 face layer, inspected both cross sections, replaced it, see Figure 2a, Step 4; Figure 3, L113]
7. "… now different groups will report on different things- I am sure each group is not going to report on all of them- what have you found? what sort of thinking are you doing?"	Jesse: [pulled off a different 3x3x1 face layer and inspected it quickly and replaced it, L113, 1:00:48]
8. "Are you seeing any patterns? Why do you think the way you are thinking about it is working- how are you working it out? …"	Jesse: [pointed one by one to the blocks on one face as though counting L113]

As Jesse had made 8-block and 27-block cubes in Lesson 1, his comments show he had developed some awareness that larger cubes can be made and had not yet considered a single block as a cube. How much he knew about how to form these larger cubes is not evident in the data.

Jesse's generalisation of this structure to the internal structure of rectangular prisms resulted from his synthesis of his ideas with the dynamic visual images generated by another group as they displayed vertical slices of a 24-block rectangular prism that was not a cube.

Jesse's mathematical activity

Jesse's mathematical activity is now examined to identify whether it fits with flow conditions (Csikszentmihalyi & Csikszentmihalyi, 1992) during mathematical problem-solving (Williams, 2002a). Is it autonomous, spontaneous, mathematically creative activity accompanied by high positive affect?

Autonomous and spontaneous mathematical activity

An unfamiliar mathematical complexity became apparent to Jesse when he had trouble making a cube with 24 blocks [Figure 1, L105]. The black arrows pointing backwards in the Interpretive Visual Displays Social Elements Column [Figures 1, 3, and 4: Column 4] show who Jesse responded to during Lesson 1, and when he focused only on his own activity. In most instances, Jesse's activity was in response to his own previous activity [black arrows pointing from Jesse back to Jesse's previous activity]. His response to Lexie was to try to make a cube as she suggested. This occurred before Jesse's spontaneous exploration commenced.

Jesse responded to four external social elements once his spontaneous exploration commenced [Fig. 1, L105]. See Figures 3 and 4, Column 4. These external Social Elements [SE] were:

1. Clara [Fig. 3, L106] queried Jesse's finding [Fig. 1, L105] and placed a cube in front of him [Fig. 3, L112]
2. Jesse attended only to a small part of RT's 'talk', where she wondered how some groups worked out how many blocks in their 'box' [Fig. 3, L113]
3. Clara queried Jesse's finding of 26 blocks in her cube [Fig. 4, L139]
4. Clara gave the answer to three nines when Jesse asked [Fig. 4, L146]

Neither Lexie's suggestion to make a cube (prior to Jesse's spontaneous activity), nor the four external social elements identified during Jesse's spontaneous exploratory activity in Lesson 1 contributed mathematically to Jesse's insight into cube structure. Although Lexie influenced what box Jesse tried to make, Jesse's own activity focused his spontaneous activity '… there must be a way to make a cube' [Fig. 1, L105]. Clara's incorrect assertion that a 24-block cube could be made [Fig. 3, C4, Social Element Subcategory 3: Clara], and the 27-block cube she presented to justify this led to Jesse's analysis of Clara's cube. He wanted to find the number of blocks within it, to justify that a 24-block cube could not be made. It was Jesse's decision to analyse the cube [Fig. 3, L113], influenced but not caused by Clara's activity.

There was no expectation that groups would focus on areas the RT drew attention to. Jesse's attention to one part only [Fig. 3, C4: Social Element Subcategory 2: Jesse] (how are people finding the number of blocks) was followed by his idiosyncratic internal analyses of Clara's cube [see Fig. 2a and Fig. 3, L113] where he appeared to be trying to find a way to count the number of blocks within. The decisions made were Jesse's, influenced but not caused by Clara [see Fig. 3, SE Subcategory 3: Clara, and 2: Jesse].

Although Clara queried Jesse's answer of 26 [Fig. 4, L139, C4 Subcategory 3: Clara], Jesse had achieved his purpose, he knew there were more than 24 blocks needed to make that cube. Even so, he checked his answer another way as indicated by his question to Clara "What's three nines?" Clara's answer [Fig. 4, C4, SE Subcategory 3: Clara] did not contribute new mathematical ideas. It provided Jesse with a faster answer to the multiplication than he could have achieved more slowly himself, enabling him to retain his focus on checking his answer another way. Clara was a non-expert other compared to Jesse in her understanding of cube structure, as demonstrated in her unsystematic deconstructing and counting of cube pieces [see Fig. 2b]. She did not contribute mathematically to Jesse's insight into cube structure. Thus, there was no external mathematical input to Jesse's insight development. Jesse's activity was autonomous and spontaneous.

<u>Creative mathematical activity accompanied by high positive affect</u>

As the complexity of Jesse's thinking (cognitive activity) increased over time [see Figures 1, 3, and 4], his positive affective also increased [more elements occurring simultaneously in Columns 3 and 5]. Jesse confirmed his interest in his exploration in his interview: "I got caught in the 27 blocks square [cube] because … *mmh* … it's interesting". As Jesse continued to try to make the 24-block cube without success, his engagement increased. Initially, Jesse used mathematical actions he was already aware of (trial and error, building with blocks to construct a familiar object—a cube). In doing so, his mathematical activity involved Recognizing (R), and Building-with (BW) familiar mathematical ideas. Jesse displayed engagement by participating in group activity with his eyes and his body directed towards the task [Figure 1, Column 5, L91-94: Ey, D, P]. He then also displayed evidence that he was unaware of what was happening around him [Figure 1, Column 5, L95-104: Ey, D, P, U. L95, 96: Clara's talk and building actions; L97-99: Mrs J question and observation of group; L100-104: group discussion of who will report]. He then articulated his finding with emphases "You can't *really* make an *actual* cube" and 'self-talk': "there *must* be a way to make a cube" [L105]. He displayed additional intensity of interest through emphases in his statements to the group: Jesse had started to realise that it may not be possible to make a cube. [Figure 1, L105: Ey, D, P, Ex]. The tentative nature of the ideas he expressed are indicated by inclusion of the words 'can't really', 'actual cube' and 'must be a way'. It is unclear what thought processes led to these tentative ideas. Evidence of whether Jesse had started to build an understanding of why this was so was not available. Jesse was probably beginning to undertake novel BW and may also have commenced Constructing (C). Question marks in the C cell of Column 3 in Figures 1, 3, and 4 represent such uncertainty in interpretation.

As Jesse twisted the cube Clara presented to him, and examined it, he undertook simple BW as he counted external features of the cube [Fig. 3, L113]. Novel BW occurred when he deconstructed the cube [Figures 2a; Fig. 3, L113] by taking off and replacing various faces after counting features within. This more complex BW involved simultaneously considering internal elements of the cube and numerical features of these (synthetic-analyses, see Williams (2007)). Jesse undertook even more complex BW (evaluative-analysis) when he made a judgment: "There's too many pieces there" [Fig. 4, L116]. Jesse articulated his findings with emphases each time he reached a new realisation [Fig. 4, Column 5, L116, 129: Fig. 4, L147 Ey, D, P, Ex]. Jesse slowly built his own 27-block cube (layer by layer) after this, inspecting it closely before adding the final layer [Fig. 4, L124]. As his thinking became more complex, his positive affect increased [Fig. 4, Columns 3, 5]. Jesse was unaware of what was happening around him [Fig. 4, Column 5, U] as he checked the reasonableness of his results [Figure 4, BW: evaluative-analysis], latched to the mathematical fact Clara provided, and developed his insight into cube structure [Fig. 4, C3, **C,** Constructing] [Fig. 4, L146, 147, C5: Ey, D, U, P, L, Ex]. In judging whether there were 26 or 27 blocks in the cube Jesse asked Clara: "What's three nines?" [L145], indicating he was considering three layers with nine blocks in each (novel BW). It appeared that he initially counted blocks on the outside rather than considered layers (to get 26 blocks) BW, because he exclaimed "*oh yeah*! I missed the middle one" (C) [L147]. Jesse synthesised numerical operations (counting numbers of blocks on a face, on layers, on edges) with the cube's physical structure (one in the middle) as he developed insight into the structure of the cube and why the initial answers (26, 27) differed by one [Fig. 4, L139, 144-147, C3].

DISCUSSION

Jesse experienced 'optimal learning conditions': 'flow' (Csikszentmihalyi, 1992)

Jesse's mathematical activity:

- was autonomous (Csikszentmihalyi, 1992; deCharms, 1976)
- was focused by a spontaneously set mathematical challenge almost out of reach (Csikszentmihalyi, 1992)
- involved creative development of mathematical insight (Dreyfus et al., 2001; Williams, 2002a), and

- was accompanied by high positive affect (Csikszentmihalyi, 1992; Williams, 2002a).

Jesse experienced flow—optimal learning conditions (Csikszentmihalyi & Csikszentmihalyi, 1992). The creative development of mathematical insight during flow did occur. The mathematical ideas that contributed to the insight were internally sourced. Even so, the development of this insight did not occur without external influences.

Scaffolding Actions

Scaffolding creative mathematical activity, for the purpose of this paper is defined as:

> enabling the student to develop mathematical insights that would be beyond his unassisted effort, where the scaffolding must be such that when removed the mathematical construction stands alone. Thus, mathematical input from external sources does not occur.

What Jesse knew about cube structure and was able to use by the end of Lesson 1, was not known to him before the lesson. Jesse developed these ideas without mathematical input from external sources, but this development was *not unassisted*.

<u>Scaffolding actions of external sources:</u>

1. *Non-action* [Mrs J not requiring response to question to group in Lesson 1, sustained autonomy] [Fig. 1, L97, 99, C6]
2. *Deflected* [RT directed Lexie back to task, avoided expert other role] [Figure 4, L133-136]
3. *Reflected Back* [Mrs J repeated Jesse's report comment (Lesson 2) to elicit elaboration]
4. *Elicited* [In a non-confrontational way, Mrs J asked Jesse to explain further in his report which led to further elaboration]
5. *Multiple Attentions* [without obligation to respond [RT, Table 2] to different exploratory directions]
6. *Implicit Indicating Task Is Possible* [Table 2]: "How are [people] doing that? How do you know how many are there?]
7. *Implicit Licence to Pursue Idiosyncratic Ideas* [Table 2] 'can you explain to the others why the way you are thinking about it is helping you'
8. *Asked Questions* [to elicit, sustain, and / or increase the complexity of creative mathematical thinking].
9. *Group Organising* [Lexie allocated making of the cube to Jesse]
10. *Non-expert querying / incorrectly justifying* [e.g., Clara's incorrect cube]

These scaffolding actions arise from sources external to the exploring student and do not contain mathematics. They include teacher actions that are the antithesis of actions undertaken by Vygotsky's (1978) expert other who demonstrates, instructs, shows, starts the task, hints, adds ideas, and asks leading questions (van der Veer & Valsiner, 1994, p. 337).

<u>Features of E2L that Scaffolded Creative Mathematical Activity</u>

1. *Accessible task* [various complexities to pursue, different representations to explore, opportunities for complex mathematical thinking]
2. *Reporting* [e.g., Jesse synthesises own ideas with those of another group: generalises cubic structure to rectangular prism structure more generally]

3. *Cycles of group work and reporting*: [progressive developing of more complex thinking] [e.g., Jesse's insight from Lesson 1 contributing to his generalising in Lesson 2]
4. *Disallowing querying and contradicting* of reporters [to keep them emotionally safe during reports] [evidenced by its non-appearance]

Teachers and students influenced Jesse's development of insight but did not contribute mathematically to his new ideas. Scaffolding actions set up by the teacher before the task include many features other researchers have identified as scaffolds (e.g., discourse Wegerif, 2006), whole class scaffolding (Smit et al., 2013) and group collaboration (Fernández et al., 2001)). The present study includes microanalyses that extend across various of these scaffolding structures rather than focus particularly within each (e.g., discourse, questioning, group collaboration). An illustration is scaffolding that occurred across several forums of discourse that provided multiple opportunities (that vary in nature) that contribute "… cultural tools … gradually appropriated by the pupils as cognitive means for regulating their personal mathematical activity" (Ohtani, 2007, p. 40). Jesse's generalisations developed gradually across two lessons through group discourse, teacher-whole class discourse, questioning and discourse between reporter and whole class including teachers.

Teacher scaffolding actions that elicited clarification, elaboration, and justification encouraged communication of students' decision making. These scaffolding actions were not articulated in demanding ways but rather wondering, ways that gave Jesse (and others) the opportunity to respond or not. Illustrations of such actions were reflecting *words bac*k: "you did mention something in there Jesse something about you can't make a cube out"; *asking for further explanation:* "Are you able to explain that a little bit more?" Such scaffolding sustained dialogue about ideas (Bakhtin 1986 in Wegerif 2006).

Links Between Bruner's Scaffolding and Creative Mathematical Thinking

Various of the scaffolding actions identified by Bruner (1986) became internal actions controlled by Jesse. He focused attention, controlled the degree of complexity, and in conjunction with the task nature, through his analyses made his solution visible (before he understood or recognised it), when he examined Clara's cube. He made the internal structure visible but at that time did not recognise how he could use it to quickly find the number of blocks in the cube.

The role of the student who creatively developed new mathematical ideas through E2L in this study differs to: a) the role of students learning in interaction with an expert other, and b) students learning through Japanese Lesson Study. These three different pedagogical approaches differ in the degree of control the teacher has over student mathematics learning. With a 'transmission' approach to mathematics learning, the teacher is the mathematical authority who presents the mathematics and takes the role of expert other—controlling the learning process. Teachers within Japanese Lesson Study guide the learning process. With open-ended problems they commence by eliciting student ideas. There are opportunities in this phase of the sequence for students to experience surprise as ideas are challenged, and /or pleasure when a well-argued idea is put forward. The teacher takes control of the mathematical ideas developed, drawing attention to those that fit the intended curriculum goal, making links within and across lessons, and summarising what has been found. Thus, part of student work with open-ended tasks is autonomous and part teacher controlled. Within the Engaged to Learn Approach the students are the proprietors of the classroom and the strength of the mathematical arguments put forward by the students are the mathematical authority students refer to. Although the role of the teacher could appear to an outsider to be almost non-existent, this role is complex as indicated by the subtle nature of the scaffolding actions identified.

CONCLUSIONS

Findings from this one case are insufficient for generalisation. Further research of approaches that promote creative mathematical activity during flow are required to confirm the generalisability of these scaffolding actions and identify other relevant scaffolding. This single case is sufficient though to demonstrate that scaffolding of creative mathematical activity is possible, and to illuminate the nature of various such scaffolding actions that occurred. Teacher actions that scaffolded but did not cause exploratory mathematical activity served various purposes including to: maintain autonomy [non-action, drawing multiple attentions, implicit licence to pursue idiosyncratic ideas], encourage deeper thinking [reflecting back, eliciting, questioning, implicit suggesting the task is possible, and implicit licence to pursue idiosyncratic ideas], retain emotional safety of students [non-confrontational questioning, reporting rule that reporter cannot be contradicted at the board] and enable student scaffolding [group member allocation of tasks, group member flawed challenge to ideas]. Differentiating between *internally sourced* and *externally sourced* mathematical ideas is crucial to teachers developing expertise in scaffolding creative mathematical activity.

These findings inform teacher actions, and pedagogy intended to deepen mathematical understandings whilst increasing student enjoyment of mathematics. They show close links between subtle teacher actions that sustain exploratory activity, and actions associated with implementing an accessible conceptual task in an environment which promotes students' autonomous development of mathematical ideas during flow in a safe learning environment. The enabling of students' idiosyncratic explorations that can develop mathematical ideas within, peripheral to, or beyond the mathematics expected to emerge from the task requires a rethinking of how teachers can monitor curriculum coverage. In addition, such scaffolding requires teacher faith in students' ability to think mathematically without explicit teacher guidance. Further research is required to find how to support teachers developing such expertise.

This study contributes methodologically to study of students' creative mathematical thinking in class, and social influences upon this, by illustrating a tool for simultaneously analysing these constructs to identify relationships between them. This Interpretive Visual Display played a crucial role in identifying flow situations, and in identifying external social elements that influenced but did not cause creative mathematical activity.

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Glossary of Terms

Term	Literature	Meaning
Autonomous student activity	(deCharms, 1976) (Rotter, 1966) (Csikszentmihalyi & Csikszentmihalyi, 1992)	'Autonomous' activity, is captured by the activity of "Origins" (deCharms, 1976; Rotter, 1966). Origins locate control internally while for "Pawns" control is located externally (Rotter, 1966). Autonomous mathematical activity is controlled by the student who makes decisions about what mathematics to explore, what mathematics is relevant to this exploration, how to use it, and whether or not mathematics generated is reasonable.
Expert others	(Vygotsky, 1978, p. 87)	An expert other, in working with a child, stimulates activity in a region associated with the child's "mental development prospectively" [Zone of Proximal Development] to extend student learning by for example, showing, instructing, stating the task, and hinting.
Creative mathematical thinking	(Williams, 2007)	Spontaneous abstracting, the development of mathematical insight through autiniomous and spontaneous student activity.
Flow	Csikszentmihalyi (1992)	Flow is a state in which creative activity is accompanied by high positive affect. During flow people lose of all sense of time, self, and the world around as all their energies are focused on the task at hand.
High positive affect	Csikszentmihalyi (1992)	Feelings of intensity, surprise, and pleasure during flow, indicated by the loss of all sense of time, self, and the world around as energies focus on the task at hand.
Scaffolding	(Wood et al., 1976, p. 90)	A metaphor for support to learning that enables the "child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts"
Scaffolding Creative Mathematical Activity	Williams, this publication	"scaffolding is enabling the student to develop mathematical insights that would be beyond his unassisted effort, where the scaffold must be such that when removed the mathematical construction remains standing: mathematical input is not provided".
Spontaneity	(Steffe & Thompson, 2000)	Types of actions not *caused* by an external source (even though they may be influenced by external sources). The student self-regulates when interacting with these external sources in ways consistent with the student's own frame of reference.

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