CHARACTERIZING THE QUALITY OF MATHEMATICS LESSONS IN JAPAN FROM THE NARRATIVE STRUCTURE OF THE CLASSROOM: "MATHEMATICS LESSONS INCORPORATING STUDENTS' 'QUESTIONS' AS A MAIN AXIS" AS A LEADING CASE

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Abstract

The organization and procedures with lesson study are widely understood internationally; however, how Japanese teachers have made efforts to enhance the quality of mathematics lessons requires clarification if Japan is to make an international contribution in this regard. In this study, to approach the quality of mathematics lessons, we focused on narrative structures in mathematics lessons as beneficial viewpoints for understanding the cultural background of lessons in Japan. Especially, we considered sequentiality and the dual nature of acting and the complementary characteristic of the dual landscape as useful in clarifying the origins and development of the narrative for mathematics lessons. As an example showing the efforts of teacher study groups to develop quality lessons, we introduced a collaborative project "mathematics lessons incorporating students' 'questions' as a main axis." We examined the project in terms of how the teacher and students created the narrative of mathematics lessons based on the students' continuous exploration of the subject. We conclude that a pioneering spirit on the part of teachers, which is diametrically opposed to the stereotyped approach to lesson study, should be introduced so that Japan can make a real international contribution in this area.

Key words: lesson study, lesson quality, narrative structure, classroom culture

INTRODUCTION

Since the TIMSS video study (Stigler and Hiebert, 1999), lesson study has drawn global attention as a means for improving the quality of mathematics lessons and teachers' knowledge in teaching (e.g., Corey et al., 2010; Murata, 2011; Takahashi, 2011). Today, "lesson study" has become established as a familiar term in international conferences about mathematics education. However, in contrast to the pervasiveness of procedures about how to implement lesson study, clarification of the nature of mathematics lessons in Japan is less advanced.

One problem is the lack of resolution about how a quality lesson should be produced: Murata (2011) stated that outside Japan, most schools and teachers are at early stages of adopting and implementing the innovation of lesson study. The quality of mathematics lessons is a global concern; however, the research methodology takes the form of interviews and questionnaires: there is a lack of research focus on the actual lesson process. Further, it is simplistic to assess lesson quality in terms of practices in, say, Asia or by making extreme comparisons between Eastern and Western approaches or between teacher- and student-centered styles (Shimizu, 2017).

Generally speaking, mathematics lessons in Japan can be characterized as following a problem-solving approach: the teachers emphasize the students' problem solving and attempt to teach the meaning and use of mathematical concepts by reflecting on their solutions. Stigler and Hiebert (1999) referred to this approach as "structured problem solving"; they identified the patterns or scripts in effectively taught mathematics lessons. However, such scripts should not be directly equated with an effective mathematics lesson: there is a range of quality from effective to ineffective teaching—even if that pattern is indeed adopted by most teachers in Japan. Every Japanese teacher tries to enhance their teaching skills and study the principles and mechanisms for producing successful lessons; they do so throughout their teaching carriers by means of lesson study beyond simply following the cultural pattern described above.

Thus, it is necessary to address the problems of what a quality lesson is and how it can be produced. Stigler and Perry (1988) and Shimizu (2009) reported that a quality mathematics lesson in Japan can be characterized as a coherent account of a sequence of events and activities that comprise the classes as if they were a story or drama. The story here does not signify the teacher's story: it is basically that of the students, in which their awareness of the issue and the solution process constitute the central part of the story; hopefully, that is consistent with the teacher's intention in mathematics teaching. It has been suggested that it may be possible to examine the quality of a lesson in terms of narratology (Noe, 2005) or theater theory (Hirabayashi, 2003).

We believe it is an important research task to clarify the philosophical and cultural backgrounds that underpin mathematics lessons in Japan beyond the organizations and procedures of lesson study; in particular, it is necessary to identify Japanese teachers' views of mathematics lessons toward determining the quality of those lessons. Specifically, it is important to have a deep understanding of the following: the perspectives possessed by mathematics teachers; how their views may be realized in mathematics lessons; how those views are interpreted by classroom students; and what type of classroom culture is developed by the teacher and students. The classroom culture may be partly seen through narratives that students construct after having taken a series of mathematics lessons on a particular topic (a teaching unit): that way, the students' modes of thought, values, beliefs, and attitudes for mathematics learning may appear.

In summary, in addition to descriptions of the lessons themselves, we believe it is possible to clarify the quality of mathematics lessons based on the narrative structure of those lessons from the teacher's perspective and the students' narratives after having completed a series of such lessons. To examine the actual process of achieving quality mathematics lessons, we introduce a collaborative project "mathematics lessons incorporating students' 'questions' as a main axis" (ML'Q') by a Japanese teacher study group. That project is one that Okamoto (one of the present authors) has developed over many years—in conjunction with academic colleagues and elementary and junior high school teachers—as appropriate for developing quality mathematics lessons (Okamoto and Shizuoka University attached Junior High School, 1998; Okamoto, 1999, 2001, 2010, 2011, 2013; Okamoto and Morozumi, 2008; Okamoto and Tsuchiya, 2014). The project places great emphasis on students' questions: they are used to develop mathematics lessons as students' problem-solving stories; they also allow insight into students' difficulties, their inner world, and their understanding process. The aim with ML'Q' is mathematics lessons that the teacher and students construct collaboratively based on the students' questions related to the core of those lessons.

CRITIQUE ON PROBLEM-SOLVING MATHEMATICS LESSONS

It is well known that in Japan, school lessons are conducted through problem solving. A school hour lasts 45 minutes at elementary school and 50 minutes at junior high school. Content related to a single mathematical topic typically comprises a teaching unit of about 10 hours. The flow of a 1-hour lesson in the structured problem-solving pattern has been characterized as follows (Stigler and Hiebert, 1999):

- "*Reviewing the previous lesson*. The review is conducted in the form of a brief introduction by the teacher, the teacher leading a discussion, or the students reciting the main points. Frequently, the day's lesson builds directly on the previous day's lesson—perhaps by using the methods that were developed the previous day to solve the current day's problem. . . .
- *Presenting the problem for the day.* Usually, there is one key problem that sets the stage for most of the work during the lesson.
- *Students working individually or in groups*. This almost always follows the presentation of the problem: it lasts from 1 to 20 minutes; often, it lasts 5-10 minutes. Students rarely work in small groups to solve problems before they have first worked by themselves.
- *Discussing solution methods*. After the students have worked on the problem, one or more solution methods are presented and discussed. Often, the teacher asks one or more students to share what they have found. . . . When students present methods, the teacher often summarizes and elaborates.
- *Highlighting and summarizing the major points*. Usually, at the end of the lesson or sometimes during the lesson, the teacher presents a brief lecture on the main point(s) of the lesson." (Stigler and Hiebert, 1999, pp. 79-80).

Stigler and Hiebert compared US teachers' views of mathematics as a set of skills with those of Japanese teachers as a set of relationships among concepts, facts, and procedures (p. 89). Those authors stated that teaching is a cultural activity and that it does not easily change owing to the cultural system. Moreover, Stigler and Hiebert emphasized the importance of gradually improving lesson quality by focusing on students' learning and through collaboration among teachers based on the longitudinal and continuous improvement model using lesson study in Japan.

We agree with the general findings of Stigler and Hiebert. However, it should be noted that lessons in Japan are not always developed as described above. In particular, problems arise if teachers just follow the above pattern as a kind of ritual and the lesson does not bring the students closer to understanding mathematics.

Further, the teachers' intentions and purposes in lessons differ according to whether the particular lesson is at the introductory, intermediate, or final stage within the teaching unit. All lessons are not always developed in the same way.

Bruner (2002) suggested that humans conventionalize the accidents they encounter in terms of genres. Thus, the purposes and methods of a mathematics lesson differ according to what aspects of the lesson the teacher considers problematic. For example, if a teacher problematizes students' difficulties in giving an answer quickly, the teacher may focus on how to acquire mathematical procedures and how to implement them promptly in the lesson. Alternatively, if the teacher problematizes students' difficulties in solving a problem using various approaches, the teacher may devote considerable lesson time so that the students became able to solve it. We believe that the key question is whether students can deepen their understanding of the meaning and significance of mathematical concepts by utilizing their own questions with respect to a series of mathematics lessons (teaching unit).

Okamoto and colleagues identified the problems with problem-solving lessons in Japan and described them as follows (Okamoto and Shizuoka university attached junior high school, 1998; Okamoto and Tsuchiya, 2014):

> "Such problem-solving lessons are clearly different from those in which the teacher uses drills to teach mathematics in a unilateral way. Therefore, it can be said that such lessons are an excellent type of student-driven lesson.
>
> However, problems arise if a mathematics lesson is constructed by the dichotomy of teacher as questioner and student as responder: that assumes a lack of intention to promote students as people who raise questions. Accordingly, it appears that the following views of teachers and students would be produced both explicitly and implicitly:
>
> - Students who are used to being questioned consider it normal for a teacher to ask them questions. The students do not wish to pose question themselves. Thus, the students lack strength in autonomous consciousness of task and purpose.
> - Teachers who regard students as good when they follow the problem-solving track that the teachers devised in advance. The teacher tries to make the students learn within the teacher's own subject territory." (Okamoto and Tsuchiya, 2014, pp. 12-13)

In this respect, ML'Q' is an attempt to break that dichotomy in mathematics lessons toward one based on the humanistic view of people asserting themselves by asking questions. In this regard, the questions referred to are mathematical questions that the students freely raise based on the following: their values; their interests; their experience; their existing knowledge stimulated by some mathematical information; and the situation created by the teacher as part of the learning activity. Each question relates to each student personally; thus, it is necessary to enhance those questions toward the learning themes that students commonly explore in mathematics lessons. Further, to assist students in actively learning mathematics, it is recommended that their questions and learning themes be utilized not only in one lesson but throughout the teaching unit.

Before introducing practical examples of mathematics lessons, we offer two theoretical perspectives toward understanding the quality of mathematics lessons in Japan. First, we examine the relationship between the flow of a well-presented mathematics lesson and a coherent narrative: that is because lessons in Japan have characteristics that resemble a narrative or drama (Stigler and Perry, 1988; Shimizu, 2009; Okazaki et al., 2015). Second, we assess the relationship between a mathematics lesson and classroom culture: that is because a good lesson demands a good classroom culture and vice versa. That constitutes reflexivity between learning and culture (Nickson, 1992; Cobb et al., 1993). We consider that the classroom culture includes the teacher's views of mathematics lessons, which are shared by the students. We then cover the genres for constructing mathematics lessons. The idea of genre is effective for characterizing the substantial efforts of Japanese teacher groups, which hold the same views for developing mathematics lesson from a cultural perspective.

NARRATIVE ACCOUNTS OF MATHEMATICS LESSONS AS BACKGROUND FOR ASSESSING LESSONS IN JAPAN

We start with the words of Dewey (1915, p. 141): "[Children's] interest is of a personal rather than of an objective or intellectual sort. Its intellectual counterpart is the story-form. . . . Their minds seek wholes, varied through episode, enlivened with action and defined in salient features—there must be go, movement, the sense of use and operation—inspection of things separated from the idea by which they are carried. Analysis of isolated detail of form and structure neither appeals nor satisfies." This reasoning suggests that even if all the parts that constitute a lesson structure are collected, the lesson will not attract the attention of children unless it is in story form.

Bruner (1986) distinguished two modes of thought for constructing reality: paradigmatic and narrative. He stated that the paradigmatic mode of thought "deals in general causes, and in their establishment, and makes use of procedures to assure verifiable reference and to test for empirical truth." By contrast, the narrative mode "deals in human or human-like intention and action and the vicissitudes and consequences that mark their course. It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place" (p. 13).

We believe that the narrative mode of thought helps a person envision their own niche and create their own world. If a mathematics lesson dealt only with mathematics and proceeded just by following the mathematical story, the talk in such a lesson would be filled with general principles and proofs or necessary, adequate explanations with respect to mathematical principles. We consider that mathematics lessons follow the paradigmatic mode, based on the concept of teaching (instruction, *kyo* in Japanese) but not that of fostering (*iku* in Japanese) (Hirabayashi, 1993). In addition to teaching mathematical knowledge and skills to students, teachers in Japan generally try to foster students' mathematical thinking and expressing their abilities and attitudes toward mathematical learning. Thus, during lessons, teachers need to take account of both students' observable behavior and their conscious minds.

A narrative is a speech act that is equipped with events, contexts, and time sequences; teaching may be regarded as a narrative act. A narrative act signifies "a speech act that plots the temporally distant events along a temporal order of beginning-middle-end" (Noe, 2005, p. 326). The general norm of a narrative is as follows (Bruner, 1996, pp. 94-95; Bruner, 2002, pp. 16-17). (1) First, there is an explicit or implicit prologue, establishing the ordinariness or legitimacy of the initial circumstances, in which a cast of characters, who are free agents with minds of their own, appear. (2) The story continues by leading to a breach, a violation of legitimate expectancy. Something goes awry: otherwise, there is no story to relate. (3) What follows is either a restitution of the initial legitimacy or a revolutionary change of affairs with a new order of legitimacy. However, the final scene of the story may be characterized by additional dilemmas; thus, trouble could lie ahead. (4) The narrator restores the listener to the present: the place where the narrative is told. The narrator also gives a coda: that is usually a hint or evaluation of what has transpired, where the story expresses the narrator's point of view or knowledge about the world, or indeed the truthfulness, objectivity, or even integrity of the story. If we replace the narrator with the teacher and the protagonists with the students (and also with the teacher), the above characteristics may apply to a mathematics lesson.

Here, we present two characteristics of narrative thought (Bruner, 1990, pp. 43-44). First is its inherent sequentiality and the dual nature of acting within that sequentiality. The dual nature of acting means that on the one hand, to make sense of each event, the interpreter has to grasp each event in the narrative in terms of the narrative's configuring plot; on the other hand, the plot configuration itself has to be extracted from the succession of events (sequentiality). Namely, the narrative shows that each event is mutually related to the plot. When the students act and talk, they do so in terms of comprehending the whole story of the mathematics lesson; the story in the mathematics lesson may be developed from each student's actions and words.

The second characteristic is the dual landscape, and that complements the first characteristic in making a classroom story. In the story, some events and actions in the real world occur concurrently with events in the minds of the protagonists. In a mathematics lesson, the teacher undertakes in advance the lesson development process: that is a hypothesis of the students' learning goals and in what situations and how the students might behave when learning about mathematics. However, in many cases, the teacher's hypothesis conflicts with the students' actual thoughts regarding one or more of the above points. Thus, the whole lesson process does not necessarily correspond to the thought process in each student's mind. If the teacher incorporates the students' ideas and thoughts when developing the mathematics lesson, that may demand the teacher changing the original lesson plan. However, we believe that when the teacher rethinks the teaching plan by accepting students' ideas, the dynamics of a quality mathematics lesson can be achieved. In that way, the students' hidden or potential ideas gradually become manifest: that could lead to a new frontier, and coping with that new frontier concludes the story in the mathematics lesson.

Teachers can try to take advantage of this dual nature of acting and the dual landscape as means to elicit students' questions: teachers can then develop such questions into shared learning themes for the whole class. We consider that this constitutes a challenging and important task for a teacher's professional development.

CLASSROOM CULTURE AS BACKGROUND FOR EXAMINING THE QUALITY OF MATHEMATICS LESSONS IN JAPAN

Culture can be described in terms of various dimensions and scales. In this paper, we address culture relative to the activities and customs in teaching and learning mathematics in the classroom space. Even with the same content in teaching mathematics, mathematics learning would be heavily influenced by culture;

that is because learning outcomes may differ according to social practices in the classroom (Sekiguchi, 2010).

Culture involves the complex interaction of ways of life related to some values; the substance of a culture includes symbolic systems, such as language, discourse modes, and logical and narrative modes of thinking (Bruner, 1990). Culture is also "the way of life and thought that we construct, negotiate, institutionalize, and finally (after it's all settled) end up calling 'reality' to comfort ourselves" (Bruner, 1996). Thus, a mathematics lesson can be considered a cultural practice, in which mathematics learning is promoted by the culture that students themselves foster. By comprehending the classroom culture, students can perceive changes or deficits in their learning and complement them accordingly.

We consider two views of learning in terms of individual and social perspectives. From the social viewpoint, learning can be described as an interpersonal interaction process in which people influence one another. It is human nature to form communities that consist of individuals who learn interactively. Human cultural groups intrinsically make their products collaboratively; they produce and sustain group solidarity. From the individual viewpoint, learning can be regarded as the process of building one's identity: someone who can accept the agency of others and experience themselves as an agent ("possible self") that is able to accomplish something (Bruner, 1996). We believe that learning is the process of shaping the self in the mind by overcoming the dialectic opposition between the past and future by narrating oneself.

Finally, we explain here the existence of genres in examining mathematics lessons in Japan. "Genre" signifies "culturally specialized ways of both envisaging and communicating about human condition" (Bruner, 1996, p. 136). We consider that a number of genres reflect the developing of mathematics lessons in Japan.

Generally speaking, mathematics lessons in Japan can be described as structured problem solving. We believe that most Japanese teachers have shared cultural similarities with respect to problem-solving lessons. However, as noted above, Japanese teachers do not attempt to follow a script: they conduct mathematics lessons so that students can behave autonomously and comprehensively acquire mathematical knowledge, skills, thinking, and attitudes for learning through mathematical activities; they do so based on their educational purposes and a deep understanding of teaching content. That is how the differences between teacher study groups and their various characteristics emerge, whereby each teacher group has its own particular view about mathematics lessons.

The classroom culture of mathematics lessons can be characterized by examining the teacher's views of developing the lesson. Of course, just an idealistic wish to develop a mathematics lesson on the part of the teacher does not constitute a culture. It is thus necessary to investigate also how the students regard the teacher's views about the lesson. That is because a mathematics lesson is constructed collaboratively between the teacher and students by fostering the classroom culture of teaching and learning mathematics.

Below, we describe the type of mathematics lesson one Japanese teacher group attempted to conduct beyond basic problem-solving lessons. We present as a leading case ML'Q', which Okamoto and colleagues developed to produce quality mathematics lessons.

MATHEMATICS LESSONS INCORPORATING STUDENTS' 'QUESTIONS' AS A MAIN AXIS

We first explain the teacher's perspective for constructing a mathematics lesson and the fundamental flow of learning. Thereafter, we present examples of classroom practice using ML'Q'.

Teacher's views of mathematics lessons

ML'Q' involves both a view of mathematics lessons and a teaching method. With ML'Q', the mathematics lesson is conceived as an organism, in which the lesson itself is not just a set of components: it is a whole, and its components are closely related to one another; students' questions are regarded as a crucial factor.

"Organism" generally refers to something with constituent parts that are differentiated morphologically and functionally but work together as a unity (Mori, 1995). The components of a lesson are as follows: (1) students' learning; (2) construction and development of the lesson; (3) the teacher teaching; (4) common views of scholastic abilities; (5) learning norms; and (6) selection and presentation of teaching materials. The teacher's educational ideas integrate those components as a whole (Fig. 1).

The central educational idea with ML'Q' is that people ask questions and that by asking questions humans open themselves to the world, and they can extend the world by answering their own questions

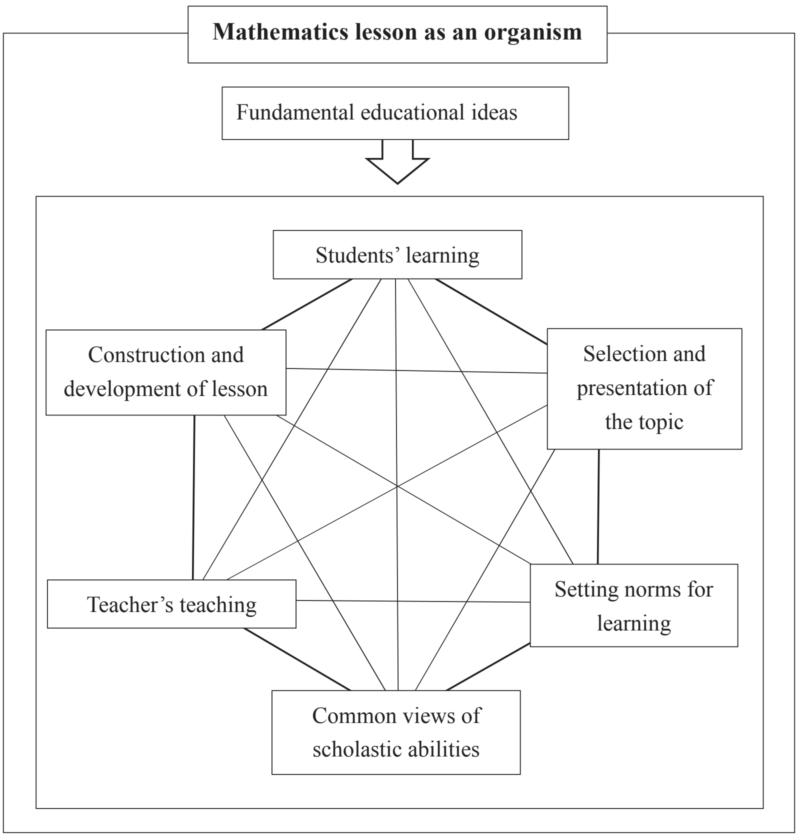

Fig. 1. View of a mathematics lessons in ML'Q'

(Bollnow, 1978). Thus, ML'Q' is an attempt to incorporate students' questions in the mathematics lesson by critically examining that lesson in terms of the dichotomy of teacher as questioner and students as responders. However, students' questions are diverse: in terms of their mathematical nature, the questions vary from irrelevant to relevant to the teaching content. Thus, to develop mathematics lessons based on students' questions, it is critical to examine the teacher's views and sense of mathematics.

We clarify here the roles of students' questions in ML'Q' (Okamoto and Tsuchiya, 2014):

- Questions proposed by each student may include the general paradigm.
- Students' questions offer the chance to stimulate the controlled, fixed status of classroom culture.
- Students' questions activate collaborative learning activity.
- Students' questions lead to forming personal identity and spiritual and moral values.

The questions proposed in mathematics lessons are intrinsically individual. However, culture acquires identity by universalizing something individual: the generality of culture does not exist by eliminating the individual's status but by connecting to very individual subjective values, beliefs, thoughts, and behaviors (Eagleton, 2000). Students' questions proposed during mathematics lessons may include what should be pursued as an essential learning theme—even if they are proposed naively. Thus, it is important for the teacher to play a role in finding the values inherent in individual questions.

Second, we consider it important to foster a classroom culture of continually changing toward higher values. Dealing with students' questions offers a good chance for the teacher who conducts a lesson according to a fixed scenario to revise the teacher's own values, beliefs, and knowledge and to reflect on the students' position. If humans' shared values, thought modes, and behavioral patterns are fixed and result in a mere facade, the stability obtained by simply obeying them results in dissatisfaction with the culture. This negative characteristic of culture can often develop in mathematics lessons where the students passively learn under the teacher's direct instruction. It is thus to be expected that students' questions will help break down the fixed cultural status. Here, it is important to note that it is not possible to judge clearly the relative merit among classroom cultures. It is essential to consider the higher value of classroom culture to which the teacher aspires and whether the culture contributes to achieving that value.

Third, communication in sharing common goals and having enthusiasm for collaboration can be achieved when the classroom members establish a common learning theme based on each student's questions. In a mathematics classroom in which questions can be freely proposed, both high- and low-ability students are able to ask questions of one another. Students who are asked a question try to respond by reflecting on their own knowledge and understanding.

Further, questions have the effect of building on a foundation that is accepted and understood by other students. The process of understanding other students is achieved through follow-up experience and follow-up construction of other students' experience and empathy with them based on their manner of expression. In a mathematics class, accepting other students' questions provides a good opportunity for follow-up construction of other students' ambivalence and difficulties by projecting oneself onto them. The questions here signify having a deep understanding of the other students and being able to explain oneself to others—rather than seeking a particular mathematical answer. This kind of communication appears to contribute to fostering a collaborative classroom culture.

Finally, asking questions leads to forming one's own identity beyond the tentative identity initially formed by others. By bringing out their own individuality, the person becomes someone who can accomplish something in the community. Regrettably, students often passively acquire much of what is valuable, beliefs toward learning, codes of conduct, and thought modes under the teacher's instruction and through exchanges with other students. Asking questions in such a situation is a signal that the student wishes to clarify vague personal doubts.

To promote the student's own identity, it is necessary to create a classroom culture that accepts and uses questions among the students. Such a class is a place and time for teaching the importance of equity, freedom, responsibility, honesty, truth, and faith and also for fostering students' spiritual and moral values.

Fundamental stages of learning

With ML'Q', a series of mathematics lessons (a teaching unit) is developed with the following stages based on the perspectives of the teacher's views of the lessons and the roles of the students' questions.

Stage 1: Mathematical activities for motivation from the teacher's orientation

In evoking and driving the students' own questions, the teacher needs to give them some information as an opportunity and to motivate them. With ML'Q', the following is implemented:

a) The teacher presents an introductory problem, and the students have to solve it. In that, the teacher makes the students aware of the central problem in the context so that they can intuitively conceive mathematics, which will be finally constructed.

b) The teacher has the students recall existing knowledge related to the new teaching content and to reflect on it. With that basis, the students explore the problematic issues.

Stage 2: Students formulate their own questions and share them with classmates

This second stage aims at capitalizing on the students asking questions. The teacher gives students the chance to propose their own questions by excluding self-limiting attitudes and without depending on others. Writing down the questions on paper may help many students in formulating their questions rather than expressing them verbally.

The questions are shared in the classroom. A list of the students' questions is made; through that, private questions may be changed to social ones, and the content may be conceived from a higher perspective. Each student can discuss the opinions about other students' questions or can see if others have the same question.

Stage 3: Setting learning themes for exploration by all classroom members based on students' questions

When the teacher tries to encourage each student's questions toward the common learning theme, adequate strategies should be taken. We adopt the following two strategies as basic teaching procedures:

1) Setting a learning theme by the teacher's lead

Using personal judgment, the teacher selects relevant items among the students' questions in terms of which are valuable and may lead to understanding the nature of mathematics. Then, it is important for the students to understand that the learning themes are selected based on their questions. That way, they feel that they have substantially contributed to the lesson and accept that the selected questions are pertinent to the learning theme.

2) Setting a learning theme by the students' lead

The teacher lets the students decide which questions should be adopted as learning themes and the order in which the themes should be addressed. The teacher acts as an advisor, giving suggestions where necessary. As a general procedure, the students first collect their questions, group them, and provide a title for each group. Next, the students decide which questions in each group should be adopted as the learning theme. The students also determine the order of the themes and prepare a learning plan.

Stage 4: Exploring and solving learning themes collaboratively and summarizing results

In this stage, learning through problem solving is conducted as a form of exploration. The difference from the standard problem-solving lesson is that the aim and reason for the students examining the learning themes is clear to them: that is because the students proposed the questions, not the teacher.

In the process of pursuing learning themes, new questions may be proposed by the students. That can lead to new learning themes: one question often produces another and then another. Owing to time constraints, it may be difficult to adopt all questions. However, we believe it is important that the teacher finds high mathematical value in the students' questions, adopts them, and enhances and deepens the students' learning.

Stage 5: Practice for establishment of procedures (explanation omitted here)

Stage 6: Dealing with remaining questions and setting new ones

After having completed all the content in a teaching unit, the teacher can direct the students' future learning. One way to develop a mathematics lesson is to use the students' remaining questions: those are ones that were not previously used owing to time considerations or the fact that they involved the content from higher-grade mathematics. We wish to utilize such questions as much as possible: otherwise, that could cause problems for the students. One way of using such questions is to have the students study them, write a report about them, and then make a presentation.

Another approach is for the teacher to use the remaining time by giving the students the chance to produce the new questions. Based on the content they have learned, the teacher asks them what they wish to explore in the next stage. That way, the learning habits of continually asking questions may be formed.

EXAMPLE OF MATHEMATICS LESSON BY ML'Q'

Backgrounds of lesson practice

The mathematics lessons we introduce here were conducted at a junior high school (hereafter, "attached school") attached to a national university in Shizuoka Prefecture, Japan. The attached school operates as a kind of experimental institution, in which the teaching and learning of mathematics are practically developed. The teachers are eager to develop professionally. The students have somewhat higher abilities than those in general public schools: they have basically acquired such abilities as reasoning, communicating, comparing ideas, and shaping their ideas following the teachers' instructions.

The teacher for our lessons was Fumito Tsuchiya. He is an experienced teacher and has conducted

ML‘Q’ continuously. He usually prepared his lessons with the following views of mathematics lessons:
Basic philosophy. The main purpose of a mathematics lesson is to cultivate students’ character building through the lesson and to foster scholastic abilities for the character building.
Student learning. Students can devise their own questions, ask them, and share them. Based on those questions, several learning themes are set and explored collaboratively toward a solution.
Lesson construction and development. The teacher constructs and develops a mathematics lesson toward clarifying the purpose of learning, i.e., for what and how the students learn. In addition to guiding them to the correct answer, the teacher orients them in thinking about the value of what they have learned.
Selection and presentation of teaching materials. The teacher sets the learning themes and teaching materials (mainly mathematical problems) by making learning plans based on the students’ questions and after gaining approval from them.
Teaching. The teacher deals with the students’ thinking flexibly without adhering to the teacher’s anticipated learning process, and the teacher activates the students’ learning activities.

We should note that the students in the attached school enjoy unfettered discussion under the school’s motto of “autonomy and independence.” The students offer their ideas and questions during lessons and have acquired the attitude of accepting and using them in the classroom. The teacher and students have developed their own classroom culture in line with the teacher’s views of mathematics lessons and the school motto.

Practice of mathematics lesson

The lessons reported here are the second section (Calculations of Expressions Using Square Roots) of a larger teaching unit (Square Roots) in ninth grade; 10 hours of lesson time are allotted for that section. The students have already learned the concept of square roots and how to compare their sizes in the first section (Basic Concept of Square Roots).

Stages 1 and 2

In the first stage of the lessons, the teacher asked as follows by way of orientation for the learning. The numbers used below are not the same as those in the textbook but are isomorphic to them.

T (teacher): The formulas show the results of calculations of square roots. What questions do you have from the results of these calculations? (Fig. 2)

Multiplication: $\sqrt{3} \times \sqrt{7} = \sqrt{21}$
Division: $\sqrt{45} \div \sqrt{3} = \sqrt{15}$
Addition: $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$, $4\sqrt{2} - 7\sqrt{2} = -3\sqrt{2}$, $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$
Subtraction: $5\sqrt{2} + \sqrt{8} = 7\sqrt{2}$

Fig. 2. Calculations used by the teacher to stimulate students’ questions

Here, it should be noted that when the students explored the above query, the teacher allowed them to refer freely to their textbook. The teacher recommends that the students use their textbook on a regular basis as an important learning resource. Likewise, the students used the textbook as needed in their learning with this teaching unit.

The following questions correspond to stage 2 (Students formulate their own questions and share them with their classmates) (Fig. 3). Below, the numbers used in the students' questions differ somewhat from those presented by the teacher; that is because the students used the numbers that appeared in the textbook. However, both sets of numbers are isomorphic with each other.

List of questions proposed by the students

1) Why does $\sqrt{2} \times \sqrt{5}$ come to be $\sqrt{10}$?
2) Why is $2\sqrt{2}$ equal to $\sqrt{8}$?
3) Why is it that $\sqrt{2} + \sqrt{3}$ cannot be $\sqrt{5}$?
4) Why does $3\sqrt{2} + 4\sqrt{2}$ come to be $7\sqrt{2}$?
5) What is the answer to $\sqrt{2} + \sqrt{3}$?
6) When we square $\sqrt{2} + \sqrt{3}$, it is $2 + 3$. When we bring it back to the square root and make it $\sqrt{5}$, what is wrong?
7) How can we add when the numbers within the square root symbol are the same?
8) How are square root calculations similar to those with algebraic expressions?
9) We cannot add the numbers within the square root symbols, but how can we multiply them?

Fig. 3. List of questions proposed by students

The students' proposed questions at this stage are very diverse. The questions are produced promptly and subjectively based on the shown results of calculations of addition, subtraction, multiplication, and division of square roots. These are the questions that should be explored: they address mathematical distinctions in calculating square roots by comparison with calculations of rational numbers, which the students have already learned. For example, exploring question 5 ("What is the answer to $\sqrt{2} + \sqrt{3}$?") may lead to understanding the dual nature of the expression of $\sqrt{2} + \sqrt{3}$ as the expression of addition as well as of the number itself.

Stage 3

In stage 3, based on the students' consensus, the teacher sets the learning plan around the learning themes dealt with in the section Calculations of Expressions Using Square Roots (Fig. 4).

Learning plan of the second section, Calculations of Expressions Using Square Roots

- Learning theme 1: Why does $\sqrt{2} \times \sqrt{5}$ come to be $\sqrt{10}$?
  - Division of square roots
  - Practice with multiplication and division of square roots
- Learning theme 2: Why is $2\sqrt{2}$ equal to $\sqrt{8}$?
- Learning theme 3: Why $\sqrt{2} + \sqrt{3}$ cannot be $\sqrt{5}$
  - Practice with addition and subtraction of square roots

(allotted time: 10 hours)

Fig. 4. Lesson plan based on students' questions

Stage 4

Learning theme 1

Under the above learning plan, the students in stage 4 started collaboratively exploring learning theme 1. In the following sections, we present all the students as "S" and the teacher as "T": in our field notes, we did not accurately record the names of the students who made the statements.

T: Why does $\sqrt{2} \times \sqrt{5}$ come to be $\sqrt{10}$?
S: $2 \times 4 = 8$. $2 = \sqrt{4}$, $4 = \sqrt{16}$, $\sqrt{4} \times \sqrt{16} = \sqrt{64}$, $\sqrt{64} = 8$. So, $\sqrt{2} \times \sqrt{5} = \sqrt{10}$
T: What do you think of that idea?
S: I am not sure. In this case, $\sqrt{2}$ and $\sqrt{5}$ are not integers. I am not sure that it comes to $\sqrt{10}$.
T: Are there any other opinions about $\sqrt{2}$ and $\sqrt{5}$?
S: (One student raised his hand.) $(\sqrt{2} \times \sqrt{5})^2 = (\sqrt{2})^2 \times (\sqrt{5})^2 = 10$. So, $\sqrt{2} \times \sqrt{5} = \sqrt{10}$
T: Can you add something to that idea?
S: I can generalize it. $(\sqrt{x} \times \sqrt{y})^2 = (\sqrt{x})^2 \times (\sqrt{y})^2 = xy$. So, $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
S: Why do you square it?
T: He expressed his opinion about "why square it?" What does everybody think about that?
S: If we square it, we can think of it without the square root.
T: Anything else?
S: To square means that we can go to the world of integers. So it is better to change it.

The teacher confirmed with the students that $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ was the summary of those exploratory process.
(Here, we omit descriptions of the exchanges between the teacher and students about learning theme 2.)

Learning theme 3

Regarding learning theme 3 ("Why $\sqrt{2} + \sqrt{3}$ cannot be $\sqrt{5}$"), the following opinions were presented by the students:

T: Today, let's hear your ideas about why $\sqrt{2} + \sqrt{3}$ cannot be $\sqrt{5}$.
S: $\sqrt{2}$ = 1.414. $\sqrt{3}$ = 1.732. 1.414 + 1.732 = 3.146. $\sqrt{5}$ = 2.236. So $\sqrt{2} + \sqrt{3}$ can't be $\sqrt{5}$.
T: Ah. What does everyone think about that?
S: $\sqrt{2}$ is a square root of 2. $\sqrt{3}$ is a square root of 3. $\sqrt{5}$ is a square root of 5. So $\sqrt{2} + \sqrt{3}$ can't be $\sqrt{5}$.
S: I don't understand that. I don't understand why $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ is right but $\sqrt{2} + \sqrt{3} = \sqrt{5}$ is not allowed.
T: Now, the opinion about why $\sqrt{2} \times \sqrt{3}$ is all right but $\sqrt{2} + \sqrt{3}$ is not allowed has been made. I think that is an important point. What do you all think about it?
S: $(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$. It is not 5. So it can't be $\sqrt{5}$.
(Several students raised their hands.)
T: Do you have anything to say?
S: A square where the lengths of the sides is $\sqrt{2} + \sqrt{3}$ has a different area from a square where the

lengths of the sides is $\sqrt{5}$. So it can't be $\sqrt{5}$. (Fig. 5)

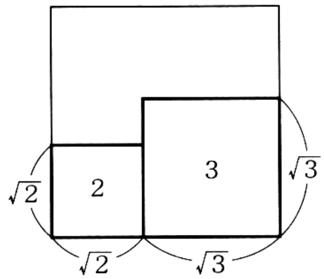

Fig. 5. Figure for explaining that $\sqrt{2}+\sqrt{3} \neq \sqrt{5}$

T: I also think that $\sqrt{2}+\sqrt{3}$ can't be $\sqrt{5}$. But do you think that the question about why $\sqrt{2}+\sqrt{3}$ can't be $\sqrt{5}$ has been fully answered?

(The teacher set a time for the students to explore the task individually.)

T: So why is it that $\sqrt{2}+\sqrt{3}$ can't be $\sqrt{5}$?

S: Perhaps it is that $\sqrt{2}$ and $\sqrt{3}$ are different things.

S: What do you mean by "different things"?

T: What does everyone think about that opinion?

S: It means that $\sqrt{2}$ and $\sqrt{3}$ don't have a common number.

S: What do you mean by a "common number"?

S: It means that one can't be adjusted to the other.

T: Now, the word "adjusted" has appeared. Does anybody have any questions related to this?

S: The reason $\sqrt{2}$ and $2\sqrt{2}$ can be added is that they can adjust to $\sqrt{2}$. That's why we can add them.

S: There is a common $\sqrt{2}$.

T: So $3\sqrt{2}$ plus $4\sqrt{2}$ is $7\sqrt{2}$. Then, what is the answer to $\sqrt{2}+\sqrt{3}$, which is one of the questions you raised?

S: $\sqrt{2}+\sqrt{3}$ is just $\sqrt{2}+\sqrt{3}$.

T: Whether there is a common number or not determines whether we can use addition with a simple number.

S: Huh? What is addition?

Discussion took place in the classroom about the question of what addition might be. Several students were puzzled by the word "addition" that the teacher had casually used. That introduced a new learning theme. So the teacher and students set learning theme 4 as "What is addition?"

<u>Learning theme 4</u>

T: $2+3$ is addition. Right? Why does this expression become the simple number 5?

S: Because there is a common number, 1.

T: What is this common number?

S: The numbers can be adjusted to the common unit, 1.

T: The words "common number" and "common unit" have appeared.

S: So does $1/2 + 1/3$ become the simple number $5/6$?

S: I think they have the common number of $1/6$.

S: Yes. Fractions can produce the common number by reduction.

T: To reduce means to create a common number. If $\sqrt{2}$ and $\sqrt{3}$ can produce a common number in reduction, it would be a simple number. However, in this case, there is no way to create a common number.

S: So why can multiplication make $\sqrt{2} \times \sqrt{5}$ equal $\sqrt{10}$ even though they have no common number?

S: I've lost track here. What do you mean by $\sqrt{5}$ "items" of $\sqrt{2}$?

S: What is multiplication?

Through exploring addition, a new learning theme appeared: learning theme 5, "What is multiplication?"

While exploring learning theme 4, the teacher and students discussed the fundamental unit when constructing integers and fractions and commensurability; they did so when they reflected on common features between additions of integers and those of fractions. Following the teacher's summary as to why $\sqrt{2}$ and $\sqrt{3}$ cannot create a common number, the incommensurability of $\sqrt{2}$ and $\sqrt{3}$ was intuitively confirmed. Further, it was also confirmed that the addition of $\sqrt{2} + \sqrt{3}$ expresses both the methods of calculation and a number as a result of the calculation, as it has already been clarified under the learning theme 3. The students' understanding of the dual nature of $\sqrt{2} + \sqrt{3}$ caused them to reflect on the meaning of the multiplication of square roots. That was represented by the comment, "What do you mean by $\sqrt{5}$ 'items' of $\sqrt{2}$?" It was an introspective question—one that encouraged the students to reflect on what they had hitherto believed to be correct.

<u>Learning theme 5</u>

T: Now, the question "What is multiplication?" emerged when we discussed the question "What is addition?" Today, let's talk about "What is multiplication?" That is something that you investigated for homework.

S: I think that $\sqrt{a}$ belongs to a different world. Suppose there is a line and a plane. The numbers we have learned in the past belong to the world of lines, which don't have any width. That's the world of $\mathrm{y} = \mathrm{ax}$. But a square root has length and width.

S: I understand. Addition is only the world of length, the world of line.

T: You related the world of numbers to the world of lines and planes.

S: Multiplication has width. It is the world of area. If the areas are 1, 2, and 3, $\sqrt{1}$, $\sqrt{2}$, and $\sqrt{3}$ are produced.

T: I see. Does anyone have any questions?

S: I understand the difference between addition and multiplication. But I am not sure why multiplication and division can be calculated as they are—even though there's no common number.

T: How about the question of the difference between multiplication and addition?

S: $(a + b)^2 \neq a^2 + b^2$. But it is because $(a \times b)^2 = a^2 \times b^2$.

S: When we square additions, we have to use the multiplication formula. But when we square multiplications, we can square them as they are.

S: $a + a = 2a$. That remains a linear expression. But $a \times a = a^2$. That changes to a quadratic expression. Square roots can change the dimension.

T: That's great. The expression "change the dimension" was marvelous.

S: We could say that multiplication is a bridge to exceed the dimension.

T: When we discussed the difference between addition and multiplication, the expressions "change the dimension" and "exceed the dimension" appeared. The opinion about $\sqrt{a}$ belonging to a different world is also very important.

Based on the discussions about learning themes 4 and 5 (2 hours for both), the following points were confirmed:

- Addition can make a single number if there is a common number.
- $2 + 3$ has the common number of 1.
- Reduction of fractions can produce the common number.
- Addition is a world of length, a world of line.
- Multiplication is a world of area.
- The reason that $\sqrt{2} \times \sqrt{5}$ can be changed into $\sqrt{10}$ despite a lack of a common number is that $(a \times b)^2 = a^2 \times b^2$ is true.

In interpreting the reason for $\sqrt{2} \times \sqrt{5} = \sqrt{10}$, the students used geometric models or provided an explanation using calculation laws that had already been recognized as correct. Of course, the idea that the calculation law $(a \times b)^2 = a^2 \times b^2$ can be applied to the extended world of numbers is analogical reasoning; it thus needs to be discussed mathematically.

The students expressed the following opinions after the final lesson in the unit dealing with square roots:

- I understood the meaning of $\sqrt{a}$. I could feel its significance. At that point, I understood the frontier spirit of pioneering mathematics. I also felt the infinite extension of numbers.
- $\sqrt{a}$ is like a huge vessel that can contain all numbers.
- Through exploring $\sqrt{a}$, I see that mathematics can be infinitely extended. The dimensions are different. I expect that there may be different numbers from these.
- The world I knew in the past was tied to the base of 1. In a world that has different bases, the numbers themselves are rather different. We have to identify where the numbers are connected and where they are not—though both are important.
- $\sqrt{a}$ are not just numbers that exist infinitely: broadly speaking, they are a bridge to connect with various formulas.
- Multiplication is a bridge between dimensions.

DISCUSSION

Narrative characteristics of the practice

In this paper, we have shown that how a mathematics lesson can be constructed as a coherent narrative (created by the teacher and students through their interactions) is a necessary viewpoint for examining the quality of problem-solving lessons in Japan. It is also essential for clarifying teachers' efforts to improve their teaching practices in terms of their cultural background.

With ML'Q', the teacher shares their own views of mathematics lessons (focused on character building) with the students. The teacher fosters an original classroom culture, and the lessons are created collaboratively by the teacher and students based on the students' questions. The outcome is that through ML 'Q', a rich story emerges. A crucial characteristic is that students' questions elicit further questions.

We observed two characteristics of narrative thinking in the lesson practice conducted using ML'Q'. One characteristic is the facet of time sequentiality and the dual nature of acting in that sequentiality. For example, when it became evident that addition can be produced if there is a common number, one student asked why multiplication could make $\sqrt{2} \times \sqrt{5} = \sqrt{10}$ despite the fact that they have no common number. Another student asked what was meant by $\sqrt{5}$ "items" of $\sqrt{2}$. We believe that those questions were posed with respect to the total story of the lesson. In other words, if the story had been lacking for the lesson, such utterances would not have been made. Conversely, triggered by those questions, a new learning theme ("What is multiplication?") emerged; through that, a whole new story for the lesson was developed. We consider that this reveals one aspect of the dual nature of acting.

The second characteristic is the facet called dual landscape. That facet is evident as the discrepancy between the teacher's expectation and the students' utterances and thinking. Although the teacher may assume some of the students' questions before conducting the lesson, it is difficult to predict all of them, in particular those which emerge from the students' exploration process at the stage 4. Thus, the gaps between the teacher's expectation of the lesson process and the students' thinking can be exposed in the mathematics lessons incorporating students' questions. If the teacher recognizes that such a gap exists, it is possible to address the teacher's expectation of the lesson process: the teacher can lead a new story by adopting the students' thinking toward a new conclusion.

These two narrative characteristics can often be found with lessons that use ML'Q'. They can be considered critical features of ML'Q'.

Mathematics lesson views and classroom culture

It is our view that a mathematics lesson is an organism: the situations in the lesson are constructed by the teacher using many components; those components are influenced by the teacher's basic, interconnected educational ideas. Those components constitute the whole, which is the teacher's view of the mathematics lesson: what a mathematics lesson should be.

According to the views of Tsuchiya (the teacher introduced here), the central component of his fundamental educational ideas is character building. The other components include asking questions related to his educational ideas, dealing with students' questions from the perspective of character building, and motivating his students. It should be particularly noted that Tsuchiya's views about mathematics lessons served as a background for developing the lessons detailed above.

We should emphasize that owing to the particular characteristics of the classroom culture of the attached school, the students were activated in their questioning; they made use of that activation at various times in the lessons. Questions play a major role when developing mathematics lessons in a classroom culture where each student can accept other students' questions and collaborate in creating a chain of questions. As detailed above, one student's personal question ("What is multiplication?") developed into a general learning theme that was explored by the whole class. We also observed that on many occasions, the students' questions contributed to developing a mathematics lesson that activated their collaborative investigations.

Enhancing mathematics lessons beyond problem solving with the cultural script

All Japanese teachers share the basic educational purposes of enhancing students' thinking, expressive abilities, and positive attitudes for learning ("formal discipline") through problem-solving lessons. However, we believe that lessons conducted procedurally by just following a cultural script lead to problems. A range of quality— from effective to ineffective teaching—results if such a pattern is adopted. Thus, we intended to show the efforts of teacher groups toward overcoming current problems and exploring a new approach to lessons using the idea of genre. In fact, various teacher study groups attempt to construct the new mathematics lessons according to their educational purposes.

ML'Q' problematizes the dichotomy of teacher as questioner and students as responders; Such a situation does not allow students to sufficiently develop their own problem-solving skills and thus defeats the educational goal. ML'Q' aims at developing mathematics lessons in which the students formulate their own questions, enhance their questions toward common learning themes in the classroom, and investigate them collaboratively. The learning at stages 2 and 3—and the process of exploring questions newly expressed from students during the solution process at stages 4—are a particular characteristic of ML'Q'. That approach is not seen in standard problem-solving lessons.

To develop this type of mathematics lesson, teachers need to have the ability to accept students' questions and incorporate them into the students' learning process so that the lesson can be a mathematically suitable classroom narrative—as evident above with ML'Q'. Though not easy, acquiring such a teaching perspective and skills will be valuable for teachers: it will allow them to promote the development of their students.

FINAL REMARKS

In this paper, we have attempted to clarify how mathematics teachers in Japan try to develop quality lessons beyond the organizational details and procedures for lesson study. In particular, we do not consider mathematics lessons good if they adhere to traditional patterns of teaching instruction and demonstration and follow-up student practices. We likewise do not regard lessons in Japan as good if they simply follow the cultural script of structured problem solving as some kind of standard procedure. Rather, Japan's mathematics teachers should aim to develop lessons where the students—as protagonists—continue to be conscious of their tasks. In that way, the students can acquire knowledge and skills; they are also able to enhance their abilities for thinking and expression and improve their attitude for learning mathematics.

We have focused on narrative structures of mathematics lessons and the classroom culture centered on the teacher's views of such lessons toward achieving high lesson quality. Creating a structure for mathematics lessons should not be based on the concept of teaching but on that of fostering. How teachers and students construct the narrative for mathematics lessons based on the students' continuous exploration of the subject is an important issue when examining mathematics teaching in Japan. We examined sequentiality and the dual nature of acting within that sequentiality; we also addressed the complementary characteristic of the dual landscape: we identified both as necessary perspectives when clarifying the genesis and development of the narrative for mathematics lessons. The framework of narrative structure appears effective toward giving lessons in Japan a more international character; it is also necessary for Japanese to recognize how their mathematics lessons may be developed.

Teacher groups in various parts of Japan make efforts to develop quality mathematics lessons. They study such lessons on a daily basis, which is a common characteristic of problem solving. But they also have original views about those lessons. The content of the lessons is disclosed to the public; the teachers conduct exchanges with one another, and they continually try to develop new lessons for students in the present age.

We believe that teacher studying and developing abilities with regard to mathematics lessons as well as their ambitions and attitudes in creating new lessons is a key characteristic in Japan. That characteristic is diametrically opposed to stereotyped approaches to lesson study. Teachers' pioneering spirit should be encouraged both within and outside Japan: they can make real progress in that direction through lesson study.

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